Principles of Knowledge Representation and Reasoning

Modal Logics

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Motivation for Studying Modal Logics

- Notions like believing and knowing require a more general semantics than e.g. propositional logic has.
- Some KR formalisms can be understood as (fragments of) a propositional modal logic.
- Application 1: spatial representation formalism RCC8
- Application 2: description logics
- Application 3: reasoning about time
- Application 4: reasoning about actions, strategies, etc.

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Often, we want to state something where we have an "embedded proposition":

- John believes that it is Sunday.
- I know that $2^{10} = 1024$.

Reasoning with embedded propositions:

- *John believes that if it is Sunday, then shops are closed.*
- *John believes that it is Sunday.*
- This implies (assuming *belief* is closed under *modus ponens*):
  John believes that shops are closed.

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Propositional logic + operators $\Box$ & $\Diamond$ (*Box & Diamond*):

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\varphi & \longrightarrow & \ldots & \textit{classical propositional formula} \\
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$\Box$ and $\Diamond$ have the same operator precedence as $\neg$.

Some possible readings of $\Box\varphi$:

- Necessarily φ (alethic)
- Always φ (temporal)
- φ should be true (deontic)
- Agent A believes φ (doxastic)
- Agent A knows φ (epistemic)

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Truth-Functional Semantics?

- Could it be possible to define the meaning of $\Box\varphi$ truth-functionally, i.e. by referring to the truth value of φ only?
- An attempt to interpret *necessity* truth-functionally:
  - If φ is false, then $\Box\varphi$ should be false.
  - If φ is true, then ...
    - ... $\Box\varphi$ should be true $\leadsto$ $\Box$ is the identity function
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- Note: There are only 4 different unary Boolean functions $\{T, F\} \to \{T, F\}$.

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Semantics: The Idea

In classical propositional logic, formulae are interpreted over single interpretations and are evaluated to *true* or *false*.

In modal logics one considers sets of interpretations: possible worlds (physically possible, conceivable, ...).

Main idea:

- Consider a world (interpretation) w and a set of worlds W which are possible with respect to w.
- A classical formula (with no modal operators) φ is true with respect to (w, W) iff φ is true in w.
- $\Box\varphi$ is true wrt (w, W) iff φ is true in all worlds in W.
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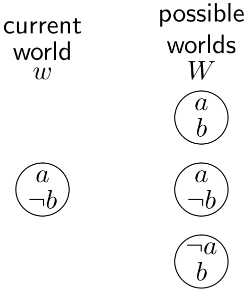

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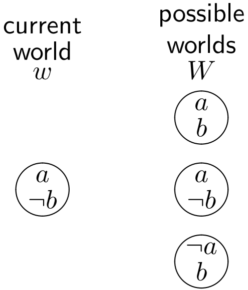

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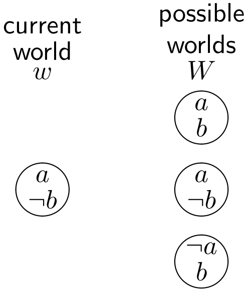

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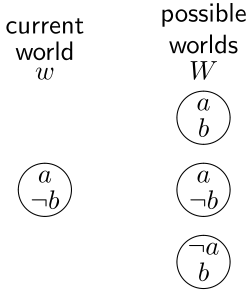

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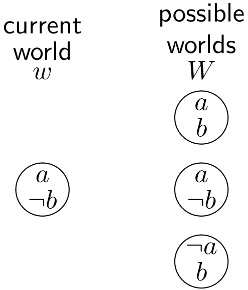

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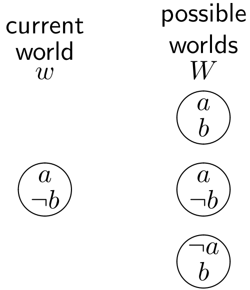

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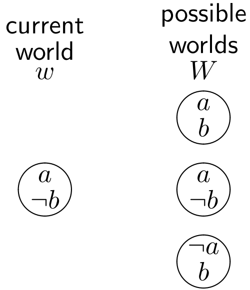

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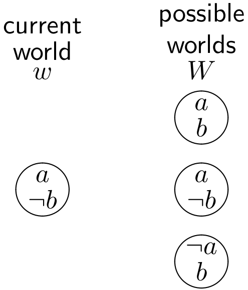

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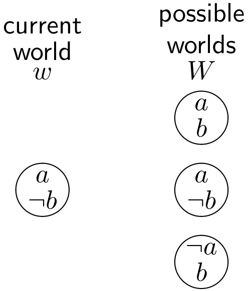

