

# Principles of Knowledge Representation and Reasoning

## Qualitative Representation and Reasoning II: Allen's Interval Calculus

Bernhard Nebel, Stefan Wöflf, and Marco Ragni

Albert-Ludwigs-Universität Freiburg

June 23, 2010

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Allen's Interval Calculus

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Motivation

Intervals and Relations Between Them

Composing Interval Relations

Reasoning in Allen's Interval Calculus

Enforcing Path Consistency

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The Endpoint Subclass

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## Allen's Interval Calculus – Outline

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## Qualitative Temporal Representation and Reasoning

Often we do not want to talk about precise times:

- ▶ **NLP** – we do not have precise time points
- ▶ **Planning** – we do not want to commit to time points too early
- ▶ **Scenario descriptions** – we do not have the exact times or do not want to state them

What are the primitives in our representation system?

- ▶ **Time points**: actions and events are instantaneous, or we consider their beginning and ending
- ▶ **Time intervals**: actions and events have duration
- ▶ Reducibility? Expressiveness? Computational costs for reasoning?

## Motivation: Example

Consider a planning scenario for multimedia generation:

- P1: *Display Picture1*
- P2: *Say "Put the plug in."*
- P3: *Say "The device should be shut off."*
- P4: *Point to Plug-in-Picture1.*

Temporal relations between events:

P2	should happen during	P1
P3	should happen during	P1
P2	should happen before or directly precede	P3
P4	should happen during or end together with	P2

- ↪ P4 happens before or directly precedes P3
- ↪ We could add the statement "P4 does not overlap with P3" without creating an inconsistency.

## Allen's Interval Calculus

- ▶ Allen's interval calculus: **time intervals** and **binary relations** over them
- ▶ **Time intervals**:  $X = (X^-, X^+)$ , where  $X^-$  and  $X^+$  are interpreted over the reals and  $X^- < X^+$  ( $\rightsquigarrow$  naïve approach)
- ▶ **Relations** between concrete intervals, e. g.:
  - $(1.0, 2.0)$  strictly before  $(3.0, 5.5)$
  - $(1.0, 3.0)$  meets  $(3.0, 5.5)$
  - $(1.0, 4.0)$  overlaps  $(3.0, 5.5)$
  - ...
- ↪ Which relations are conceivable?

## The Base Relations

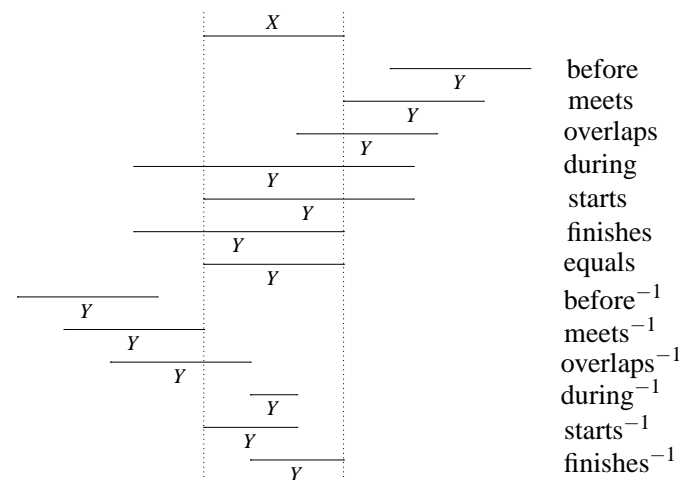
How many ways are there to order the four points of two intervals?

Relation	Symbol	Name
$\{(X, Y) : X^- < X^+ < Y^- < Y^+\}$	$<$	before
$\{(X, Y) : X^- < X^+ = Y^- < Y^+\}$	$\text{m}$	meets
$\{(X, Y) : X^- < Y^- < X^+ < Y^+\}$	$\text{o}$	overlaps
$\{(X, Y) : X^- = Y^- < X^+ < Y^+\}$	$\text{s}$	starts
$\{(X, Y) : Y^- < X^- < X^+ = Y^+\}$	$\text{f}$	finishes
$\{(X, Y) : Y^- < X^- < X^+ < Y^+\}$	$\text{d}$	during
$\{(X, Y) : Y^- = X^- < X^+ = Y^+\}$	$\equiv$	equal

and the **converse** relations (obtained by exchanging  $X$  and  $Y$ )

- ↪ These relations are JEPD.

