# 2 The Dictionary Problem: Search Trees

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## The Dictionary Problem

The dictionary problem can be described as follows:

Given: a set of objects (data) where each element can be identified by a unique ***key*** (integer, string, ... ).

Goal: a structure for storing the set of objects such that at least the following operations (methods) are supported:
- search (find, access)
- insert
- delete



## The Dictionary Problem (2)

The following conditions can influence the choice of a solution to the dictionary problem:

- The place where the data are stored: main memory, hard drive, tape, WORM (write once read multiple)
- The frequency of the operations:
  – mostly insertion and deletion (dynamic)
  – mostly search (static)
  – approximately the same frequencies
  – not known
- Other operations to be implemented:
  – Enumerate the set in a certain order (e.g. ascending by key)
  – Set operations: union, intersection, difference quantity, ...
  – Split
  – construct
- Measure for estimating the solution: average case, worst case, amortized worst case
- Order of executing the operations:
  – sequential
  – concurrent



## The Dictionary Problem (3)

Different approaches to the dictionary problem:
- Structuring the complete universe of all possible keys: hashing
- Structuring the set of the actually occurring keys: lists, trees, graphs, ...



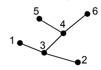
## Trees (1)

Trees are
- generalized lists
  (each list element can have more than one successor)
- special graphs:
  – in general, a graph $G = (V,E)$ consists of a set $V$ of vertices and a set $E \subseteq V \times V$ of edges.
  – the edges are either directed or undirected.
  – vertices and edges can be labelled (they contain further information).
  A tree is a connected acyclic graph, where:
  \# vertices = \# edges + 1
- A general and central concept for the hierarchical structuring of information:
  – decision trees
  – code trees
  – syntax trees

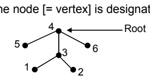


## Trees (2)

Several kinds of trees can be distinguished:
- Undirected tree (with no designated root)




- Rooted tree (one node [= vertex] is designated as the root)




– From each node $k$ there is exactly one path (a sequence of pairwise neighbouring edges) to the root

– the parent (or: direct predecessor) of a node $k$ is the first neighbour on the path from $k$ to the root

– the children (or: direct successors) are the other neighbours of $k$

– the rank (or: outdegree) of a node $k$ is the number of children of $k$

## Trees (3)

- Rooted tree:
  – root: the only node that has no parent
  – leaf nodes (leaves): nodes that have no children
  – internal nodes: all nodes that are not leaves
  – order of a tree $T$: maximum rank of a node in $T$
  – The notion *tree* is often used as a synonym for *rooted tree*.
- Ordered (rooted) tree: the children of each node are somehow ordered, i.e., there is the "leftmost child", the "second child from the left", ..., the "rightmost child".
  - In the graphical representation of a tree, this inevitably so.
  - In the formal definition of a tree, this is a-priori not the case. We have to explicitly state it.
- Binary tree: ordered tree of order 2; the children of a node (if there are 2) are referred to as left child and right child.
- Multiway tree: ordered tree of order > 2



## Trees (4)

A more precise definition of the set $M_d$ of the ordered rooted trees of order ($d \geq 1$):

Consider a set of nodes V.

- Each node in V is in $M_d$
- Let $t_1, \ldots, t_d \in M_d$ and $w$ a node in V. Then $w$ with the roots of $t_1, \ldots, t_d$ as its children (from left to right) is a tree $t \in M_d$. The $t_i$ are subtrees of $t$.

– According to this definition each node has rank $d$ (or rank 0).

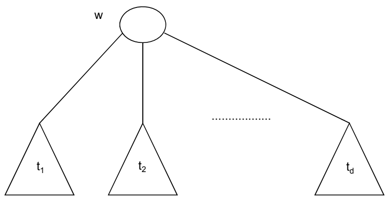
Nodes of binary trees either have 0 or 2 children.

– Variation of the definition: allowing for rank $\leq d$.

For binary trees, nodes with exactly 1 child could then also be permitted.

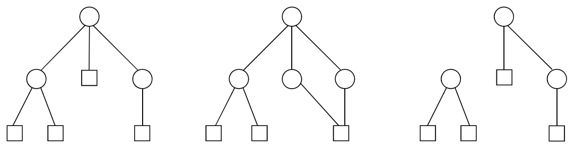


## Illustration of the Definition






## Examples



tree | not a tree | not a tree (but two trees!)

Note that we chose to depict inner nodes as circles and leaves as boxes.



## Structural Properties of Trees

- Depth of a node $k$: # edges from the tree root until $k$ (distance of $k$ to the root)
- Height $h(t)$ of a tree $t$: maximum depth of a leaf in $t$.
  Alternative (recursive) definition:
  – $h$(leaf) = 0
  – $h(t) = 1 + \max\{t_i \mid \text{root of } t_i \text{ is a child of the root of } t\}$ ($t_i$ is a subtree of $t$)
- Level $i$: all nodes of depth $i$
- Complete tree: tree where each non-empty level has the maximum number of nodes.
  → all leaves have the same depth.



