9 Dynamic tables

Summer Term 2011

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Dynamic tables

Problem:

Maintenance of a table under the operations insert and delete such that

- the table size can be adjusted to the number of elements
- a fixed portion of the table is always filled with elements
- the costs for *n* insert or delete operations are in $O(n)$.

Organisation of the table: hash table, heap, stack, etc.

Load factor α_T : fraction of table spaces of *T* which are occupied.

Cost model:

Insertion or deletion of an element causes cost 1, if the table is not filled yet. If the table size is changed, all elements must be copied.

Initialisation

```
class dynamicTable {
       private int [] table;
       private int size;
       private int num;
       dynamicTable () {
           table = new int [1];       // initialize empty table
           size = 1;
           num = 0;
       }
```

Expansion strategy: insert

Double the table size whenever an element is inserted in the fully occupied table!

```
public void insert (int x) {
      if (num == size) {
          int[] newTable = new int[2*size];
          for (int i=0; i < size; i++)
              insert table[i] in newTable;
          table = newTable;
          size = 2*size;
      }
      insert x in table;
      num = num + 1;
   }
```

Insert operation in an initially empty table

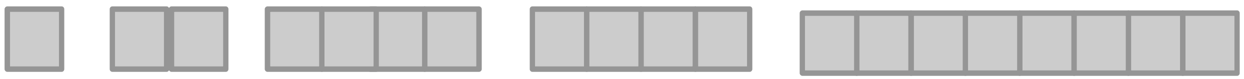
t_i = cost of the i-th insert operation

Worst case:

t_i = 1, if the table was not full before operation i
$t_i = (i - 1) + 1$, if the table was full before operation i
Hence, n insert operations require costs of at most

$$\sum_{i=1}^{n}(i) = O(n^2)$$

Amortized worst case:
Aggregate analysis, accounting method, potential method



Amortized cost

- Let the real cost of the ith insertion be t_i. We want to define the amortized cost a_i such that

$$\sum a_i \geq \sum t_i \in O(n).$$

Potential method

T table with

- $k = T.num$ elements and
- $s = T.size$ spaces

Potential function

$$\phi(T) = 2k - s$$

We also write ϕ_i *for* $\phi(T)$ after the ith insert operation.

We now define $a_i = \phi_i - \phi_{i-1} + t_i$.

Properties of the potential function

Properties

- $\phi_0 \;=\; \phi(T_0) = \phi$ (empty table) = -1

- For all $i \geq 1$: $\phi_i = \phi\,(T_i) \geq 0$
  Since $\phi_n - \phi_0 \geq 0$, $\Sigma\, a_i$ is an upper bound for $\Sigma\, t_i$

- Directly before an expansion, $k = s$,
  hence $\phi(T) = k = s.$

- Directly after an expansion, $k = s/2$,
  hence $\phi(T) = 2k - s = 0$.

Amortized cost of insert (1)

k_i = # elements in T after the i-th operation

s_i = table size of T after the i-th operation

Case 1: [i-th operation does not trigger an expansion]

$$a_i = \phi_i - \phi_{i-1} + t_i = (2k_i - s_i) - (2k_{i-1} - s_{i-1}) + t_i =$$

$$2(k_i - k_{i-1}) - (s_i - s_{i-1}) + t_i =$$

$$2 - 0 + 1 \le 3.$$

Amortized cost of insert (2)

Case 2: [*i*-th operation triggers an expansion]

$$a_i = \phi_i - \phi_{i-1} + t_i = (2k_i - s_i) - (2k_{i-1} - s_{i-1}) + t_i =$$

$$2(k_i - k_{i-1}) - (s_i - s_{i-1}) + i =$$

$$2(1) - (2s_{i-1} - s_{i-1}) + i =$$

$$2 - s_{i-1} + i =$$

$$2 - (I - 1) + i \leq 3.$$

Insertion and deletion of elements

Now: contract table, if the load is too small!

Goals:

(1) Load factor is always bounded below by a constant

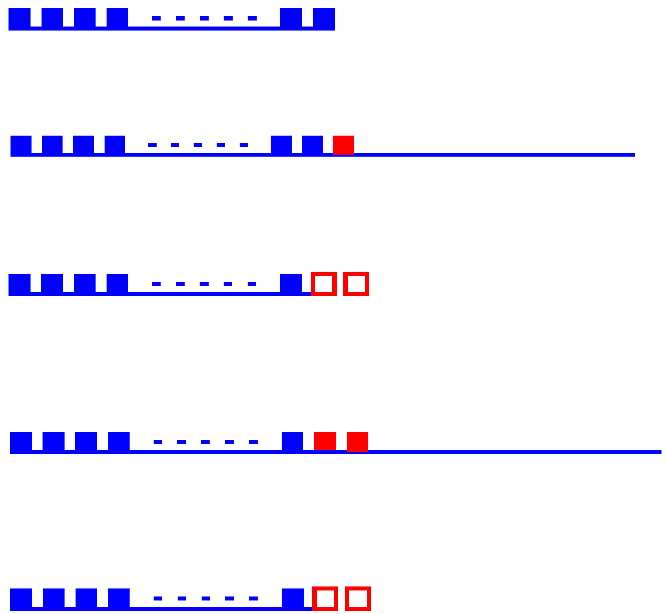
(2) Amortized cost of a single insert or delete operation is constant.

