4 Balanced Trees, AVL Trees

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Balanced trees

- A class of binary search trees is balanced, if each of the three dictionary operations
  - find
  - insert
  - delete

  of keys for a tree with n keys can always (in the worst case) be carried out in $O(\log n)$ steps.

- Possible balancing conditions:

  - height condition →AVL-Trees
  - weight condition → BB[α] -Trees
  - structural conditions → B-Tree

- Goal: Height of a tree with n keys is always in $O(\log n)$.

AVL trees

Developed by Adelson-Velskii and Landis (1962)

- Search, insertion and deletion of a key in a randomly created standard search tree with *n* keys can be done, on average, in $O(\log_2 n)$ steps.

- However, the worst case complexity is $\Omega(n)$.

- Idea of AVL trees: modified procedures for insertion and deletion, which prevents the tree from degenerating.

- Goal of AVL trees: height is in $O(\log_2 n)$ and search, insertion and deletion can be carried out in logarithmic time.

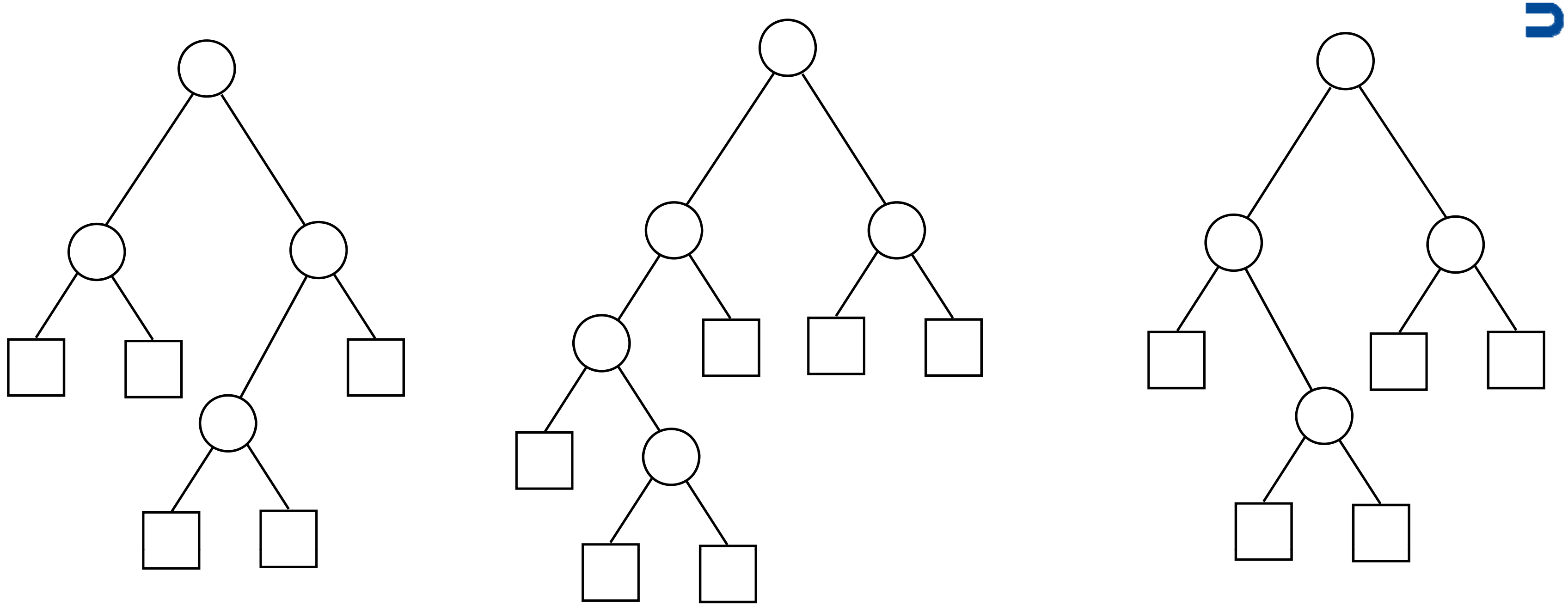
Definition of AVL trees

Definition: A binary search tree is called AVL tree or height-balanced tree, if for each node v the height of the right subtree $h(T_r)$ of v and the height of the left subtree $h(T_l)$ of v differ by at most 1.