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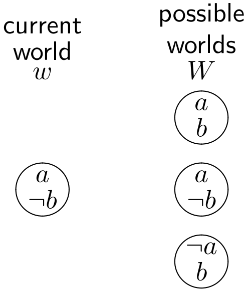

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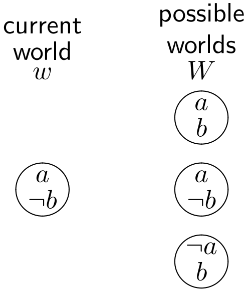

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Frames, Interpretations, and Worlds

A (Kripke, relational) frame is a pair $\mathcal{F} = \langle W, R \rangle$ where W is a non-empty set (of worlds) and $R \subseteq W \times W$ (the accessibility relation).

For $(w, v) \in R$ we write also $w\,R\,v$.

We say that v is an R-successor of w and that v is reachable (or R-reachable) from w.

A (Σ)-interpretation (or model) based on the frame $\mathcal{F} = \langle W, R \rangle$ is a triple $\mathcal{I} = \langle W, R, \pi \rangle$, where π is a function from worlds to truth assignments:

$$\pi\colon W \to (\Sigma \to \{T, F\}).$$

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Semantics: Truth in a World

A formula φ is true in world w of an interpretation $\mathcal{I} = \langle W, R, \pi \rangle$ under the following conditions:

$$
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\mathcal{I}, w \models a & \textit{iff } \pi(w)(a) = T \\
\mathcal{I}, w \models \top & \\
\mathcal{I}, w \not\models \bot & \\
\mathcal{I}, w \models \neg\varphi & \textit{iff } \mathcal{I}, w \not\models \varphi \\
\mathcal{I}, w \models \varphi \land \psi & \textit{iff } \mathcal{I}, w \models \varphi \text{ and } \mathcal{I}, w \models \psi \\
\mathcal{I}, w \models \varphi \lor \psi & \textit{iff } \mathcal{I}, w \models \varphi \text{ or } \mathcal{I}, w \models \psi \\
\mathcal{I}, w \models \varphi \rightarrow \psi & \textit{iff } \text{if } \mathcal{I}, w \models \varphi \text{ then } \mathcal{I}, w \models \psi \\
\mathcal{I}, w \models \varphi \leftrightarrow \psi & \textit{iff } \mathcal{I}, w \models \varphi \text{ if and only if } \mathcal{I}, w \models \psi \\
\mathcal{I}, w \models \Box\varphi & \textit{iff } \mathcal{I}, u \models \varphi \text{ for all } u \text{ s.t. } wRu \\
\mathcal{I}, w \models \Diamond\varphi & \textit{iff } \mathcal{I}, u \models \varphi \text{ for at least one } u \text{ s.t. } wRu
\end{array}
$$

Semantics: Truth in a World

A formula φ is true in world w of an interpretation $\mathcal{I} = \langle W, R, \pi \rangle$ under the following conditions:

$\mathcal{I}, w \models a$	*iff* $\pi(w)(a) = T$
$\mathcal{I}, w \models \top$	
$\mathcal{I}, w \not\models \bot$	
$\mathcal{I}, w \models \neg\varphi$	*iff* $\mathcal{I}, w \not\models \varphi$
$\mathcal{I}, w \models \varphi \wedge \psi$	*iff* $\mathcal{I}, w \models \varphi$ and $\mathcal{I}, w \models \psi$
$\mathcal{I}, w \models \varphi \vee \psi$	*iff* $\mathcal{I}, w \models \varphi$ or $\mathcal{I}, w \models \psi$
$\mathcal{I}, w \models \varphi \rightarrow \psi$	*iff* if $\mathcal{I}, w \models \varphi$ then $\mathcal{I}, w \models \psi$
$\mathcal{I}, w \models \varphi \leftrightarrow \psi$	*iff* $\mathcal{I}, w \models \varphi$ if and only if $\mathcal{I}, w \models \psi$
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Satisfiability and Validity

A formula φ is satisfiable in an interpretation $\mathcal{I}$ (or in a frame $\mathcal{F}$, or in a class of frames $\mathcal{C}$) if there exists a world in $\mathcal{I}$ (or an interpretation $\mathcal{I}$ based on $\mathcal{F}$, or an interpretation $\mathcal{I}$ based on a frame contained in the class $\mathcal{C}$, respectively) such that $\mathcal{I}, w \models \varphi$.

A formula φ is true in an interpretation $\mathcal{I}$ (symbolically $\mathcal{I} \models \varphi$) if φ is true in all worlds of $\mathcal{I}$.

