Introduction to Modal Logic

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May 2, 2011 — Introduction

Introduction

- Modalities
- History of modal logic
- What is modal logic?
- Examples of modal logics
- What is modal logic? (rev'd)
- Modal logic in CS and AI
- Topics

Organization

- Time, Location, Web
- Lecturers
- Exercises
- Literature
- Course goals

Modalities and modal logic

What is a modality?

1: . . .
2: the classification of logical propositions according to their asserting or denying the possibility, impossibility, contingency, or necessity of their content
3/4: . . .

(from *Merriam-Webster's Online Dictionary*)

Modalities and modal logic

Modal logic?

Modal logic can be viewed broadly as the logic of different sorts of modalities, or modes of truth: alethic ("necessarily"), epistemic ("it is known that"), deontic ("it ought to be the case that"), or temporal ("it has been the case that") among others ... In the strict sense however, the term "modal logic" is reserved for the logic of the alethic modalities.

(from *Stanford Encyclopedia of Philosophy*)

History of modal logic – 1

- Modal-logical principles presumably first discussed in a systematic way by Aristotle in *De Interpretatione*
  For example:
  - "Necessity implies possibility"
  - Inter-definability of "it is possible that" and "it is necessary that"
  - Logical laws of modalities
- Contributions by the Megarians, the Stoics, Ockham, and Pseudo-Scotus
- Leibniz introduced the concept of possible world

History of modal logic – 2

Anselm's argument for the existence of God (*Proslogion*, 1077/78):

And indeed we believe you are something greater than which cannot be thought ("*id, quo nihil maius cogitari potest*"). Or is there no such kind of thing, for "the fool said in his heart, 'there is no God"'. . . ? But certainly that same fool, having heard what I just said, "something greater than which cannot be thought," understands what he heard, and what he understands is in his thought, even if he does not think it exists . . .
In fact, it so undoubtedly exists that it cannot be thought of as not existing. For one can think there exists something that cannot be thought of as not existing, and that would be greater than something which can be thought of as not existing. For if that greater than which cannot be thought can be thought of as not existing, then that greater than which cannot be thought is not that greater than which cannot be thought, which does not make sense. Thus that greater than which nothing can be thought so undoubtedly exists that it cannot even be thought of as not existing.

History of modal logic – 3

Thomas on determinism
(interpretation by Plantinga, *The nature of necessity*, 1974):

In *Summa Contra Gentiles*, St. Thomas considers the question whether God's foreknowledge of human action — a foreknowledge that consists, according to St. Thomas, in God's simply seeing the relevant action's taking place — is consistent with human freedom. In this connection he inquires into the truth of

(13) What is seen to be sitting is necessarily sitting.

For suppose at t_1 God sees Theatetus is sitting at t_2. If (13) is true, then presumably Theatetus is necessarily sitting at t_2, in which case he was not free, at that time, to do anything but sit. St. Thomas concludes that (13) is true taken de dicto but false taken de re; that is

(13') It is necessarily true that whatever is seen to be sitting is sitting

is true but

(13") Whatever is seen to be sitting has the property of sitting necessarily or essentially

is false. The deterministic argument, however, requires the truth of (13"); and hence that argument fails.

History of modal logic – 4

- In the 1910s C. I. Lewis investigated modal logic as a possibility to introduce a strengthening of material implication.
- First semantics of modal logic introduced by A. Tarski (topological semantics)
- R. Carnap's suggestion (1942, 1947): Let M be a set of state descriptions. Then 'Necessarily p' is true in s if and only if p is true in every state description in M.
- A. Prior (1957): 'It was once the case that p' is true at instant t if and only if there is an instant t' earlier than t s. t. p is true at t'.
- S. Kripke (1959) introduced the concept of accessibility relation defined on (possible) worlds. 'Necessarily p' is true in w if and only if p is true in every world w' accessible from w.