Balance factor:

$$bal(v) = h(T_r) - h(T_l) \in \{-1, 0, +1\}$$

Examples



AVL tree | not an AVL tree | AVL tree

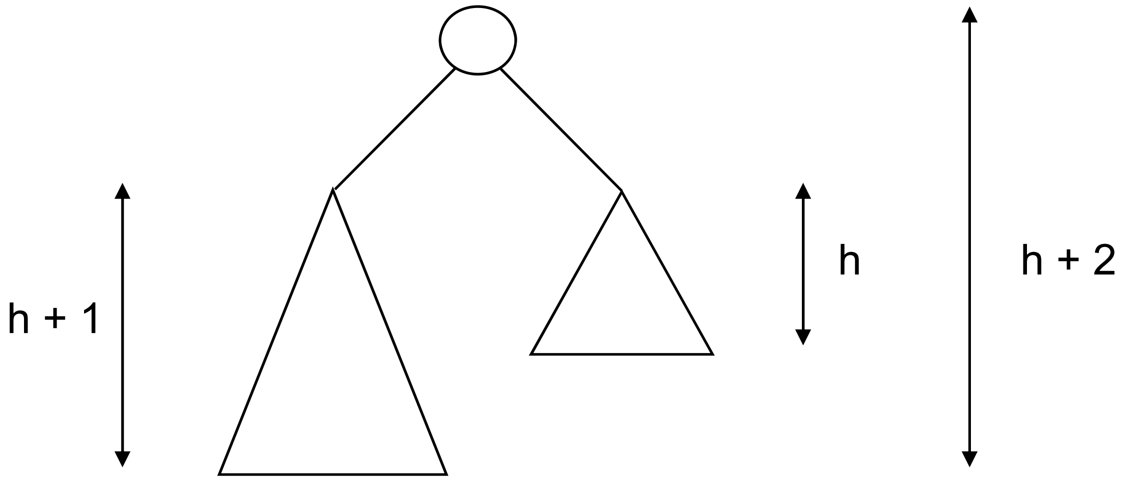
Properties of AVL trees

- AVL trees cannot degenerate into linear lists.
- AVL trees with n nodes have a height in $O(\log n)$.

Apparently:

- An AVL tree of height 0 has 1 leaf
- An AVL tree of height 1 has 2 leaves
- An AVL tree of height 2 with a minimal number of leaves has 3 leaves
- ...
- How many leaves does an AVL tree of height h with minimal number of leaves have?