A formula φ is valid in a frame $\mathcal{F}$ or $\mathcal{F}$-valid (symbolically $\mathcal{F} \models \varphi$) if φ is true in all interpretations based on $\mathcal{F}$.

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K is the class of all frames – named after Saul Kripke, who invented this semantics.

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3. $\Box\varphi$, if φ is a classical tautology
4. $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$ (axiom schema K)

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If $\Box\varphi$ is false in w, then $(\Box\varphi \to \Box\psi)$ is true and K is true in w.
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Proposition

$\Diamond\top$ *is not* **K**-*valid.*

Proof.

A counterexample is the following interpretation:

$$\mathcal{I} = \langle \{w\}, \emptyset, \{w \mapsto (a \mapsto T)\} \rangle.$$

We have $\mathcal{I}, w \not\models \Diamond\top$ because there is no u such that wRu. □

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Let us consider the following axiom schemata:

T:	$\Box\varphi \rightarrow \varphi$	(knowledge axiom)
4:	$\Box\varphi \rightarrow \Box\Box\varphi$	(positive introspection)
5:	$\Diamond\varphi \rightarrow \Box\Diamond\varphi$	(or $\neg\Box\varphi \rightarrow \Box\neg\Box\varphi$: negative introspection)
B:	$\varphi \rightarrow \Box\Diamond\varphi$	
D:	$\Box\varphi \rightarrow \Diamond\varphi$	(or $\Box\varphi \rightarrow \neg\Box\neg\varphi$: disbelief in the negation)

. . . and the following classes of frames, for which the accessibility relation is restricted as follows:

- T: reflexive (wRw for each world w)
- 4: transitive (wRu and uRv implies wRv)
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Connection between Accessibility Relations and Axiom Schemata (1)

Theorem

Axiom schema T $(4, 5, B, D)$ is **T**- *valid (***4-, 5-, B-***, or* **D***-valid, respectively).*

Proof.

For T and **T**:Let $\mathcal{F}$ be a frame from class **T**. Let $\mathcal{I}$ be an interpretation based on $\mathcal{F}$ and let w be an arbitrary world in $\mathcal{I}$. If $\Box\varphi$ is not true in world w, then axiom T is true in w. If $\Box\varphi$ is true in w, then φ is true in all accessible worlds. Since the accessibility relation is reflexive, w is among the accessible worlds, i.e., φ is true in w. This means that also in this case T is true in w. This means, T is true in all worlds in all interpretations based on **T**-frames. □

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Proof.

For T and **T**:Let $\mathcal{F}$ be a frame from class **T**. Let $\mathcal{I}$ be an interpretation based on $\mathcal{F}$ and let w be an arbitrary world in $\mathcal{I}$. If $\Box\varphi$ is not true in world w, then axiom T is true in w. If $\Box\varphi$ is true in w, then φ is true in all accessible worlds. Since the accessibility relation is reflexive, w is among the accessible worlds, i.e., φ is true in w. This means that also in this case T is true in w. This means, T is true in all worlds in all interpretations based on **T**-frames. □

Connection between Accessibility Relations and Axiom Schemata (2)

Theorem

If T $(4, 5, B, D)$ is valid in a frame $\mathcal{F}$, then $\mathcal{F}$ is a **T***-Frame (***4-, 5-, B-***, or* **D***-frame, respectively).*

Proof.

For T and **T**: Assume that $\mathcal{F}$ is not a **T**-frame. We will construct an interpretation based on $\mathcal{F}$ that falsifies T.

Because $\mathcal{F}$ is not a **T**-frame, there is a world w such that not wRw. Construct an interpretation $\mathcal{I}$ such that $w \not\models p$ and $v \models p$ for all v such that wRv.

Now $w \models \Box p$ and $w \not\models p$, and hence $w \not\models \Box p \rightarrow p$. □

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Different Modal Logics

Name	Property	Axiom schema
K	–	$\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$
T	reflexivity	$\Box\varphi \rightarrow \varphi$
4	transitivity	$\Box\varphi \rightarrow \Box\Box\varphi$
5	euclidicity	$\Diamond\varphi \rightarrow \Box\Diamond\varphi$
B	symmetry	$\varphi \rightarrow \Box\Diamond\varphi$
D	seriality	$\Box\varphi \rightarrow \Diamond\varphi$

Some basic modal logics:

$$\begin{array}{rcl} K & & \\ KT4 & = & S4 \\ KT5 & = & S5 \\ \vdots & & \end{array}$$

Different Modal Logics

logics	$\Box$	$\Diamond = \neg\Box\neg$	K	T	4	5	B	D
alethic	necessarily	possibly	Y	Y	Y	Y	Y	Y
epistemic	known	possible	Y	Y	Y	Y	Y	Y
doxastic	believed	possible	Y	N	Y	Y	N	Y
deontic	obligatory	permitted	Y	N	Y?	Y?	N	Y
temporal	always in the future	sometimes	Y	Y/N	Y	N	N	N/Y

Proof Methods

- How can we show that a formula is $\mathcal{C}$-valid?
- In order to show that a formula is not $\mathcal{C}$-valid, one can construct a counterexample (= an interpretation that falsifies it).
- When trying out all ways of generating a counterexample without success, this counts as a proof of validity.

