3 Trees: traversal and analysis of standard search trees

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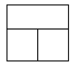
Jan-Georg Smaus

Albert-Ludwigs-Universität Freiburg

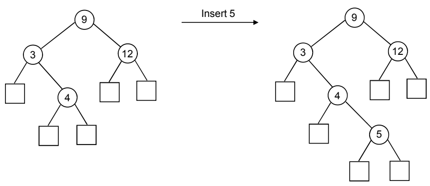
Binary Search Trees

- Binary trees for storing sets of keys (in the internal nodes of trees), such that the operations
  - find
  - insert
  - delete (remove)

  are supported.
- Search tree property: All keys in the left subtree of a node *p* are smaller than the key of *p*, and the key of *p* is smaller than all keys in the right subtree of *p*.
- Implementation:



Standard binary search trees (8)



- Tree structure depends on the order of insertions into the initially empty tree
- Height can increase linearly, but it can also be in O(log *n*), more precisely $\lceil \log 2\ (n+1) \rceil$.

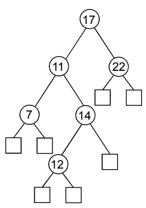
Traversal of trees

Traversal of the nodes of a tree
- for output
- for calculating the sum, average, number of keys ...
- for changing the structure

Most important traversal orders:
1. Preorder = NLR (Node-Left-Right)
   first visit the root, then recursively the left and right subtree (if existent)
2. Postorder = LRN
3. Inorder = LNR
4. The mirror image versions of 1-3

Preorder

Preorder traversal is recursively defined as follows: Let p be the root.
- Visit *p*,
- traverse the left subtree of *p* in preorder,
- traverse the right subtree of *p* in preorder.



Preorder implementation

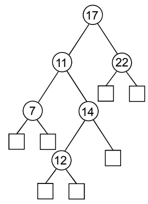
```
// Preorder Node-Left-Right
  void preOrder (){
      preOrder(root);
      System.out.println ();
  }
  void preOrder(SearchNode n){
      if (n == null) return;
      System.out.print (n.content+" ");
      preOrder(n.left);
      preOrder(n.right);
  }

  // Postorder Left-Right-Node
  void postOrder(){
      postOrder(root);
      System.out.println ();
  }
  // ...
```

Inorder

- The traversal order is: first the left subtree, then the root, then the right subtree:

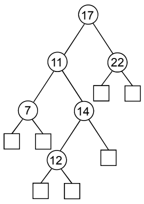
```
 // Inorder Left-Node-Right
void inOrder(){
    inOrder(root);
    System.out.println ();
}
// ...
```


Example



Preorder:
17, 11, 7, 14, 12, 22
Postorder:
7, 12, 14, 11, 22, 17
Inorder:
7, 11, 12, 14, 17, 22

Example for Search, Insertion, Deletion



Sorting with standard search trees

Idea: Create a search tree for the input sequence and output the keys by an inorder traversal.

Remark: Depending on the input sequence, the search tree may degenerate.

Complexity: Depends on internal path length

Worst case: Sorted input: $\Rightarrow \Omega(n^2)$ steps.

Best case: We get a complete search tree of minimal height of about log *n*. Then *n* insertions and outputs are possible in time O(*n* log *n*).

Average case: Later ...

Analysis of search trees

Two possible approaches to determine the internal path length:

1. Random tree analysis, i.e. average over all possible permutations of keys to be inserted (into the initially empty tree).

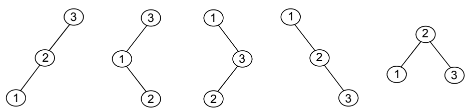
2. Shape analysis, i.e. average over all structurally different trees with *n* keys .

Difference of the expected values for the internal path:

1. $\approx 1.386\, n \log_2 n - 1.846 \cdot n + O(\log n)$

2. $\approx n \cdot \sqrt{\pi n} + O(n)$

Reason for the difference



3,2,1 3,1,2 1,3,2 1,2,3 2,1,3 and 2,3,1

$\longrightarrow$ Random tree analysis counts more balanced trees more often.

Internal path length

Internal path length *I*: measure for judging the quality of a search tree *t*.
Recursive definition:

1. If *t* is empty, then $I(t) = 0.$
2. For a tree *t* with left subtree t_l and right subtree t_r :

$I(t) = I(t_l) + I(t_r) + \#$ inner nodes of *t*

Apparently:

$$I(t) = \sum_{p} (depth(p) + 1)$$

p internal node in t

Average search path length

For a tree *t* the average search path length is defined by:

$$D(t) := I(t)/n, n = \# \text{ nodes in } t.$$

Question: What is the size of *D*(*t*) in the

- best
- worst
- average

case for a tree *t* with *n* internal nodes?

Internal path: best case

We obtain a complete binary tree

Internal path: worst case

Random trees

- Without loss of generality, let {1,…,*n*} be the keys to be inserted.
- Let $s_1,\ldots, s_n$ be a random permutation of these keys.
- Hence, the probability that s_1 has the value *k*, $P(s_1=k) = 1/n$.
- If *k* is the first key, *k* will be stored in the root.
- Then the left subtree contains *k*-1 elements (the keys 1, …, *k*-1) and the right subtree contains *n*-*k* elements (the keys *k*+1, …,*n*).