Minimal number of leaves of AVL trees of height *h*

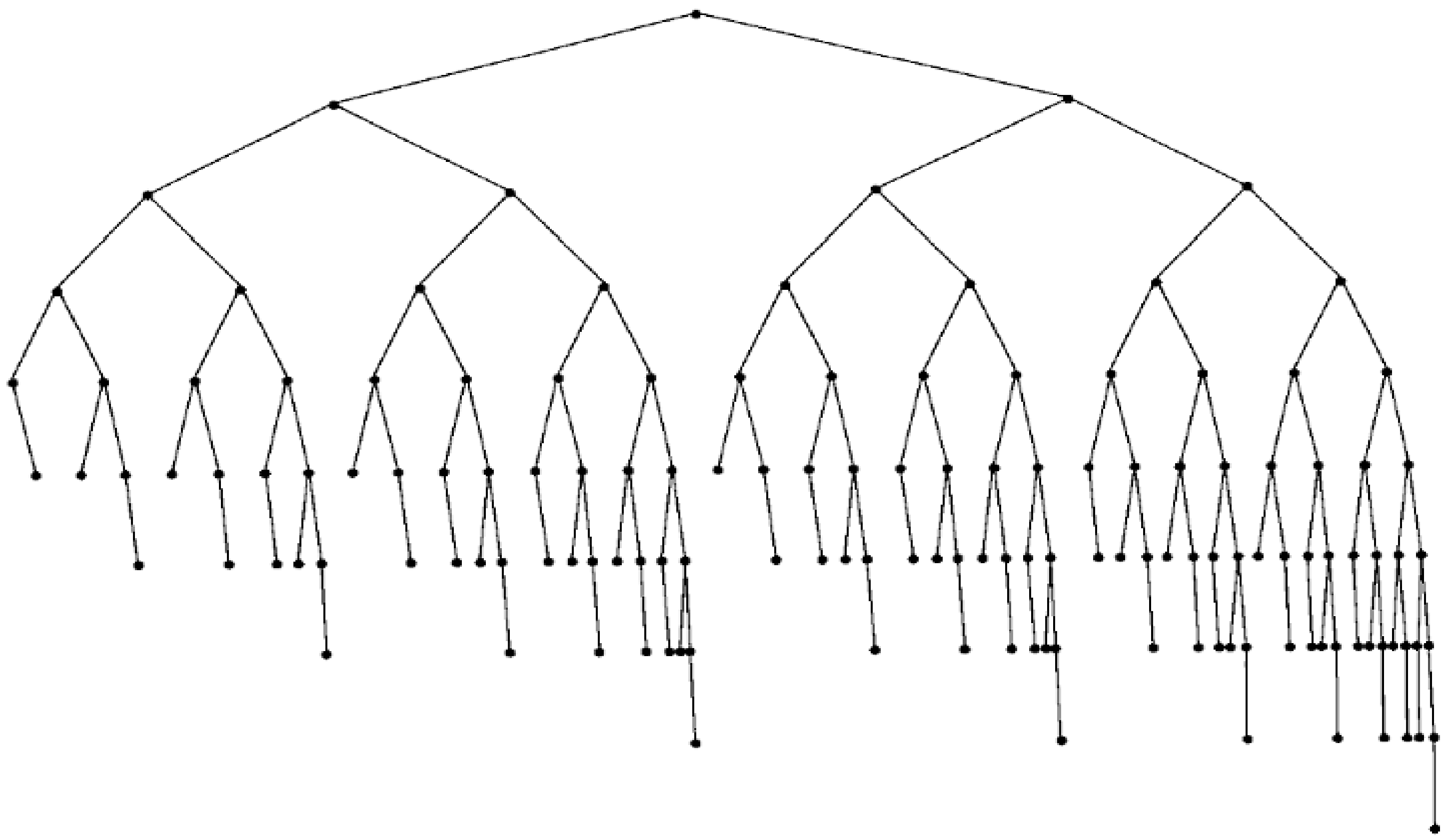


Hence: An AVL tree of height h has at least F_{h+2} leaves, where

$$F_0 = 0$$
$$F_1 = 1$$
$$[\ F_2 = 1 \text{ (height 0)}$$
$$F_3 = 2 \text{ (height 1)}\ ]$$
$$F_{i+2} = F_{i+1} + F_i$$

F_i is the i-th Fibonacci number.

Minimal AVL tree of height 9



55 leaves are shown, but all nodes with just one child (there are 34 such nodes) would need another leaf. This gives 89 leaves in total. In fact $F_{11} = 89$.

Height of an AVL tree

Theorem: The height *h* of an AVL tree with *n* leaves (and *n*-1 internal nodes) is at most $c \cdot log_2\ n + 1$, i.e.

$$h \leq c \cdot log_2\ n + 1, \text{with a constant c.}$$

Proof: For the Fibonacci numbers we know

$$F_h = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{h+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{h+1}\right) \approx 0.7236... \cdot 1.618...^h$$

Since $\quad n \geq F_{h+2} \approx 1.894... * 1.618...^h$

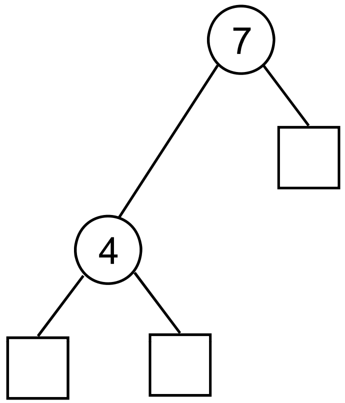
we get

$$h \leq \frac{1}{\log_2 1.618\ldots} * \log_2 n - \frac{\log_2 1.894\ldots}{\log_2 1.618\ldots} \leq 1.44\ldots \log_2 n + 1$$

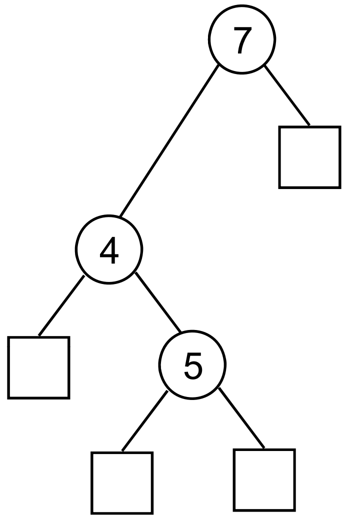
Insertion in an AVL tree

- For each modification of the tree we have to guarantee that the AVL property is maintained.

Original situation:



After inserting key 5:



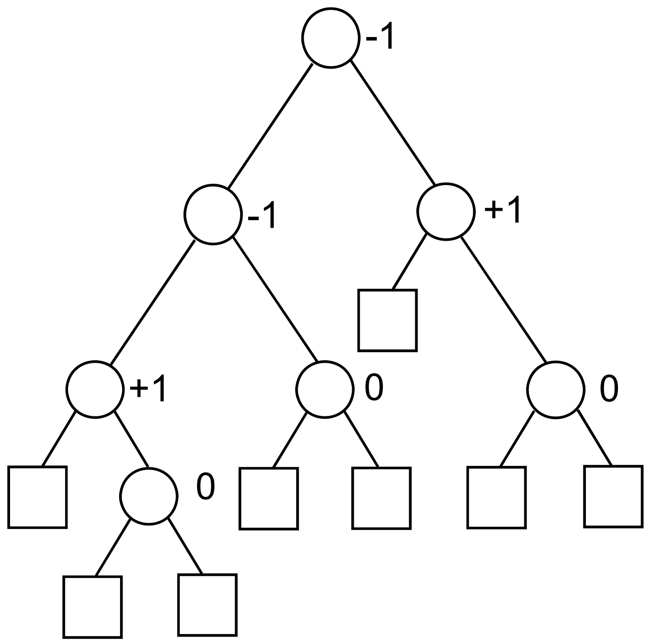
Problem: How can we modify the new tree such that it will be an AVL tree?

