Principles of Knowledge Representation and Reasoning

Nonmonotonic Reasoning II: Minimal Models and Nonmonotonic Logic Programs

Bernhard Nebel, Stefan Wölfl, and Marco Ragni

Albert-Ludwigs-Universität Freiburg

May 19, 2010

Principles of Knowledge Representation and Reasoning

May 19, 2010 — Nonmonotonic Reasoning II:
Minimal Models and Nonmonotonic Logic Programs

Minimal Model Reasoning
- Motivation
- Definition
- Example
- Embedding in DL

Nonmonotonic Logic Programs
- Motivation
- Answer Sets
- Complexity
- Stratification
- Applications
- Literature

Minimal Model Reasoning

- Conflicts between defaults in default logic lead to multiple extensions
- Each extension corresponds to a maximal set of non-violated defaults
- Reasoning with defaults can also be achieved by a simpler mechanism: predicate or propositional logic + minimize the number of cases where a default (expressed as a conventional formula) is violated $\Longrightarrow$ minimal models
- Notion of minimality: cardinality vs. set-inclusion

Entailment with respect to Minimal Models

Definition

Let A be a set of atomic propositions. Let Φ be a set of propositional formulae on A, and $B \subseteq A$ a set (called abnormalities).

Then ψ B-minimally follows from Φ ($\Phi \models_B \psi$) if $\mathcal{I} \models \psi$ for all interpretations $\mathcal{I}$ such that

- $\mathcal{I} \models \Phi$ and
- there is no $\mathcal{I}'$ such that $\mathcal{I}' \models \Phi$ and $\{b \in B \,|\, \mathcal{I}' \models b\} \subsetneq \{b \in B \,|\, \mathcal{I} \models b\}$.

Minimal models: example

$$\Phi = \left\{ \begin{array}{ll} \text{student} \land \neg\text{ABstudent} \rightarrow \neg\text{earnsmoney}, & \text{student}, \\ \text{adult} \land \neg\text{ABadult} \rightarrow \text{earnsmoney}, & \text{student} \rightarrow \text{adult} \end{array} \right\}$$

Φ has the following models:

$$\begin{array}{l}
\mathcal{I}_1 \models \text{student} \land \text{adult} \land \text{earnsmoney} \land \text{ABstudent} \land \text{ABadult} \\
\mathcal{I}_2 \models \text{student} \land \text{adult} \land \neg\text{earnsmoney} \land \text{ABstudent} \land \text{ABadult} \\
\mathcal{I}_3 \models \text{student} \land \text{adult} \land \text{earnsmoney} \land \text{ABstudent} \land \neg\text{ABadult} \\
\mathcal{I}_4 \models \text{student} \land \text{adult} \land \neg\text{earnsmoney} \land \neg\text{ABstudent} \land \text{ABadult}
\end{array}$$

Relation to Default Logic

We can embed propositional minimal model reasoning in the propositional default logic.

Theorem

Let A be a set of atomic propositions. Let Φ be a set of propositional formulae on A, and $B \subseteq A$.
Then $\Phi \models_B \psi$ if and only if ψ follows from $\langle D, W \rangle$ skeptically, where

$$D = \left\{ \frac{: \neg b}{\neg b} \middle| \, b \in B \right\} \text{ and } W = \Phi.$$

Relation to Default Logic: Proof

Proof sketch.

"$\Rightarrow$": Assume there is an extension E of $\langle D, W \rangle$ such that $\psi \notin E$. Hence there is an interpretation $\mathcal{I}$ such that $\mathcal{I} \models E$ and $\mathcal{I} \models \neg\psi$.
By the fact that there is no extension F such that $E \subsetneq F$, $\mathcal{I}$ is a B-minimal model of Φ. Hence ψ does not B-minimally follow from Φ.
"$\Leftarrow$": Assume ψ does not B-minimally follow from Φ. Hence there is a B-minimal model $\mathcal{I}$ of Φ such that $\mathcal{I} \not\models \psi$. Define

$$E = \mathrm{Th}(\Phi \cup \{\neg b | b \in B, \mathcal{I} \models \neg b\}).$$

Now $\mathcal{I} \models E$ and because $\mathcal{I} \not\models \psi$, $\psi \notin E$.
We can show that E is an extension of $\langle D, W \rangle$.
Because there is an extension E such that $\psi \notin E$, ψ does not skeptically follow from $\langle D, W \rangle$. □

Nonmonotonic Logic Programs: Background

- Answer set semantics: a formalization of negation-as-failure in logic programming (Prolog)
- Other formalizations: well-founded semantics, perfect-model semantics, inflationary semantics, ...
- Can be viewed as a simpler variant of default logic
- A better alternative to propositional logic in some applications

Nonmonotonic Logic Programs

Let $A = \{a_1, \ldots, a_n\}$ be a set of propositions.

Rules:

$$c \leftarrow b_1, \ldots, b_m, \text{not } d_1, \ldots, \text{not } d_k$$

where $\{c, b_1, \ldots, b_m, d_1, \ldots, d_k\} \subseteq A$

- Meaning similar to default logic:
  If
  1. **we have derived** $b_1, \ldots, b_m$ **and**
  2. **cannot derive any of** $d_1, \ldots, d_k$**,**

  then derive c**.**
- Rules without right-hand side (facts): $c \leftarrow$
- Rules without left-hand side (constraints):
  $$\leftarrow b_1, \ldots, b_m, \text{not } d_1, \ldots, \text{not } d_k$$

Answer Sets – Formal Definition

Definition

Let P be a set of rules without **not**, $\Delta \subseteq A$.
The closure $\mathrm{dcl}(P) \subseteq A$ of P is defined by iterative application of the rules in the obvious way. Δ is an answer set of P if $\Delta = \mathrm{dcl}(P)$ and there is no constraint in P violated by Δ.