First attempt:

- Expansion: same as before
- Contraction: halve the table size as soon as table is less than ½ occupied (after the deletion)!

„Bad“ sequence of insert and delete operations

	Cost
n/2 times insert (table fully occupied)	3 *n/2*
I: expansion	*n/2* + 1
D, D: contraction	*n/2* + 1
I, I : expansion	*n/2* + 1
D, D: contraction	



Total cost of the sequence
In/2,I,D,D,I,I,D,D,... of length n: $\Omega(n^2)$

Second attempt

Expansion: (as before) double the table size, if an element is inserted in the full table.

Contraction: As soon as the load factor is below ¼, halve the table size.

Hence:

At least ¼ of the table is always occupied, i.e.

$$\tfrac{1}{4} \leq \alpha(T) \leq 1$$

Cost of a sequence of insert and delete operations?

Analysis: insert and delete

k = T.num, s = T.size, α *= k/s*

Potential function ϕ

$$\boxed{\phi(T) = \begin{cases} 2k - s, \text{if } \alpha \geq 1/2 \\ s/2 - k, \text{if } \alpha < 1/2 \end{cases}}$$

Homework: Check $\sum a_i \geq \sum t_i$

Analysis: insert and delete

$$\phi(T) = \begin{cases} 2k - s, \text{if } \alpha \geq 1/2 \\ s/2 - k, \text{if } \alpha < 1/2 \end{cases}$$

Directly after an expansion or contraction of the table:

$s = 2k$, hence $\phi(T) = 0$

insert

i-th operation: $k_i = k_{i-1} + 1$

Case 1: $\alpha_{i-1} \geq ½$

Case 2: $\alpha_{i-1} < ½$

Case 2.1: $\alpha_i < ½$

Case 2.2: $\alpha_i \geq ½$

insert

Case 2.1: $\alpha_{i-1} < ½$, $\alpha_i < ½$ (no expansion)

Potential function ϕ

$$\phi(T) = \begin{cases} 2k - s, \text{ if } \alpha \geq 1/2 \\ s/2 - k, \text{ if } \alpha < 1/2 \end{cases}$$

insert

Case 2.2: α_{i-1} < ½, $\alpha_i \geq$ ½ (no expansion)

Potential function ϕ

$$\phi(T) = \begin{cases} 2k - s, \text{if } \alpha \geq 1/2 \\ s/2 - k, \text{if } \alpha < 1/2 \end{cases}$$

delete

$k_i = k_{i-1} - 1$

Case 1: $\alpha_{i-1} < ½$

Case 1.1: deletion causes no contraction

$s_i = s_{i-1}$

Potential function ϕ

$$\phi(T) = \begin{cases} 2k - s, \text{if } \alpha \geq 1/2 \\ s/2 - k, \text{if } \alpha < 1/2 \end{cases}$$

delete

$k_i = k_{i-1} - 1$

Case 1: $\alpha_{i-1} < ½$

Case 1.2: $\alpha_{i-1} < ½$ deletion causes a contraction

$2s_i = s_{i-1}$

$k_{i-1} = s_{i-1}/4$

Potential function ϕ

$$\phi(T) = \begin{cases} 2k - s, \text{if } \alpha \geq 1/2 \\ s/2 - k, \text{if } \alpha < 1/2 \end{cases}$$

delete

Case 2: $\alpha_{i-1} \geq ½$ no contraction

$$s_i = s_{i-1} \quad k_i = k_{i-1} - 1$$

Case 2.1: $\alpha_{i-1} \geq ½$

Potential function ϕ

$$\phi(T) = \begin{cases} 2k - s, \text{if } \alpha \geq 1/2 \\ s/2 - k, \text{if } \alpha < 1/2 \end{cases}$$

delete

Case 2: $\alpha_{i-1} \geq ½$ no contraction

$$s_i = s_{i-1} \quad k_i = k_{i-1} - 1$$

Case 2.2: $\alpha_i < ½$

Potential function ϕ

$$\phi(T) = \begin{cases} 2k - s, \text{if } \alpha \geq 1/2 \\ s/2 - k, \text{if } \alpha < 1/2 \end{cases}$$