Storing the balance factors in the node

In order to restore the AVL property, it is useful to store the balance factor in each node:

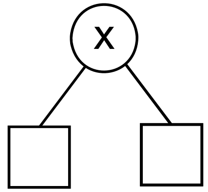
$$bal(p) = h(p.right) - h(p.left) \in \{-1, 0, +1\}$$

- Example:



Different situations of insertion in an AVL tree

1. The tree is empty: create a single node with two leaves, store *x* in it. Done!



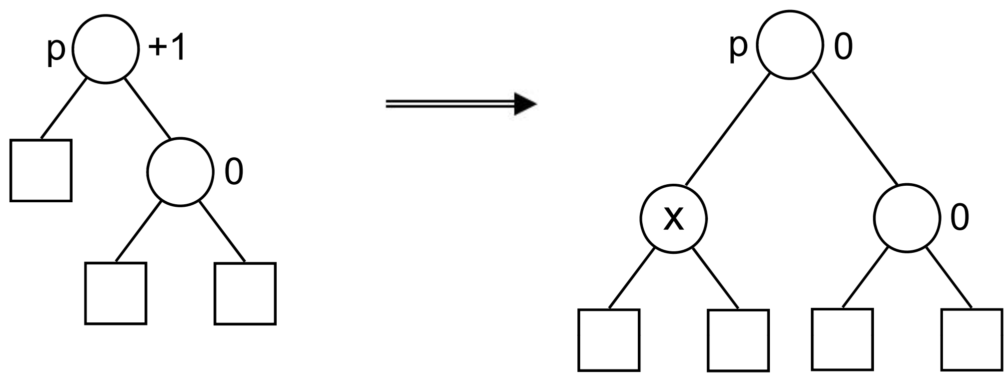
2. The tree is not empty and the search ends in a leaf.
   Let node *p* be the parent of the leaf where the search ended.

   Since bal(p) $\in$ {-1,0,1}, we know that either

- the left child of *p* is a leaf, the right child is not (case 1, “1”) or
- the right child of *p* is a leaf, the left child is not (case 2, “-1”) or
- both children of *p* are leaves (case 3, “0”).

Overall height unchanged (1)

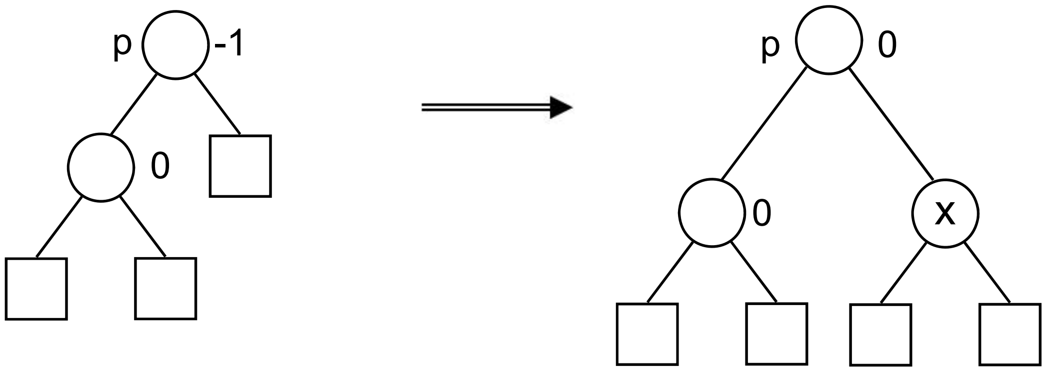
- Case 1: [*bal*(*p*) = +1] and *x* < *p.key*, since the search ends at a leaf with parent *p*.



done!

Overall height unchanged (2)

- Case 2: [*bal*(*p*) = -1] and *x* > *p.key*, since the search ends at a leaf with parent *p*.



done!

Both cases are uncritical:

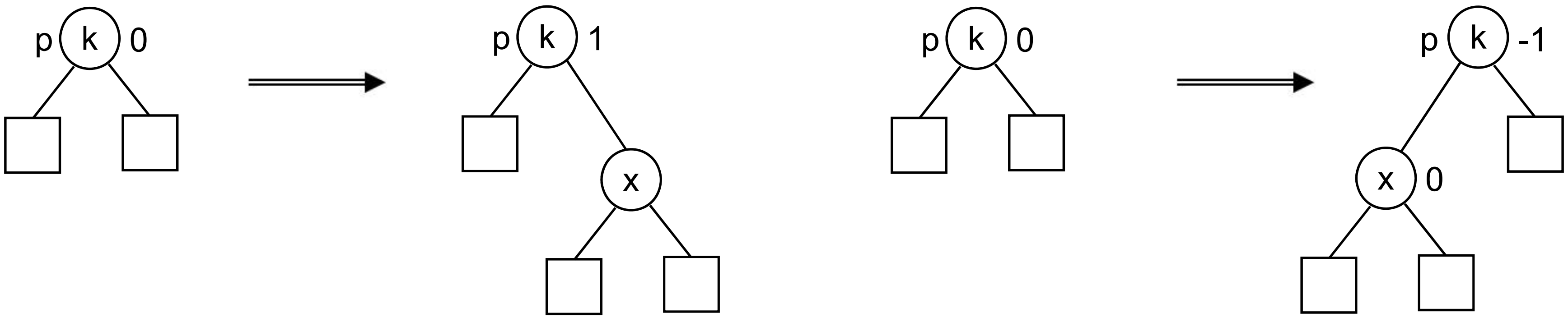
The height of the subtree containing *p* does not change.