## Labelled Vertices

- We mentioned that vertices (and edges) in a graph may be labelled, but in the above definitions on trees, we did not talk about labels yet – the set of nodes V was a "black box".
- We now consider trees where either the inner nodes or the leaves or both are labelled.

# Applications of Trees

Use of trees for the dictionary problem:

- Node: stores one data object
- Tree: stores a set of data
- Advantage (compared to hash tables): enumeration of the complete set of data (e.g. in ascending order) can be accomplished easily.



# Standard Binary Search Trees (1)

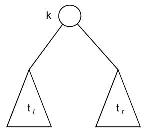
Goal: Storage, retrieval of data (more general: dictionary problem)

Two alternative ways of storage:

- Search trees: keys are stored in internal nodes
  leaf nodes are empty (usually = *null*), they represent intervals between the keys
- Leaf search trees: keys are stored in the leaves
  internal nodes contain information in order to direct the search for a key

Search tree condition:

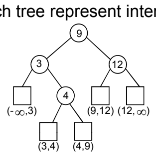
For each internal node *k*: all keys in the left subtree $t_l$ of *k* are less (<) than the key in *k* and all keys in the right subtree $t_r$ of *k* are greater (>) than the key in *k*






# Standard Binary Search Trees (2)

Leaves in the search tree represent intervals between keys of the internal nodes




How can the search for key *s* be implemented? (leaf ≅ null)

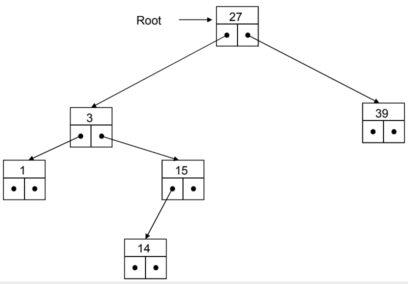
```
k = root;
while (k != null) {
    if (s == k.key)      return true;
    if (s < k.key)       k = k.left;
    else          k = k.right
}
return false;
```



# Example

Search for key *s* ends in the internal node *k* with *k.key* == *s*

or in the leaf whose interval contains *s*






# Standard Binary Search Trees (3)

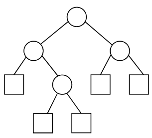
Leaf search tree:

- Keys are stored in leaf nodes
- Clues (routers) are stored in internal nodes, such that $s_l \le s_k < s_r$ ($s_l$ : key in left subtree, $s_k$ : router in *k*, $s_r$ : key in right subtree)
- Choice of *s*: maximum key in $t_l$ .
- Alternative convention (less common): require $s_l < s_k \le s_r$ and choose *s* as minimum key in $t_r$.



# Example: leaf search tree

Leaf nodes store keys, internal nodes contain routers.



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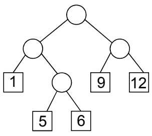




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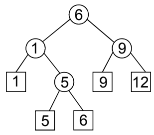
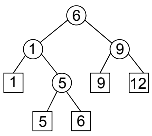





## Standard Binary Search Trees (4)

How is the search for key *s* implemented in a leaf search tree?
(leaf = node with 2 *null* pointers)

```
k = root;
if (k == null) return false;
while (k.left != null) {          // thus also k.right != null
    if (s <= k.key) k = k.left;
    else k = k.right;
}                                  // now in the leaf
return s==k.key;
```






## From now on ...

- In the following we always talk about search trees (not leaf search trees).



## Standard Binary Search Trees (5)

```
class SearchNode {
      int content;
      SearchNode left;
      SearchNode right;
      SearchNode (int c){ // Constructor for a node
          content = c;    // without successor
          left = right = null;
      }
  } //class SearchNode
  class SearchTree {
      SearchNode root;
      SearchTree () {      // Constructor for empty tree
          root = null;
      }
      // ...
  }
```



## Standard Binary Search Trees (6)

```
/* Search for c in the tree */
  boolean search (int c) {
      return search (root, c);
  }
  boolean search (SearchNode n, int c){
      while (n != null) {
          if (c == n.content) return true;
          if (c < n.content) n = n.left;
          else n = n.right;
      }
      return false;
  }
```

## Standard Binary Search Trees (7)

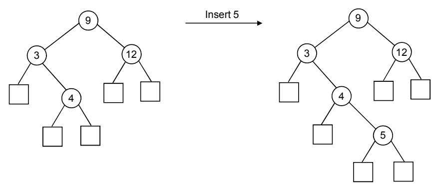
Insertion of a node with key *s* in search tree *t*:

- Search for *s* ends in a node with *s*: don‘t insert (otherwise, there would be duplicated keys)
- Search ends in leaf *b*: make *b* an internal node with *s* as its key and two new leaves.

  □ Tree remains a search tree!



## Standard Binary Search Trees (8)




- Tree structure depends on the order of insertions into the initially empty tree
- Height can increase linearly, but it can also be in O(log *n*), more precisely $\lceil \log_2 (n+1) \rceil$.