## The 13 Base Relations Graphically



## Disjunctive Descriptions

- Assumption: We don't have precise information about the relation between  $X$  and  $Y$ , e. g.:

$$X \circ Y \text{ or } X \text{ m } Y$$

- ... modelled by sets of base relations (meaning the union of the relations):

$$X \{o, m\} Y$$

$\rightsquigarrow 2^{13}$  imprecise relations (incl.  $\emptyset$  and  $\mathbf{B}$ )

Example of an indefinite qualitative description:

$$\left\{ X \{o, m\} Y, Y \{m\} Z, X \{o, m\} Z \right\}$$

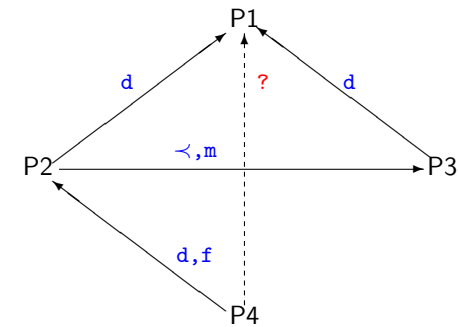
## Our Example... Formally

P1: *Display* Picture1

P2: *Say* "Put the plug in."

P3: *Say* "The device should be shut off."

P4: *Point* to Plug-in-Picture1.



Compose the constraints:  $P4 \{d, f\} P2$  and  $P2 \{d\} P1$

$\rightsquigarrow P4 \{d\} P1$ .

## Composition of Base Relations

	λ	Y	d	d <sup>-1</sup>	o	o <sup>-1</sup>	B	B <sup>-1</sup>	s	s <sup>-1</sup>	f	f <sup>-1</sup>
λ	λ	B	Y	B	Y	Y	Y	Y	Y	Y	Y	Y
Y	B	Y	B	Y	Y	Y	Y	Y	Y	Y	Y	Y
d	Y	Y	B	Y	Y	Y	Y	Y	Y	Y	Y	Y
d <sup>-1</sup>	Y	Y	Y	B	Y	Y	Y	Y	Y	Y	Y	Y
o	Y	Y	Y	Y	B	Y	Y	Y	Y	Y	Y	Y
o <sup>-1</sup>	Y	Y	Y	Y	Y	B	Y	Y	Y	Y	Y	Y
B	Y	Y	Y	Y	Y	Y	B	Y	Y	Y	Y	Y
B <sup>-1</sup>	Y	Y	Y	Y	Y	Y	Y	B	Y	Y	Y	Y
s	Y	Y	Y	Y	Y	Y	Y	Y	B	Y	Y	Y
s <sup>-1</sup>	Y	Y	Y	Y	Y	Y	Y	Y	Y	B	Y	Y
f	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	B	Y
f <sup>-1</sup>	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	B

## Outlook

- Using the **composition table** and the rules about operations on relations, we can **deduce** new relations between time intervals.
- What would be a **systematic** approach?
- How costly is that?
- Is that **complete**?
- If not, could it be complete on a subset of the relation system?

## Reasoning in Allen's Interval Calculus

## Allen's Interval Calculus

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Enforcing Path Consistency  
 NP-Hardness Example  
 The Continuous Endpoint Class  
 Completeness for the CEP Class

## A Maximal Tractable Sub-Algebra

## Literature

## Constraint Propagation – The Naive Algorithm

Enforcing path consistency using the straight-forward method:  
 Let  $Table[i, j]$  be an array of size  $n \times n$  ( $n$ : number of intervals) in which we record the constraints between the intervals.

EnforcePathConsistency1( $\mathcal{C}$ )

*Input:* a (binary) CSP  $\mathcal{C} = \langle V, D, C \rangle$

*Output:* an equivalent, but path consistent CSP  $\mathcal{C}'$

**repeat**

**for** each pair  $(i, j)$ ,  $1 \leq i, j \leq n$

**for** each  $k$  with  $1 \leq k \leq n$

$Table[i, j] := Table[i, j] \cap (Table[i, k] \circ Table[k, j])$

**until** no entry in  $Table$  is changed

↪ terminates;

↪ needs  $O(n^5)$  intersections and compositions.

