

# Principles of Knowledge Representation and Reasoning

Qualitative Representation and Reasoning: Introduction

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# Quantitative vs. Qualitative

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**Spatio-temporal configurations** can be described **quantitatively** by specifying the coordinates of the relevant objects:

**Example:** *At time point 10.0 object A is at position (11.0, 1.0, 23.7), at time point 11.0 at position (15.2, 3.5, 23.7). From time point 0.0 to 11.0, object B is at position (15.2, 3.5, 23.7). Object C is at time point 11.0 at position (300.9, 25.6, 200.0) and at time point 35.0 at (11.0, 1.0, 23.7).*

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# Quantitative vs. Qualitative

Often, however, a **qualitative** description (using a finite vocabulary) is more adequate:

**Example:** *Object A hit object B. Afterwards, object C arrived.*

Sometimes we want to reason with such descriptions, e.g.:

*Object C was not close to object A when it hit object B.*

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# Representation of Qualitative Knowledge

**Intention:** Description of configurations using a finite vocabulary and reasoning about these descriptions

- Specification of a **vocabulary**: usually a finite set of relations (often binary) that are **pairwise disjoint** and **exhaustive**
- Specification of a **language**: often sets of atomic formulae (constraint networks), perhaps restricted disjunction
- Specification of a formal **semantics**
- Analysis of computational properties and design of **reasoning methods** (often constraint propagation)
- Perhaps, specification of **operational semantics** for verifying whether a relation holds in a given quantitative configuration

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# Applications in ...

- Natural language processing
- Specification of abstract spatio-temporal configurations
- Query languages for spatio-temporal information systems
- Layout descriptions of documents (and learning of such layouts)
- Action planning
- ...

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# Qualitative Temporal Relations: *Point Calculus*

We want to talk about **time instants** (points) and binary **relations** over them.

- *Vocabulary:*

- $X$  equals  $Y$ :  $X = Y$
- $X$  before  $Y$ :  $X < Y$
- $X$  after  $Y$ :  $X > Y$

- *Language:*

- Allow for **disjunctions** of basic relations to express **indefinite information**. Use set of relations to express that. For instance,  $\{<, =\}$  expresses  $\leq$ .
- $2^3$  different relations (including the *impossible* and the *universal* relation)
- Use **sets of atomic formulae** with these relations to describe **configurations**. For example:

$$\{x\{=\}y, y\{<, >\}z\}$$

- *Semantics:* Interpret the time point symbols and relation symbols over the **rational** (or real) numbers.

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# Some Reasoning Problems

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$$\{x\{<, =\}y, y\{<, =\}z, v\{<, =\}y, w\{>\}y, z\{<, =\}x\}$$

- **Satisfiability**: Are there values for all time points such that all formulae are satisfied?
- **Satisfiability** with  $v\{=\}w$ ?
- Finding a satisfying **instantiation** of all time points
- **Deduction**: Does  $x\{=\}y$  logically follow?  
Does  $v\{<, =\}w$  follow?
- Finding a **minimal description**: What are the most constrained relations that describe the same set of instantiations?

# From a Logical Point of View . . .

In general, qualitatively described configurations are simple logical theories:

- Only sets of atomic formulae to describe the configuration
- Only existentially quantified variables (or constants)
- A fixed background theory that describes the semantics of the relations (e.g., dense linear orders)
- We are interested in **satisfiability**, **model finding**, and *deduction*
- *Constraint Satisfaction Problems*

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## Definition

A **constraint satisfaction problem (CSP)** is given by

- a set  $V$  of  $n$  **variables**  $\{v_1, \dots, v_n\}$ ,
- for each  $v_i$ , a **value domain**  $D_i$
- **constraints** (relations over subsets of the variables)

## Tasks:

Find one (or all) **solution(s)**, i. e., tuples

$$(d_1, \dots, d_n) \in D_1 \times \dots \times D_n$$

such that the assignment  $v_i \mapsto d_i$  ( $1 \leq i \leq n$ ) satisfies all constraints.

# CSP – Example

**$k$ -colorability:** Can we color the nodes of a graph with  $k$  colors in a way such that all nodes connected by an edge have different colors?