Intensional connectives

A first slogan

Modal logic is the theory of non truth-functional connectives.

$$
\begin{tikzcd}
\mathsf{Fml}(\mathcal{L})^n \arrow[r, "\text{connective}"] \arrow[d, "\text{truth assignment}^n"'] \arrow[dr, phantom, "\%"] & \mathsf{Fml}(\mathcal{L}) \arrow[d, "\text{truth assignment}"] \\
\{\mathsf{t}, \mathsf{f}\}^n \arrow[r, "\text{truth function}"'] & \{\mathsf{t}, \mathsf{f}\}
\end{tikzcd}
$$

Figure: Truth-functional connectives

Example: Basic modal logic

Modal operators:

$\Box\varphi$	it is necessarily the case that φ
$\Diamond\varphi$	it is possibly the case that φ

Interpretation:

- relational structures
- topological spaces
- modal algebras

Example: Doxastic / epistemic logic

Modal operators:

$\mathsf{B}_a\varphi$	agent *a* holds φ true
$\mathsf{B}_a(\varphi,\psi)$	ψ is a reason for *a* to believe φ
$\mathsf{K}_a\varphi$	agent *a* knows φ
$\mathsf{K}\varphi$	it is known that φ is true
$\mathsf{K}_\gamma\varphi$	among the members of group γ it is common knowledge that φ

Interpretation:

- relational structures
- belief sets (set filters)
- probability functions

Example: Deontic logic

Modal operators:

$O_a\varphi$	agent *a* is obliged to make φ true
$O_a(\varphi, \psi)$	given ψ, *a* is obliged to make φ true
$O\varphi$	φ should be true
$A\varphi$	φ is allowed
$F\varphi$	φ is forbidden

Interpretation:

- relational structures (normatively optimal worlds?)
- utility functions

Example: Temporal logic

Modal operators:

$\mathsf{F}\varphi$	sometimes in the future φ will be true
$\mathsf{G}\varphi$	always in the future φ will be true
$\mathsf{X}\varphi$	in the next state φ will be true
$\mathsf{U}(\varphi, \psi)$	until φ will be true, ψ will be true

Interpretation:

- discrete / continuous time
- linear / branching time
- concrete flows of time $\langle \mathbb{N}, <_{\mathbb{N}} \rangle$, $\langle \mathbb{Q}, <_{\mathbb{Q}} \rangle$, $\langle \mathbb{R}, <_{\mathbb{R}} \rangle$

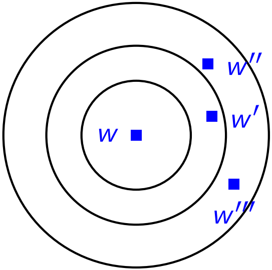
Example: Conditional logics

Modal operators:

$\varphi \mathrel{\Box\!\!\rightarrow} \psi$ if φ, then ψ

Interpretation:

- sphere models based on similarity relation between words (Lewis, 1973)



Example: Probabilistic modal logics

Modal operators:

$\Box_p \varphi$	in the next step, φ will be true with probability $\geq p$

Interpretation (Larsen and Skou, 1991):

- probabilistic transition systems

Example: Agent logics

Modal operators:

$[\gamma]\,\varphi$ the group of agents, γ, has a joint strategy to guarantee that φ becomes true

Interpretation:

- non-deterministic transition systems

Example: Dynamic logics

Modal operators:

$[\alpha]\,\varphi$ after executing α, φ will be true

$[\alpha;\beta]\,\varphi$ after executing α followed by β, φ will be true

...

$[(\psi?;\alpha)\cup(\neg\psi?;\beta)]\,\varphi$

after executing the program "If ψ, then α; else β", φ will be true

...