An  $O(n^3)$  AlgorithmEnforcePathConsistency2( $\mathcal{C}$ )

*Input:* a (binary) CSP  $\mathcal{C} = \langle V, D, C \rangle$

*Output:* an equivalent, but path consistent CSP  $\mathcal{C}'$

$Paths(i, j) = \{(i, j, k) : 1 \leq k \leq n\} \cup \{(k, i, j) : 1 \leq k \leq n\}$

$Queue := \bigcup_{i, j} Paths(i, j)$

**while**  $Queue \neq \emptyset$

select and delete  $(i, k, j)$  from  $Queue$

$T := Table[i, j] \cap (Table[i, k] \circ Table[k, j])$

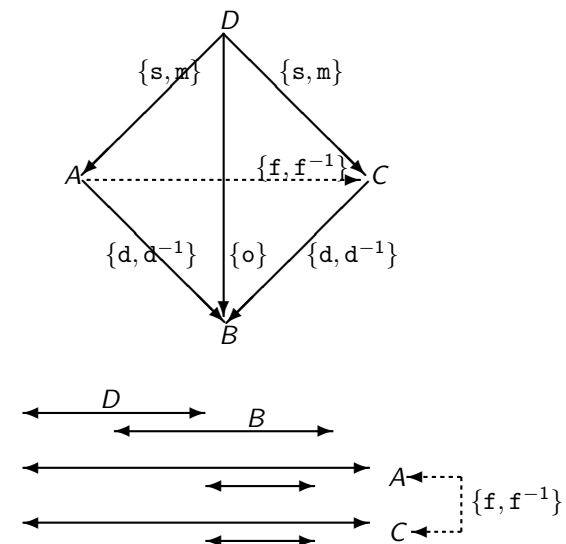
**if**  $T \neq Table[i, j]$

$Table[i, j] := T$

$Table[j, i] := T^{-1}$

$Queue := Queue \cup Paths(i, j)$

## Example for Incompleteness



## NP-Hardness

### Theorem (Kautz & Vilain)

CSAT is NP-hard for Allen's interval calculus.

#### Proof.

Reduction from 3-colorability (original proof using 3Sat).

Let  $G = (V, E)$ ,  $V = \{v_1, \dots, v_n\}$  be an instance of 3-colorability.

Then we use the intervals  $\{v_1, \dots, v_n, 1, 2, 3\}$  with the following constraints:

$$\begin{array}{lll} 1 & \{m\} & 2 \\ 2 & \{m\} & 3 \\ v_i & \{m, \equiv, m^{-1}\} & 2 \quad \forall v_i \in V \\ v_i & \{m, m^{-1}, <, >\} & v_j \quad \forall (v_i, v_j) \in E \end{array}$$

This constraint system is satisfiable iff  $G$  can be colored with 3 colors.  $\square$

## Looking for Special Cases

- **Idea:** Let us look for polynomial special cases. In particular, let us look for sets of relations (subsets of the entire set of relations) that have an easy CSAT problem.
- **Note:** Interval formulae  $X R Y$  can be expressed as clauses over atoms of the form  $a \text{ op } b$ , where:
  - $a$  and  $b$  are endpoints  $X^-, X^+, Y^-$  and  $Y^+$  and
  - $\text{op} \in \{<, >, =, \leq, \geq\}$ .
- **Example:** All base relations can be expressed as unit clauses.

### Lemma

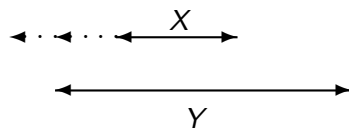
Let  $\pi(\Theta)$  be the translation of  $\Theta$  to clause form.  $\Theta$  is satisfiable over intervals iff  $\pi(\Theta)$  is satisfiable over the rational numbers.

## The Continuous Endpoint Class

**Continuous Endpoint Class  $\mathcal{C}$ :** This is a subset of  $\mathcal{A}$  such that there exists a clause form for each relation containing only unit clauses where  $\neg(a = b)$  is forbidden.

**Example:** All basic relations and  $\{d, o, s\}$ , because

$$\pi(X \{d, o, s\} Y) = \begin{cases} X^- < X^+, Y^- < Y^+, \\ X^- < Y^+, X^+ > Y^-, \\ X^+ < Y^+ \end{cases}$$



## Why Do We Have Completeness?

The set  $\mathcal{C}$  is closed under intersection, composition, and converse (it is a sub-algebra wrt. these three operations on relations). This can be shown by using a computer program.

### Lemma

Each 3-consistent interval CSP over  $\mathcal{C}$  is globally consistent.