## Standard Binary Search Trees (9)

```
int height() {
        return height(root);
   }
   int height(SearchNode n){
        if (n == null) return 0;
        else return 1 + Math.max(height(n.left),
   height(n.right));
   }
   /* Insert c into tree; return true if successful
      and false if c was in tree already */
   boolean insert (int c) {       // insert c
        if (root == null){
             root = new SearchNode (c);
             return true;
        } else return insert (root, c);
   }
```



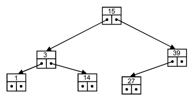
## Standard Binary Search Trees (10)

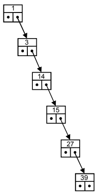
```
boolean insert (SearchNode n, int c){
        while (true){
             if (c == n.content) return false;
             if (c < n.content){
                  if (n.left == null) {
                       n.left = new SearchNode (c);
                       return true;
                  } else n = n.left;
             } else {  // c > n.content
                  if (n.right == null) {
                       n.right = new SearchNode (c);
                       return true;
                  } else n = n.right;
             }
        }
   }
```



## Special cases

- The structure of the resulting tree depends on the order in which the keys are inserted. The minimal height is $\lceil \log_2 (n+1) \rceil$ and the maximal height is *n*.
- Resulting search trees for the sequences 15, 39, 3, 27, 1, 14 and 1, 3, 14, 15, 27, 39:






## Standard Binary Search Trees (11)

A standard tree is created by iterative insertions in an initially empty tree.

- Which trees are more frequent/typical: the balanced or the degenerate ones?
- How costly is an insertion?

We will address these questions in the next chapter.

## Standard Binary Search Trees (11)

- Deletion of a node with key *s* from a tree (while retaining the search tree property)
- Search for *s*.
  If search fails: done.
  Otherwise search ends in node *k* with *k.key* == *s* and
- *k* has no child, one child or two children:
  (a) no child: done (set the parent's pointer to *null* instead of *k*)
  (b) only one child: let *k*'s parent *v* point to *k*'s child instead of *k*
  (c) two children: search for the smallest key in *k*'s right subtree, i.e. go right and then to the left as far as possible until you reach *p* (the symmetrical successor of *k*); copy *p.key* to *k*, delete *p* (which has at most one child, so follow step (a) or (b))



## Symmetrical successor

Definition: A node *q* is called the symmetrical successor of a node *p* if *q* contains the smallest key greater than or equal to the key of *p*.

Observations:

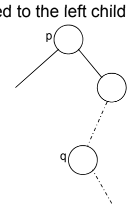
- The symmetrical successor *q* of *p* is the leftmost node in the right subtree of *p*.
- The symmetrical successor has at most one child, which is the right child.



## Finding the symmetrical successor

Observation: If *p* has a right child, the symmetrical successor always exists.

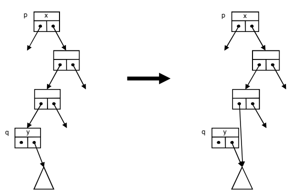
- First go to the right child of *p*.
- From there, always proceed to the left child until you find a node without a left child.






## Idea of the *delete* operation

- Delete *p* by replacing its content with the content of its symmetrical successor *q*. Then delete *q*.
- Deletion of *q* is easy because *q* has at most one child.

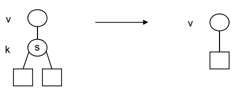




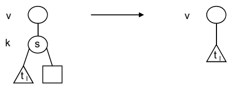

## Illustration

*k* has no internal child or one internal child:

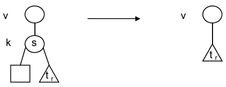
a)




b) (left)




b) (right)

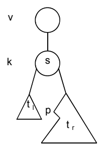





## Illustration (2)

*k* has two internal children:

c)



## Standard Binary Search Trees (12)

```
boolean delete(int c) {
       return delete(null, root, c);
   }
   // delete c from the tree rooted in n, whose parent is vn
   boolean delete(SearchNode vn, SearchNode n, int c) {
       if (n == null) return false;
       if (c < n.content) return delete(n, n.left, c);
       if (c > n.content) return delete(n, n.right, c);
       // now we have: c == n.content
       if (n.left == null) {
           point (vn, n, n.right);
           return true;
       }
       if (n.right == null) {
           point (vn, n, n.left);
           return true;
       }
       // ...
```



## Standard Binary Search Trees (13)

```
// now n.left != null and n.right != null
       SearchNode q = pSymSucc(n);
       if (n == q) {   // right child of q is pSymSucc(n)
           n.content = q.right.content;
           q.right = q.right.right;
           return true;
       } else {        // left child of q is pSymSucc(n)
           n.content = q.left.content;
           q.left = q.left.right;
           return true;
       }
   } // boolean delete(SearchNode vn, SearchNode n, int c)
```



## Standard Binary Search Trees (14)

```
// let vn point to m instead of n;
   // if vn == null, set root pointer to m
   void point(SearchNode vn, SearchNode n, SearchNode m) {
       if (vn == null) root = m;
       else if (vn.left == n) vn.left = m;
       else vn.right = m;
   }
   // returns the parent of the symmetrical successor
   SearchNode pSymSucc(SearchNode n) {
       if (n.right.left != null) {
           n = n.right;
           while (n.left.left != null) n = n.left;
       }
       return n;
   }
```