The critical case

Case 3: [*bal*(*p*) = 0] Then both children of *p* are leaves. The height increases!

We distinguish the cases whether the new key x must be inserted as the right or left child of p:

[*bal(p) =0 and x > p.key*] [*bal(p) =0 and x < p.key*]



The balance of p has changed and so *bal(p)* must be updated.

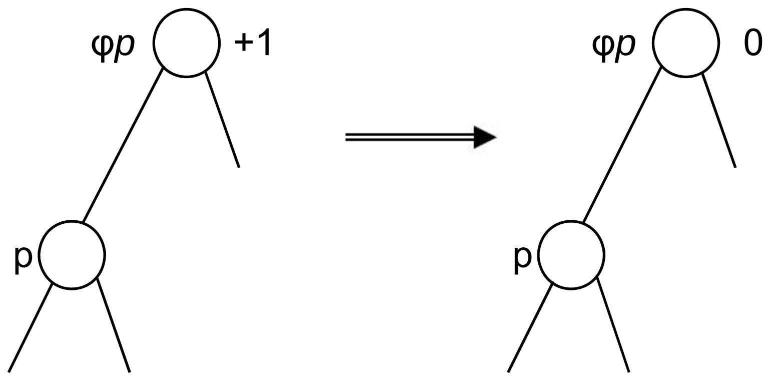
- We need a procedure *upin*(*p*) which traces back the search path, checks the balance factors and carries out restructuring operations (so-called rotations or double rotations).

The procedure *upin(p)*

- When *upin*(*p*) is called, we always have *bal*(*p*) $\in$ {-1, +1} and the height of the subtree rooted in *p* has increased by 1.
- *upin*(*p*) starts at *p* and goes upwards stepwise (until the root if necessary).
- In each step it tries to restore the AVL property.
- In the following we concentrate on the situation where *p* is the left child of its parent φ*p*.
- The situation where *p* is the right child of its parent φ*p* is handled analogously.

Case 1: $bal(\varphi p) = 1$

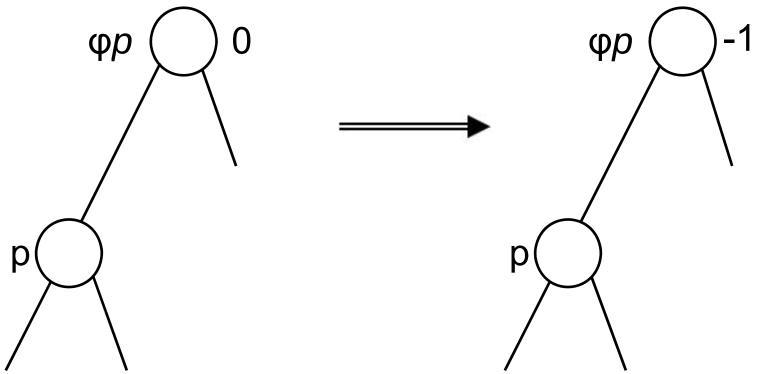
1. The parent φp has balance factor +1. Since the height of the subtree rooted in p (the left child of φp) has increased by 1, it is sufficient to set the balance factor of φp to 0:



done!

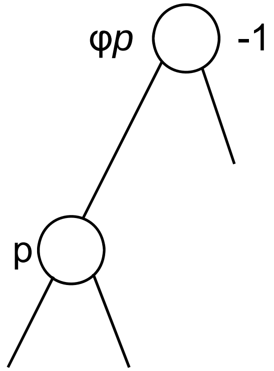
Case 2: $bal(\varphi p) = 0$

2. The parent φp has balance factor 0. Since the height of the subtree rooted in p (the left child of φp) has increased by 1, the balance factor of φp changes to -1. Since the height of the subtree rooted in φp has also changed, we must call *upin* recursively with φp as the argument.