### Theorem (van Beek)

Path consistency solves  $\text{CMIN}(\mathcal{C})$  and decides  $\text{CSAT}(\mathcal{C})$ .

(Proof: Follows from the above lemma and the fact that a strongly  $n$ -consistent CSP is minimal.)

### Corollary

A path consistent interval CSP consisting of base relations only is satisfiable.

## Helly's Theorem

### Definition

A set  $M \subseteq \mathbf{R}^n$  is **convex** iff for all pairs of points  $a, b \in M$ , all points on the line connecting  $a$  and  $b$  belong to  $M$ .

### Theorem (Helly)

Let  $F$  be a finite family of at least  $n + 1$  convex sets in  $\mathbf{R}^n$ . If all sub-families of  $F$  with  $n + 1$  sets have a non-empty intersection, then  $\bigcap F \neq \emptyset$ .

## Strong $n$ -Consistency (1)

### Proof (Part 1).

We prove the claim by induction over  $k$  with  $k \leq n$ .

**Base case:**  $k = 1, 2, 3$     ✓

**Induction assumption:** Assume strong  $(k - 1)$ -consistency (and non-emptiness of all relations)

**Induction step:** From the assumption, it follows that there is an instantiation of  $k - 1$  variables  $X_i$  to pairs  $(s_i, e_i)$  satisfying the constraints  $R_{ij}$  between the  $k - 1$  variables.

We have to show that we can extend the instantiation to any  $k$ th variable.

## Strong $n$ -Consistency (2): Instantiating the $k$ th Variable

### Proof (Part 2).

The instantiation of the  $k - 1$  variables  $X_i$  to  $(s_i, e_i)$  restricts the instantiation of  $X_k$ .

**Note:** Since  $R_{ij} \in \mathcal{C}$  by assumption, these restrictions can be expressed by inequalities of the form:

$$s_i < X_k^+ \wedge e_j \geq X_k^- \wedge \dots$$

Such inequalities define convex subsets in  $\mathbf{R}^2$ .

↪ Consider sets of 3 inequalities (= 3 convex sets).

## Strong $n$ -Consistency (3): Using Helly's Theorem

### Proof (Part 3).

**Case 1:** All 3 inequalities mention only  $X_k^-$  (or mention only  $X_k^+$ ). Then it suffices to consider only 2 of these inequalities (the strongest). Because of 3-consistency, there exists at least 1 common point satisfying these 2 inequalities.

**Case 2:** The inequalities mention  $X_k^-$  and  $X_k^+$ , but do not contain the inequality  $X_k^- < X_k^+$ . Then there are at most 2 inequalities with the same variable and we have the same situation as in Case 1.

**Case 3:** The set contains the inequality  $X_k^- < X_k^+$ . In this case, only three intervals (incl.  $X_k$ ) can be involved and by 3-consistency there exists a common point.

↪ With Helly's Theorem, there exists an instantiation consistent with **all** inequalities.

↪ Strong  $k$ -consistency for all  $k \leq n$ . □

## Outlook

- ▶  $\text{CMIN}(\mathcal{C})$  can be computed in  $O(n^3)$  time (for  $n$  being the number of intervals) using the path consistency algorithm.
- ▶  $\mathcal{C}$  is a set of relations occurring “naturally” when observations are uncertain.
- ▶  $\mathcal{C}$  contains 83 relations (incl. the impossible and the universal relations).
- ▶ Are there larger sets such that path consistency computes minimal CSPs? **Probably not.**
- ▶ Are there larger sets of relations that permit polynomial satisfiability testing? **Yes.**

## A Maximal Tractable Sub-Algebra

### Allen's Interval Calculus

### Reasoning in Allen's Interval Calculus

### A Maximal Tractable Sub-Algebra

The Endpoint Subclass

The ORD-Horn Subclass

Maximality

Solving Arbitrary Allen CSPs

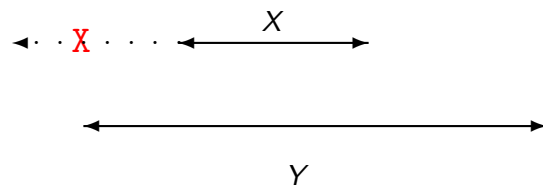
### Literature

## The EP-Subclass

**End-Point Subclass:**  $\mathcal{P} \subseteq \mathcal{A}$  is the subclass that permits a clause form containing only **unit** clauses ( $a \neq b$  is allowed).