Interpretation:

- labeled transition systems

Example: Description logic

"Modal" operators:

$\exists r.\varphi$	true for all objects x such that there exists an object y that is in relation r to x and satisfies φ
$\forall r.\varphi$	true for all objects x such that for each object y in relation r to x, φ is true for y
...	
$\exists r^{-1}.\varphi$	(converse relation)
$\exists r^{*}.\varphi$	(reflexive and transitive closure)
$\exists (r \circ s).\varphi$	(composition)

Interpretation:

- relational structures / first-order structures

Relational structures

Slogan

Modal languages are simple yet expressive languages for talking about relational structures. [BRV02]

- Modal languages extend propositional logic by further connectives (boxes $\Box$ and diamonds $\Diamond$)
- ... which are semantically characterized in terms of relational structures (e.g., linear orders, transition systems)

Locality

Slogan

Modal languages provide an internal, local perspective on relational structures. [BRV02]

- Modal formulae are evaluated at a state / possible world in a relational structure / transition system
- ... and for evaluating a formulae only those states are relevant that are accessible by a transition in the system

Modal logics and other logics

Slogan

Modal languages are not isolated formal systems. [BRV02]

- Modal languages have corresponding (first- / second-order) languages that describe the same class of relational structures / transition systems
- . . . which enables import and export of results (correspondence theory)

Modal logics in computer science

- Specification languages for properties of reactive and distributed programs:
  temporal logics (LTL, CTL, CTL*), dynamic logics, ...
- Logics for different topics in AI:
  temporal logics, logics of belief/knowledge, logics of space
- Language for data and knowledge representation:
  description logics

Model checking (verification)

- possible worlds = states of the system that we are verifying
- accessibility/reachability relation = transitions the system can make
- Modal logics are specification languages for expressing the correctness properties the system has to satisfy
  - p will hold in every future state of the system: $\mathsf{G}p$
  - p will hold infinitely often: $\mathsf{GF}p$
  - Whenever p is true, q will be true later: $\mathsf{G}(p \rightarrow \mathsf{F}q)$

Artificial Intelligence

- Applications similar to those in philosophical logic, except that the goal is to formalize and solve problems by automated reasoning (complexity issues)
- How to reason about knowledge and beliefs?
- How to reason about time and change?
- How to reason about space?

Topics

1. Overview on diverse modal logics
2. The modal logic S5: From propositional to modal logic
3. Modal logic, language and semantics: the general setting
4. Bisimulation and expressivity
5. Completeness and compactness
6. Complexity of reasoning in modal logics
7. Decision procedures for modal logics
8. Doxastic and epistemic modal logics
9. Temporal logics
10. Propositional dynamic logic
11. Fixpoint logics, model μ-calculus
12. Description logics
13. CTL and model checking
14. Modal first-order logics

Lectures: Where, When, Web Page

Where

101, Room 01-018

When

Monday: 10:15–12:00
Wednesday: 10:15–11:00 (+ exercises: 11:15–12:00)

Summer Term Holidays

June 13, 2011 – June 17, 2011

Web Page

`http://www.informatik.uni-freiburg.de/~ki/teaching/ss11/ml/`

Lecturers

PD Dr. Jan-Georg Smaus

Room 52-00-042
Consultation: by appointment
Phone: 0761/203-8251
Email: smaus@informatik.uni-freiburg.de

Dr. Stefan Wölfl

Room 52-00-043
Consultation: by appointment
Phone: 0761/203-8228
Email: woelfl@informatik.uni-freiburg.de

Exercises

Who

Robert Mattmüller
Room 52-00-045
Consultation: by appointment
Phone: 0761/203-8229
Email: mattmuel@informatik.uni-freiburg.de

Where

101, Room 01-018

When

Wednesday, 11:15-12:00

Examination

Exams

- Oral exams in September 2011

Literature

- Patrick Blackburn, Maarten de Rijke, and Yde Venema.
  Modal Logic,
  Cambridge University Press, 2002

- M.J. Cresswell and G.E. Hughes.
  A New Introduction to Modal Logic,
  Routledge, 1996

- Further readings will be given during the lecture.

Course Prerequisites & Goals

Goals

- Acquiring in-depth knowledge on modal logics and related families of logics, "applications" of modal logics in AI, proof techniques, relationship between expressiveness and complexity of logical formalisms, ...
- Understanding the principles behind techniques for solving reasoning problems in modal logics
- Being able to read and understand research literature
- Being able to complete a thesis in this research area

Prerequisites

- Basic knowledge in the area of AI
- Basic knowledge in formal logic
- Basic knowledge in theoretical computer science