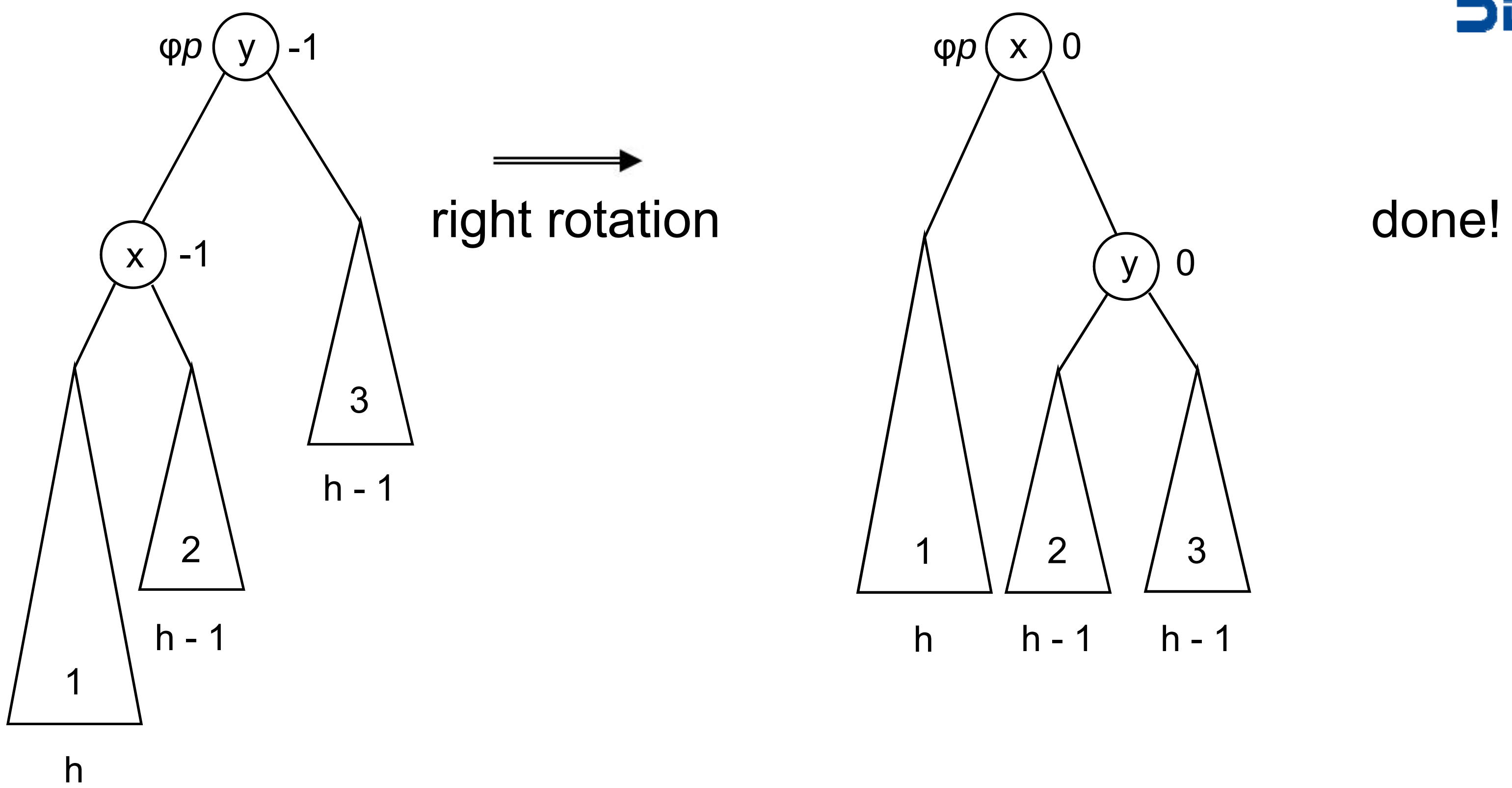
upin(φp)

The critical case 3: *bal*(φ*p*) = -1



- If *bal*(φ*p*) = -1 and the height of the left subtree of φ*p* (rooted in *p*) has increased by 1, the AVL property is now violated in φ*p*.
- In this case we have to restructure the tree.
- Again we distinguish two cases: *bal*(*p*) = -1 (case 3.1) and *bal*(*p*) = +1 (case 3.2).
- The invariant for the call of *upin(p)* is *bal*(*p*) ≠ 0. The case *bal*(*p*) = 0 can therefore not occur!

Case 3.1: $bal(\varphi p) = -1$, $bal(p) = -1$



Is the resulting tree still a search tree?

We must guarantee that the resulting tree fulfils the

1. search tree condition and the

2. AVL property.

Search tree condition: Since the original tree was a search tree, we know that

all keys in tree 1 are smaller than x.

all keys in tree 2 are greater than x and smaller then y.

all keys in tree 3 are greater than y (and x).

Hence, the resulting tree also fulfils the search tree condition.

Is the resulting tree balanced?