**Example:** all basic relations and  $\{d, o\}$  since

$$\pi(X \{d, o\} Y) = \{X^- < X^+, Y^- < Y^+, \\ X^- < Y^+, X^+ > Y^-, X^- \neq Y^-, \\ X^+ < Y^+\}$$



**Theorem (Vilain & Kautz 86, Ladkin & Maddux 88)**

*Enforcing path consistency decides  $\text{CSAT}(\mathcal{P})$ .*

## The ORD-Horn Subclass

**ORD-Horn Subclass:**  $\mathcal{H} \subseteq \mathcal{A}$  is the subclass that permits a clause form containing only **Horn clauses** where only the following **literals** are allowed:

$$a \leq b, a = b, a \neq b$$

$\neg a \leq b$  is not allowed!

**Example:** all  $R \in \mathcal{P}$  and  $\{o, s, f^{-1}\}$ :

$$\pi(X \{o, s, f^{-1}\} Y) = \left\{ \begin{array}{l} X^- \leq X^+, X^- \neq X^+, \\ Y^- \leq Y^+, Y^- \neq Y^+, \\ X^- \leq Y^-, \\ X^- \leq Y^+, X^- \neq Y^+, \\ Y^- \leq X^+, X^+ \neq Y^-, \\ X^+ \leq Y^+, \\ X^- \neq Y^- \vee X^+ \neq Y^+ \end{array} \right\}.$$

## Partial Orders: The *ORD* Theory

Let *ORD* be the following theory:

$$\begin{aligned} \forall x, y, z: \quad & x \leq y \wedge y \leq z \rightarrow x \leq z && \text{(transitivity)} \\ \forall x: \quad & x \leq x && \text{(reflexivity)} \\ \forall x, y: \quad & x \leq y \wedge y \leq x \rightarrow x = y && \text{(anti-symmetry)} \\ \forall x, y: \quad & x = y \rightarrow x \leq y && \text{(weakening of =)} \\ \forall x, y: \quad & x = y \rightarrow y \leq x && \text{(weakening of =)}. \end{aligned}$$

- ▶ *ORD* describes partially ordered sets,  $\leq$  being the ordering relation.
- ▶ *ORD* is a **Horn theory**
- ▶ What is missing wrt. *dense* and *linear* orders?

## Satisfiability over Partial Orders

### Proposition

Let  $\Theta$  be a CSP over  $\mathcal{H}$ .  $\Theta$  is satisfiable over interval interpretations iff  $\pi(\Theta) \cup \text{ORD}$  is satisfiable over arbitrary interpretations.

### Proof.

$\Rightarrow$ : Since the reals form a partially ordered set (i. e., satisfy *ORD*), this direction is trivial.

$\Leftarrow$ : Each extension of a partial order to a linear order satisfies all formulae of the form  $a \leq b$ ,  $a = b$ , and  $a \neq b$  which have been satisfied over the original partial order.  $\square$

## Complexity of $\text{CSAT}(\mathcal{H})$

Let  $\text{ORD}_{\pi(\Theta)}$  be the propositional theory resulting from instantiating all axioms with the endpoints occurring in  $\pi(\Theta)$ .

### Proposition

$\text{ORD} \cup \pi(\Theta)$  is satisfiable iff  $\text{ORD}_{\pi(\Theta)} \cup \pi(\Theta)$  is so.

**Proof idea:** Herbrand expansion!

### Theorem

$\text{CSAT}(\mathcal{H})$  can be decided in polynomial time.

### Proof.

$\text{CSAT}(\mathcal{H})$  instances can be translated into a propositional Horn theory with blowup  $O(n^3)$  according to the previous Prop., and such a theory is decidable in polynomial time.  $\square$

$\mathcal{C} \subset \mathcal{P} \subset \mathcal{H}$  with  $|\mathcal{C}| = 83$ ,  $|\mathcal{P}| = 188$ ,  $|\mathcal{H}| = 868$

## Path Consistency and the OH-Class

### Lemma

Let  $\Theta$  be a path-consistent set over  $\mathcal{H}$ . Then

$$(X \{ \} Y) \notin \Theta \text{ iff } \Theta \text{ is satisfiable}$$

**Proof idea:** One can show that  $\text{ORD}_{\pi(\Theta)} \cup \pi(\Theta)$  is closed wrt. **positive unit resolution**. Since this inference rule is refutation complete for Horn theories, the claim follows.