- The node set is the set of variables
- The domain of each variable is  $\{1, \dots, k\}$
- The constraints are that nodes connected by an edge must have a different value

**Note:** This CSP has a particular restricted form:

- Only **binary** constraints
- The domains are **finite**

**Other examples:** Many problems (e.g. cross-word puzzle,  $n$ -queens problem, configuration, ...) can be cast as a CSP (and solved this way)

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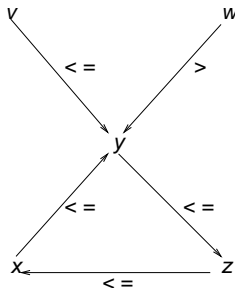
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# Our Example: Point relations

- Our point relation CSP is a binary CSP with **infinite domains**.
- It can be represented as a **constraint graph**:



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# Computational Complexity

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## Theorem

*It is NP-hard to decide solvability of CSPs, even binary CSPs.*

## Proof.

Since  $k$ -colorability is NP-complete (even for fixed  $k \geq 3$ ), solvability of CSPs in general must be NP-hard. □

**Question:** Is CSP solvability *in* NP?

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# Solving CSP

- **Enumeration** of all assignments and testing
  - ↪ ... too costly
- **Backtracking** search
  - ↪ 1001 different strategies, often “dead” search paths are explored extensively
- **Constraint propagation**: elimination of obviously impossible values followed by backtracking search
- Many other search methods, e.g., local search, stochastic search, etc.
- ↪ How do we solve CSP with infinite domains?

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# General Assumptions

- Only at most **binary** constraints (i.e., we can use **constraint graph**)
- Uniform domain  $D$  for all variables
- Unary constraints  $D_i$  and binary constraints  $R_{ij}$  are **sets** of values or sets of pairs of values, resp.
- We assume that for all nodes  $i, j$ :

$$(x, y) \in R_{ij} \Rightarrow (y, x) \in R_{ji}$$

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# Local Consistency

- A CSP is *locally consistent* if for particular subsets of the variables, solutions of the restricted CSP can be extended to solutions of a larger set of variables.
- ↪ Methods to transform a CSP into a tighter, but “equivalent” problem.

## Definition

A binary CSP  $\langle V, D, C \rangle$  is **arc consistent** (or **2-consistent**) if for all nodes  $1 \leq i, j \leq n$ ,

$$x \in D_i \Rightarrow \exists y \in D_j \text{ s. t. } (x, y) \in R_{ij}$$

- ↪ When a CSP is **arc consistent**, each one variable assignment  $\{v_i\} \rightarrow D$  that satisfies all (unary) constraints in  $v_i$ , i. e.,  $D_i$ , can be extended to a two variable assignment  $\{v_i, v_j\} \rightarrow D$  that satisfies all unary/binary constraints in these variables, i. e.,  $D_i$ ,  $D_j$ , and  $R_{ij}$ .

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# Arc Consistency

## EnforceArcConsistency ( $\mathcal{C}$ ):

*Input:* a (binary) CSP  $\mathcal{C} = \langle V, D, C \rangle$

*Output:* an equivalent, but arc consistent CSP  $\mathcal{C}'$

**repeat**

**for** each arc  $(v_i, v_j)$  with  $R_{ij} \in C$

$D_i := D_i \cap \{x \in D : \text{ex. } y \in D_j \text{ s. t. } (x, y) \in R_{ij}\}$

**endfor**

**until** no domain is changed

- Terminates in time  $O(n^3 \cdot k^3)$  if we have finite domains (where  $k$  is the number of values)
- ↪ There exist different (more efficient) algorithms for enforcing arc consistency.

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## Lemma

- *Enforcing arc consistency yields an arc consistent CSP.*
- *Enforcing arc consistency is solution invariant, i. e. it does not change the set of solutions.*

↪ Arc consistent CSPs need not be consistent, and vice versa.

# Arc Consistency – Example

$$D_1 = \{1, 2, 3\}$$

$$D_2 = \{2, 3\}$$

$$D_3 = \{2\}$$

$$R_{ij} = \neq \text{ for } i \neq j$$

①  $D_1 := D_1 \cap \{x : y \in D_3 \wedge (x, y) \in R_{13}\} = \{1, 3\}$

②  $D_2 := D_2 \cap \{x : y \in D_3 \wedge (x, y) \in R_{23}\} = \{3\}$

③  $D_1 := D_1 \cap \{x : y \in D_2 \wedge (x, y) \in R_{12}\} = \{1\}$

④ CSP is now **arc consistent**

- Since all unary constraints are singletons, this defines a **solution** of the CSP.
- Since enforcing arc consistency does not change the set of solutions, this is a unique solution of the original CSP.