AVL property: Since the original tree was an AVL tree, we know:

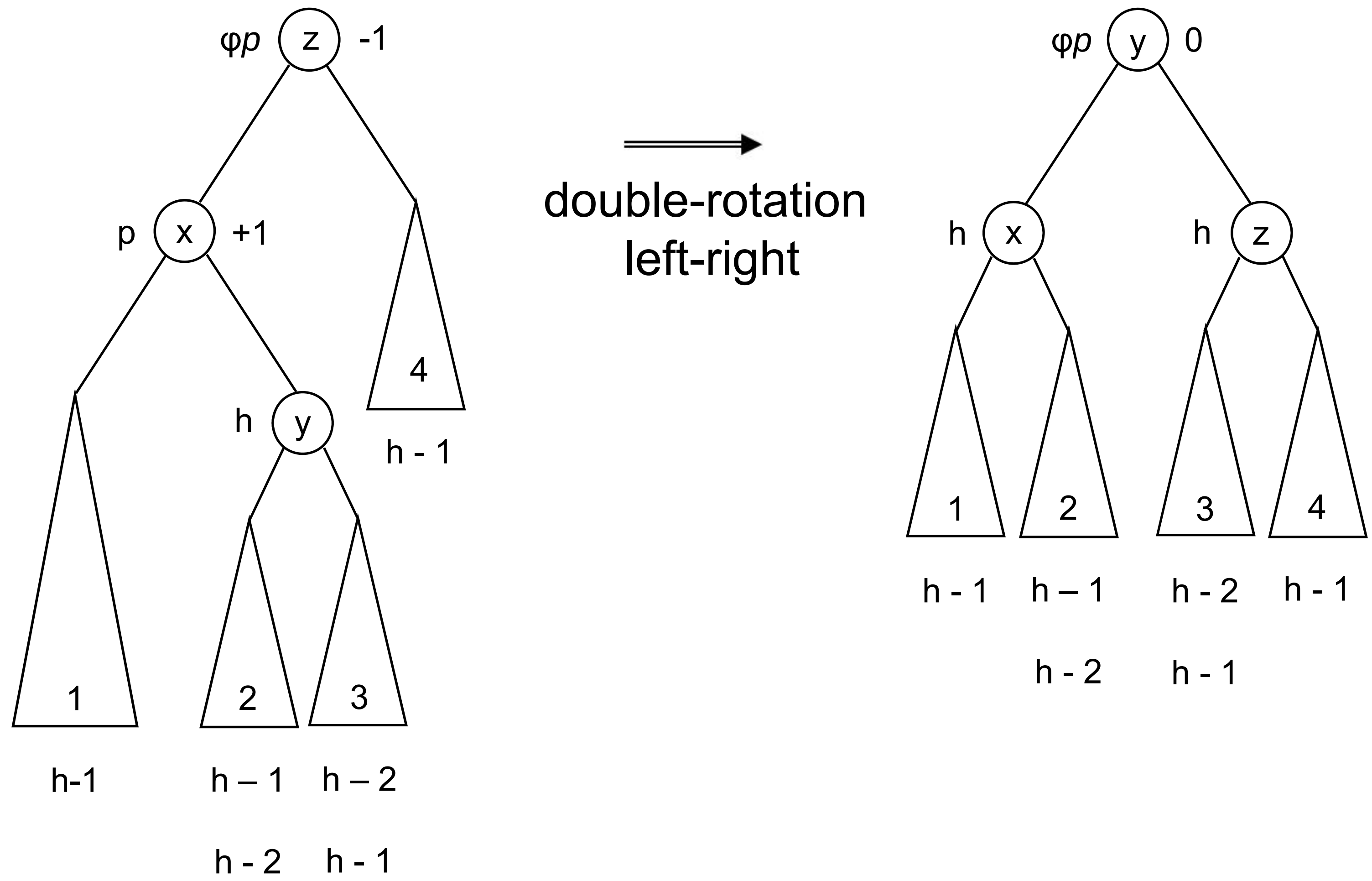
- since $bal(\varphi p) = -1$, tree 2 and tree 3 have the same height h-1.
- since $bal(p) = -1$ after the insertion, tree 1 has height h, while tree 2 has height h-1.

  Hence, after the rotation:

- The node containing y has balance factor 0.
- Node φp has balance factor 0.

Thus, the AVL property has been restored.

Case 3.2: $bal(\varphi p) = -1$, $bal(p) = +1$



Properties of the subtrees

1. The new key must have been inserted into the right subtree of *p*.

2. Trees 2 and 3 must have different height, since otherwise the method *upin* would not have been called.

3. The only possible combinations of heights in trees 2 and 3 are therefore (h-1, h-2) and (h-2, h-1), unless they are empty.

4. Since *bal*(*p*) = 1, tree 1 must have height h-1

5. Finally, tree 4 also must have height h-1 (because *bal*(φ*p*) = -1).

Hence, the resulting tree also fulfils the AVL property

Search tree condition

We have:

1. All keys in tree 1 are smaller than x.
2. All keys in tree 2 are smaller than y but greater than x.
3. All keys in tree 3 are greater than y and x but smaller than z.
4. All keys in tree 4 are greater than x, y and z.

Hence, the tree resulting from the double rotation is also a search tree.

Remarks

- We have only considered the case where p is the left child of its parent φp.
- The case where p is the right child of its parent φp is handled analogously.
- For an efficient implementation of the method *upin*(p), we have to create a list of all visited nodes during the search for the insert position.
- Then we can use this list during the recursive calls to proceed to the parent and carry out the necessary rotations or double rotations.