Expected internal path length

- *EI*(*n*) : Expectation for the internal path length of a randomly generated binary search tree with *n* nodes
- Apparently we have:

$$\begin{aligned} EI(0) &= 0 \\ EI(1) &= 1 \\ EI(n) &= \frac{1}{n}\sum_{k=1}^{n}(EI(j-1) + EI(n-k) + n) \\ &= n + \frac{1}{n}\left(\sum_{k=1}^{n} EI(k-1) + \sum_{k=1}^{n} EI(n-k)\right) \end{aligned}$$

- Assume: *EI*(*n*) = 1.386*n* $\log_2 n$ - 1.846*n* + *O*(log*n*).

Proof (1)

$$EI(n+1) = (n+1) + \frac{2}{n+1} * \sum_{k=0}^{n} (EI(k)$$

and hence

$$(n+1) * EI(n+1) = (n+1)^2 + 2 * \sum_{k=0}^{n} (EI(k)$$

$$n * EI = n^2 + 2 * \sum_{k=0}^{n-1} (EI(k)$$

From the last two equations it follows that

$$\begin{aligned}(n+1)\, EI(n+1) - n * EI(n) &= 2n + 1 + 2 * RI(n)\\ (n+1)\, EI(n+1) &= (n+2)\, EI(n) + 2n + 1\\ EI(n+1) &= \frac{2n+1}{n+1} + \frac{n+2}{n+1} EI(n).\end{aligned}$$

Proof (2)

By induction over n it is possible to show that for all $n \geq 1$:

$$EI(n) = 2(n+1)H_n - 3n$$

$H_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n}$ is the n-th harmonic number,

which can be estimated as follows:

$$H_n = \ln n + \gamma + \frac{1}{2n} + O\left(\frac{1}{n^2}\right)$$

where $\gamma = 0.5772\ldots$ the so-called Euler constant.

Proof (3)

Thus,

$$EI(n) = 2n \ln n - (3 - 2\gamma) * n + \ln n + 1 + 2\gamma + O\left(\frac{1}{n}\right)$$

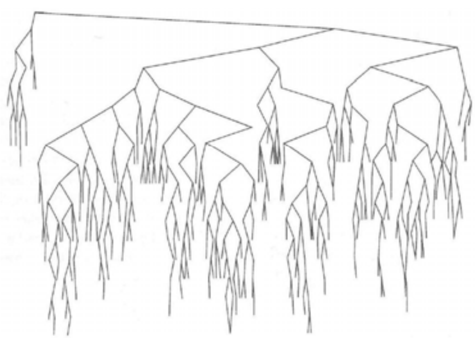
and hence,

$$\begin{aligned}\frac{EI(n)}{n} &= 2 \ln n - (3 - 2\gamma) + \frac{2 \ln n}{n} + \ldots\\ &= \frac{2}{\log_2 e} * \log_2 n - (3 - 2\gamma) + \frac{2 \ln n}{n} + \ldots\\ &= \frac{2 \log_{10} 2}{\log_{10} e} * \log_2 n - (3 - 2\gamma) + \frac{2 \ln n}{n} + \ldots\\ &= \frac{2}{\log_2 e} * \log_2 n - (3 - 2\gamma) + \frac{2 \ln n}{n} + \ldots\\ &\approx 1.386 \log_2 n - (3 - 2\gamma) + \frac{2 \ln n}{n} + \ldots\end{aligned}$$

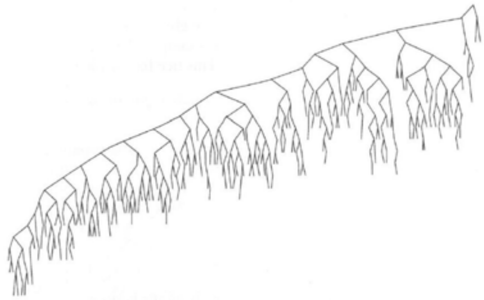
Observation

- Search, insertion and deletion of a key in a randomly generated binary search tree with n keys can be done, on average, in $O(\log_2 n)$ steps.
- In the worst case, the complexity can be $\Omega(n)$.
- One can show that the average distance of a node from the root in a randomly generated tree is only about 40% above the optimal value.
- However, by the restriction to the symmetrical successor, the behaviour becomes worse.
- If n^2 update operations are carried out in a randomly generated search tree with n keys, the expected average search path is only $\Theta(\sqrt{n})$.

Typical binary tree for a random sequence of keys



Resulting binary tree after n^2 updates



Structural analysis of binary trees

Question: What is the average search path length of a binary tree with *N* internal nodes if the average is made over all structurally different binary trees with *N* internal nodes?

Answer: Let

I_N = total internal path length of all structurally different binary trees with *N* internal nodes

B_N = number of all structurally different trees with *N* internal nodes

Then I_N/B_N =

Number of structurally different binary trees

Total internal path length of all trees with N nodes

- For each tree *t* with left subtree t_l and right subtree t_r :

Summary

The average search path length in a tree with *N* internal nodes (averaged over all structurally different trees with *N* internal nodes) is:

$1/N \cdot I_N/B_N$