Definition (Reduct)

The reduct of a program P with respect to a set of atoms $\Delta \subseteq A$ is defined as:

$$P^{\Delta} := \{c \leftarrow b_1, \ldots, b_m | \\ (c \leftarrow b_1, \ldots, b_m, \mathrm{not}\, d_1, \ldots, \mathrm{not}\, d_k) \in P, \{d_1, \ldots, d_k\} \cap \Delta = \emptyset\}$$

Definition (Answer set)

$\Delta \subseteq A$ is an answer set of P if Δ is an answer set of P^{Δ}.

Examples

- $P_1 = \{a \leftarrow, \quad b \leftarrow a, \quad c \leftarrow b\}$
- $P_2 = \{a \leftarrow b, \quad b \leftarrow a\}$
- $P_3 = \{p \leftarrow \text{not } p\}$
- $P_4 = \{p \leftarrow \text{not } q, \quad q \leftarrow \text{not } p\}$
- $P_5 = \{p \leftarrow \text{not } q, \quad q \leftarrow \text{not } p, \quad \leftarrow p\}$

Complexity: existence of answer sets is NP-complete

1. Membership in NP: Guess $\Delta \subseteq A$ (*nondet. polytime*), compute P^{Δ}, compute its closure, compare to Δ (*everything det. polytime*).
2. NP-hardness: Reduction from 3SAT: an answer set exists iff clauses are satisfiable:

$$p \leftarrow \text{not } \hat{p}$$
$$\hat{p} \leftarrow \text{not } p$$

for every proposition p occurring in the clauses, and

$$\leftarrow \text{not } l'_1, \text{not } l'_2, \text{not } l'_3$$

for every clause $l_1 \vee l_2 \vee l_3$, where $l'_i = p$ if $l_i = p$ and $l'_i = \hat{p}$ if $l_i = \neg p$.

Programs for Reasoning with Answer Sets

- smodels (Niemelä & Simons), dlv (Eiter et al.), ...
- Schematic input:

```
p(X) :- not q(X).
q(X) :- not p(X).
r(a).
r(b).
r(c).
```

```
anc(X,Y) :- par(X,Y).
anc(X,Y) :- par(X,Z), anc(Z,Y).
par(a,b). par(a,c). par(b,d).
female(a).
male(X) :- not(female(X)).
forefather(X,Y) :-
            anc(X,Y), male(X).
```

Difference to the Propositional Logic

- The *ancestor* relation is the transitive closure of the *parent* relation.
- Transitive closure cannot be (concisely) represented in propositional/predicate logic.

  $$par(X,Y) \rightarrow anc(X,Y)$$
  $$par(X,Z) \land anc(Z,Y) \rightarrow anc(X,Y)$$

  The above formulae only guarantee that *anc* is a *superset* of the transitive closure of *par*.
- For transitive closure one needs the minimality condition in some form: nonmonotonic logics, fixpoint logics, ...

Stratification

The reason for multiple answer sets is the fact that a may depend on b and simultaneously b may depend on a.
The lack of this kind of circular dependencies makes reasoning easier.

Definition

A logic program P is stratified if P can be partitioned to $P = P_1 \cup \cdots \cup P_n$ so that for all $i \in \{1, \ldots, n\}$ and $(c \leftarrow b_1, \ldots, b_m, \text{not}\, d_1, \ldots, \text{not}\, d_k) \in P_i$,

1. there is no $\text{not}\, c$ in P_i and
2. there are no occurrences of c anywhere in $P_1 \cup \cdots \cup P_{i-1}$.

Stratification

Theorem

A stratified program P has exactly one answer set. The unique answer set can be computed in polynomial time.

Example

Our earlier examples with more than one or no answer sets:

$$P_3 = \{p \leftarrow \text{not}\, p\}$$
$$P_4 = \{p \leftarrow \text{not}\, q, \quad q \leftarrow \text{not}\, p\}$$

Applications of Logic Programs

1. Simple forms of default reasoning (e.g., inheritance networks, see later)
2. A solution to the frame problem: instead of using frame axioms, use defaults

$$a_{t+1} \leftarrow a_t, \text{not}\, \neg a_{t+1}$$

   By default, truth-values of facts stay the same.
3. deductive databases (Datalog$^{\neg}$)
4. et cetera: Everything that can be done with propositional logic can also be done with propositional nonmotononic logic programs.

Literature

M. Gelfond and V. Lifschitz.
The stable model semantics for logic programming.
Proceedings of the Fifth International Conference on Logic Programming, The MIT Press, 1988.

I. Niemelä and P. Simons.
Smodels - an implementation of the stable model and well-founded semantics for normal logic programs.
Proceedings of the 4th International Conference on Logic Programming and Non-monotonic Reasoning, 1997.

T. Eiter, W. Faber, N. Leone, and G. Pfeifer.
Declarative problem solving using the dlv system.
In J Minker, editor, Logic Based AI, Kluwer Academic Publishers, 2000.