### Theorem

Enforcing path consistency decides  $\text{CSAT}(\mathcal{H})$ .

$\rightsquigarrow$  Maximality of  $\mathcal{H}$ ?

$\rightsquigarrow$  Do we have to check all 8192 – 868 extensions?



## Complexity of Sub-Algebras

Let  $\hat{S}$  be the closure of  $S \subseteq \mathcal{A}$  under converse, intersection, and composition (i.e., the carrier of the least sub-algebra generated by  $S$ ).

### Theorem

$CSAT(\hat{S})$  can be polynomially transformed to  $CSAT(S)$ .

### Proof Idea.

All relations in  $\hat{S} - S$  can be modeled by a fixed number of compositions, intersections, and conversions of relations in  $S$ , introducing perhaps some fresh variables.  $\square$

- $\rightsquigarrow$  Polynomiality of  $S$  extends to  $\hat{S}$ .
- $\rightsquigarrow$  NP-hardness of  $\hat{S}$  is inherited by all generating sets  $S$ .
- $\rightsquigarrow$  Note:  $\mathcal{H} = \hat{\mathcal{H}}$ .

## Minimal Extensions of the $\mathcal{H}$ -Subclass

A computer-aided case analysis leads to the following result:

### Lemma

There are only two minimal sub-algebras that strictly contain  $\mathcal{H}$ :  $\mathcal{X}_1, \mathcal{X}_2$

$$N_1 = \{d, d^{-1}, o^{-1}, s^{-1}, f\} \in \mathcal{X}_1$$

$$N_2 = \{d^{-1}, o, o^{-1}, s^{-1}, f^{-1}\} \in \mathcal{X}_2$$

The clause form of these relations contain “proper” disjunctions!

### Theorem

$CSAT(\mathcal{H} \cup \{N_i\})$  is NP-complete.

**Question:** Are there other maximal tractable subclasses?

## “Interesting” Subclasses

Interesting subclasses of  $\mathcal{A}$  should contain all basic relations.

A computer-aided case analysis reveals:

For  $S \supseteq \{\{B\} : B \in \mathbf{B}\}$  it holds that

1.  $\hat{S} \subseteq \mathcal{H}$ , or
2.  $N_1$  or  $N_2$  is in  $\hat{S}$ .

In case 2, one can show:  $CSAT(S)$  is NP-complete.

- $\rightsquigarrow$   $\mathcal{H}$  is the **only interesting** maximal tractable subclass.

If we include non-interesting subalgebras, there exist exactly 18 tractable classes.

## Relevance?

**Theory:**  $\oplus$  We now know the boundary between polynomial and NP-hard reasoning problems along the dimension *expressiveness*.

**Practice:**  $\ominus$  All known applications either need only  $\mathcal{P}$  or they need more than  $\mathcal{H}$ !

Backtracking methods might profit from the result by **reducing the branching factor**.

- $\rightsquigarrow$  How difficult is  $CSAT(\mathcal{A})$  in practice?
- $\rightsquigarrow$  What are the relevant branching factors?

## Solving General Allen CSPs

- ▶ Backtracking algorithm using **path consistency** as a **forward-checking method**
  - ▶ Relies on tractable fragments of Allen's calculus: split relations into relations of a tractable fragment, and backtrack over these.
  - ▶ Refinements and evaluation of different heuristics
- ↪ Which tractable fragment should one use?





## Branching Factors

- ▶ If the labels are split into **base relations**, then on average a label is split into  
**6.5 relations**
  - ▶ If the labels are split into **pointizable relations** ( $\mathcal{P}$ ), then on average a label is split into  
**2.955 relations**
  - ▶ If the labels are split into **ORD-Horn relations** ( $\mathcal{H}$ ), then on average a label is split into  
**2.533 relations**
- ↪ A difference of **0.422**
- ↪ This makes a difference for "hard" instances.

## Summary

- ▶ Allen's interval calculus is often adequate for describing relative orders of events that have duration.
- ▶ The satisfiability problem for CSPs using the relations is NP-complete.
- ▶ For the **continuous endpoint class**, minimal CSPs can be computed using the path-consistency method.
- ▶ For the larger **ORD-Horn class**, CSAT is still decided by the path-consistency method.
- ▶ Can be used in practice for backtracking algorithms.

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