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## Local Consistency (2): Path Consistency

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### Definition

A binary CSP  $\langle V, D, C \rangle$  is said to be **path consistent** (or **3-consistent**) if for all nodes  $1 \leq i, j, k \leq n$ ,

$$x \in D_i, y \in D_j, (x, y) \in R_{ij} \Rightarrow \\ \exists z \in D_k \text{ s. t. } (x, z) \in R_{ik} \text{ and } (y, z) \in R_{jk}$$

- ↪ When a CSP is **path consistent**, each two variable assignment  $\{v_i, v_j\} \rightarrow D$  satisfying all constraints in  $v_i$  and  $v_j$  can be extended to any three variable assignment  $\{v_i, v_j, v_k\} \rightarrow D$  such that all constraints in these variables are satisfied.

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# Path Consistency

## EnforcePathConsistency ( $\mathcal{C}$ ):

*Input:* a (binary) CSP  $\mathcal{C} = \langle V, D, C \rangle$  of size  $n$

*Output:* an equivalent, but path consistent CSP  $\mathcal{C}'$

**repeat**

**for** all  $1 \leq i, j, k \leq n$

$R_{ij} := R_{ij} \cap$

$\{(x, y) : \text{ex. } z \in D_k \text{ s. t. } (x, z) \in R_{ik} \text{ and } (y, z) \in R_{jk}\}$

**endfor**

**until** no binary constraint is changed

- ↪ Terminates in time  $O(n^5 \cdot k^5)$  if we have finite domains (where  $k$  is the number of values)
- ↪ Enforcing path consistency is solution invariant.

# Local Consistency (3): $k$ -Consistency and Strong $k$ -Consistency

## Definition

- A binary CSP  $\langle V, D, C \rangle$  is  **$k$ -consistent** if, given variables  $x_1, \dots, x_k$  and an assignment  $a : \{x_1, \dots, x_{k-1}\} \rightarrow D$  that satisfies all constraint in these variables,  $a$  can be extended to an assignment  $a' : \{x_1, \dots, x_k\} \rightarrow D$  that satisfies all constraints in these  $k$  variables.
- A binary CSP  $\langle V, D, C \rangle$  is **strongly  $k$ -consistent** if it is  $k'$ -consistent for each  $k' \leq k$ .
- A binary CSP  $\langle V, D, C \rangle$  is **globally consistent** if it is strongly  $n$ -consistent where  $n$  is the size of  $V$ .

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# Local Consistency (3)

- $k$ -consistency: The computation costs grow exponentially with  $k$ .
- If a CSP is globally consistent, then
  - a solution can be constructed in polynomial time,
  - its constraints are **minimal**,
  - and it has a solution iff there is no empty constraint.
- $k$ -consistent  $\not\Rightarrow k - 1$ -consistent

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# Qualitative Reasoning with CSP

If we want to use CSPs for qualitative reasoning, we have

- **infinite** domains
- mostly only **finitely many** relations (basic relations and their unions)
- **arc consistent** CSPs (usually)

Questions:

- How do we achieve  **$k$ -consistency** (for some fixed  $k$ )?
- Is  $k$ -consistency (for some fixed  $k$ ) enough to guarantee **global consistency**?

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# Operations on Binary Relations

## Composition:

$$R_1 \circ R_2 = \{(x, y) \in D^2 : \exists z \in D \text{ s.t. } (x, z) \in R_1 \text{ and } (z, y) \in R_2\}$$

## Converse:

$$R^{-1} = \{(x, y) \in D^2 : (y, x) \in R\}$$

## Intersection:

$$R_1 \cap R_2 = \{(x, y) \in D^2 : (x, y) \in R_1 \text{ and } (x, y) \in R_2\}$$

## Union:

$$R_1 \cup R_2 = \{(x, y) \in D^2 : (x, y) \in R_1 \text{ or } (x, y) \in R_2\}$$

## Complement:

$$\overline{R} = \{(x, y) \in D^2 : (x, y) \notin R\}$$

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# Conditions on Vocabulary for Qualitative Reasoning

- Let  $\mathbf{B}$  be a finite set of (binary) **base relations**.
- ↪ The relations in  $\mathbf{B}$  should be **JEPD**, i. e., jointly exhaustive and pairwise disjoint.
- $\mathbf{B}$  should be **closed under converse**.
- Let  $\mathbf{A}$  be the set of relations that can be built by taking the unions of relations from  $\mathbf{B}$  ( $\rightsquigarrow 2^{|\mathbf{B}|}$  different relations).
- ↪  $\mathbf{A}$  is closed under converse, complement, intersection and union.
- $\mathbf{A}$  should be **closed under composition of base relations**, i. e., for all  $B, B' \in \mathbf{B}$ ,  $B \circ B' \in \mathbf{A}$ .
- ↪  $\mathbf{A}$  is closed under composition of arbitrary relations.
- ↪ This condition does not hold necessarily.  
Example:  $\mathbf{B} = \{<, =, >\}$  interpreted over the integers is not closed under composition (and has no finite closure):

$$< \circ < = < \setminus \{(i, j) : i = j - 1\} \subsetneq <$$

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# Computing Operations on Relations

Let  $\mathbf{A}$  be a relation system over the set of **base relations**  $\mathbf{B}$  that satisfies the conditions spelled out above.