Insertion in a non-empty AVL tree

Search for *x* ends in a leaf with parent *p*

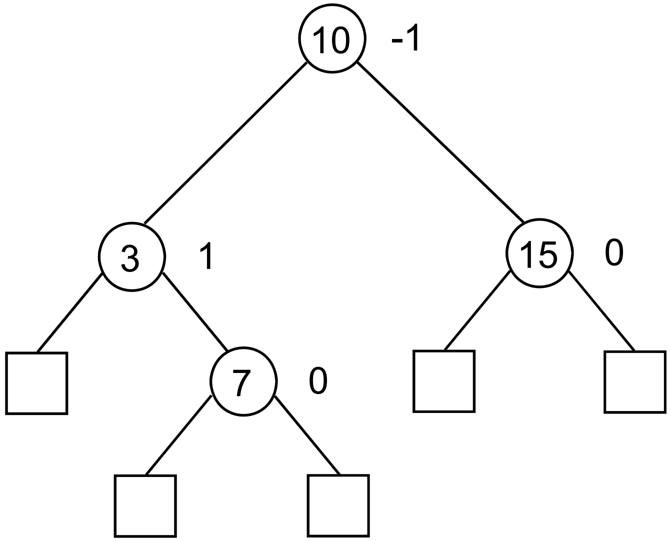
1. Right child of *p* not a leaf, $x < p.key$ → append as left child of *p*, done.
2. Left child of *p* not a leaf, $x > p.key$ → append as right child of *p*, done.
3. Both children of *p* are leaves: append *x* as child of *p* and call *upin*(*p*).

The method *upin*(*p*):

1. *p* is left child of φp
   (a) $bal(\varphi p) = +1$ → $bal(\varphi p) = 0$, done.
   (b) $bal(\varphi p) = 0$ → $bal(\varphi p) = -1$, $upin(\varphi p)$
   (c) i. $bal(\varphi p) = -1$ und $bal(p) = -1$ right rotation, done.
   ii. $bal(\varphi p) = -1$ und $bal(p) = +1$ double rotation left-right, done.
2. *p* is right child of φp.
   ...

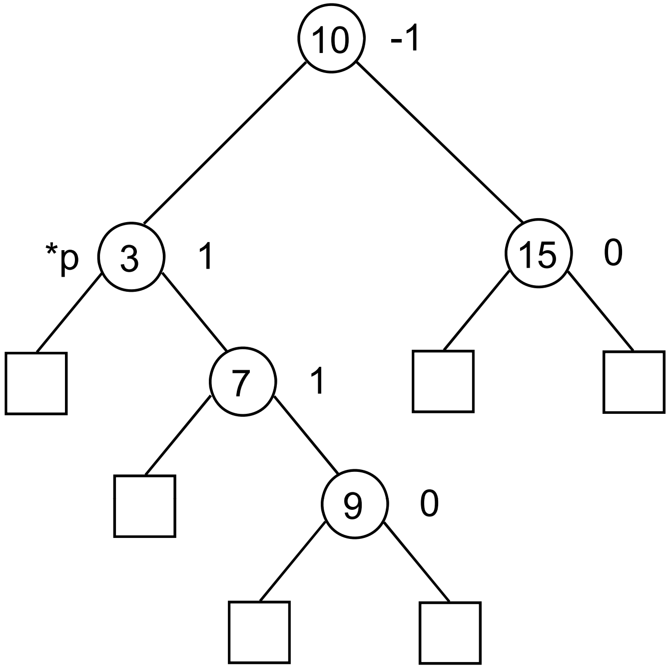
An example (1)

Original situation:



An example (2)

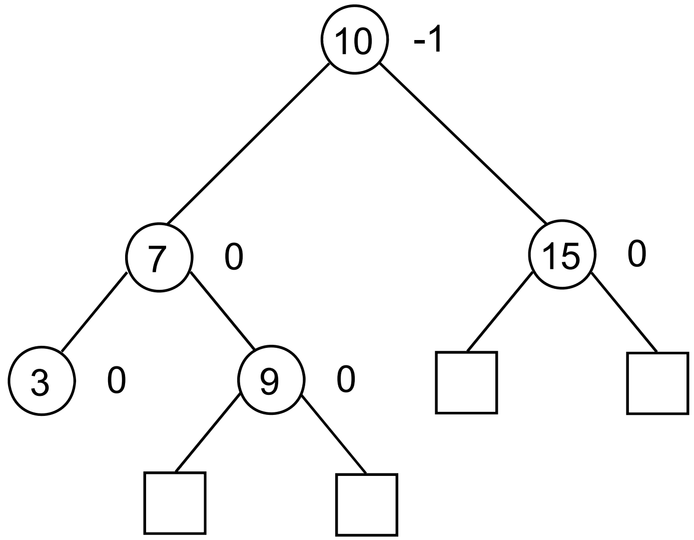
Insert key 9:



AVL property is violated!

An example (3)

Left rotation at **p* yields:



An example (4)

Insertion of 8 followed by double rotation (left-right):