$\rightsquigarrow$  We may write relations as *sets* of base relations:

$$B_1 \cup \dots \cup B_n \sim \{B_1, \dots, B_n\}$$

Then the operations on the relations can be *computed* as follows:

**Composition:**

$$\{B_1, \dots, B_n\} \circ \{B'_1, \dots, B'_m\} = \bigcup_{i=1}^n \bigcup_{j=1}^m (B_i \circ B'_j)$$

**Converse:**

$$\{B_1, \dots, B_n\}^{-1} = \{B_1^{-1}, \dots, B_n^{-1}\}$$

**Complement:**

$$\overline{\{B_1, \dots, B_n\}} = \{B \in \mathbf{B} : B \neq B_i, \text{ for each } 1 \leq i \leq n\}$$

**Intersection** and **union** are defined set-theoretically.

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# Reasoning Problems

Given a qualitative CSP:

**CSP-Satisfiability (CSAT):**

- Is the CSP satisfiable/solvable?

**CSP-Entailment (CENT):**

- Given in addition  $xRy$ : Is  $xRy$  satisfied in each solution of the CSP?

**Computation of an equivalent minimal CSPs (CMIN):**

- Compute for each pair  $x, y$  the strongest constrained (minimal) relation entailed by the CSP.

↪ These problems are **equivalent** under **Turing reductions**

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# Reductions between CSP Problems

## Theorem

*CSAT, CENT and CMIN are equivalent under polynomial Turing reductions.*

## Proof.

$CSAT \leq_T CENT$  and  $CENT \leq_T CMIN$  are obvious.

$CENT \leq_T CSAT$ : We solve  $CENT$  ( $CSP \models xRy?$ ) by testing satisfiability of the CSP extended by  $x\{B\}y$  where  $B$  ranges over all base relations. Let  $B_1, \dots, B_k$  be the relations for which we get a positive answer. Then  $x\{B_1, \dots, B_k\}y$  is entailed by the CSP.

$CMIN \leq_T CENT$ : We use entailment for computing the minimal constraint for each pair. Starting with the universal relation, we remove one base relation until we have a minimal relation that is still entailed. □

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# Path Consistency for Qualitative CSPs

Given a qualitative CSP with  $R_{ij} = R_{ji}^{-1}$ . Then **path consistency** can be enforced by doing the following:

$$R_{ij} := R_{ij} \cap (R_{ik} \circ R_{kj}).$$

Path consistency **guarantees** ...

- sometimes **minimality**
  - sometimes **satisfiability**
  - however sometimes the CSP is **not satisfiable**, even if the CSP contains only **base relations**
- ↪ All this depends on the vocabulary.

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# Example: Point Relations

Composition table:

	<	=	>
<	<	<	<, =, >
=	<	=	>
>	<, =, >	>	>

**Figure:** Composition table for the point algebra. For example:  
 $\{<\} \circ \{=\} = \{<\}$

- $\{<, =\} \circ \{<\} = \{<\}$
- $\{<, >\} \circ \{<\} = \{<, =, >\}$
- $\{<, =\}^{-1} = \{>, =\}$
- $\{<, =\} \cap \{>, =\} = \{=\}$

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# Some Properties of the Point Relations

## Theorem

*A path consistent CSP over the point relations is consistent.*

## Corollary

*CSAT, CENT and CMIN are polynomial problems for the point relations.*

## Theorem

*A path consistent CSP over all point relations without  $\{<, >\}$  is minimal.*

Proofs later ...

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# Outlook

- **Qualitative representation and reasoning** usually starts with a finite vocabulary (a finite set of relations).
- Qualitative descriptions are usually simply logical theories consisting of sets of atomic formulae (and some background theory).
- **Reasoning problems** are (as usual) satisfiability, model finding, and deduction.
- Can be addressed with **CSP methods** (but note: **infinite domains**).
- **Path consistency** is the basic reasoning step ... sometimes this is enough.
- Usually, path-consistent atomic CSPs are satisfiable. However, there exist some pathological relation systems.
- Can be taken further  $\rightsquigarrow$  **relation algebra**

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(Contains a pathological set of relations.)