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$\rightsquigarrow$ Method of (analytic/semantic) tableaux

Tableau Method

A tableau is a tree with nodes marked as follows:

- $w \models \varphi$,
- $w \not\models \varphi$, and
- wRv.

A branch that contains nodes marked with $w \models \varphi$ and $w \not\models \varphi$ is closed. All other branches are open. If all branches are closed, the tableau is called closed.

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Tableau Rules for the Propositional Logic

$$\frac{w \models \varphi \lor \psi}{w \models \varphi \mid w \models \psi} \qquad \frac{w \not\models \varphi \lor \psi}{\begin{array}{c} w \not\models \varphi \\ w \not\models \psi \end{array}} \qquad \frac{w \models \neg\varphi}{w \not\models \varphi}$$

$$\frac{w \models \varphi \land \psi}{\begin{array}{c} w \models \varphi \\ w \models \psi \end{array}} \qquad \frac{w \not\models \varphi \land \psi}{w \not\models \varphi \mid w \not\models \psi} \qquad \frac{w \not\models \neg\varphi}{w \models \varphi}$$

$$\frac{w \models \varphi \rightarrow \psi}{w \not\models \varphi \mid w \models \psi} \qquad \frac{w \not\models \varphi \rightarrow \psi}{\begin{array}{c} w \models \varphi \\ w \not\models \psi \end{array}}$$

Additional Tableau Rules for the Modal Logic **K**

$$\frac{w \models \Box\varphi}{v \models \varphi}$$ if wRv is on the branch already

$$\frac{w \not\models \Box\varphi}{\begin{array}{c} wRv \\ v \not\models \varphi \end{array}}$$ for new v

$$\frac{w \models \Diamond\varphi}{\begin{array}{c} wRv \\ v \models \varphi \end{array}}$$ for new v

$$\frac{w \not\models \Diamond\varphi}{v \not\models \varphi}$$ if wRv is on the branch already

Properties of **K** Tableaux

Proposition

If a K-tableau is closed, the truth condition at the root cannot be satisfied.

Theorem (Soundness)

*If a K-tableau with root $w \not\models \varphi$ is closed, then φ is **K**-valid.*

Theorem (Completeness)

*If φ is **K**-valid, then there is a closed tableau with root $w \not\models \varphi$.*

Proposition (Termination)

*There are strategies for constructing **K**-tableaux that always terminate after a finite number of steps, and result in a closed tableau whenever one exists.*

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Tableau Rules for Other Modal Logics

Proofs within more restricted classes of frames allow the use of further tableau rules.

- For reflexive (**T**) frames we may extend any branch with wRw.
- For transitive (**4**) frames we have the following additional rule:
  - If wRv and vRu are in a branch, wRu may be added to the branch.
- For serial (**D**) frames we have the following rule:
  - If there is $w \models \dots$ or $w \not\models \dots$ on a branch, then add wRv for a new world v.
- Similar rules for other properties...

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Testing Logical Consequence with Tableaux

- Let Θ be a set of formulas. When does a formula φ follow from Θ: $\Theta \models_{\mathbf{X}} \varphi$?
- Test whether in all interpretations on $\mathbf{X}$-frames in which Θ is true, also φ is true.
- Wouldn't there be a deduction theorem we could use?
- Example: $a \models_{\mathbf{K}} \Box a$ holds, but $a \to \Box a$ is not **K**-valid.
- There is no deduction theorem as in the propositional logic, and logical consequence cannot be directly reduced to validity!

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Tableaux and Logical Implication

For testing logical consequence, we can use the following tableau rule:

- If w is a world on a branch and $\psi \in \Theta$, then we can add $w \models \psi$ to our branch.
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Connection between propositional modal logic and FOL?

- There are similarities between the predicate logic and propositional modal logics:
  1. $\Box$ vs. $\forall$
  2. $\Diamond$ vs. $\exists$
  3. the possible worlds vs. the objects of the universe
- In fact, we can show for many propositional modal logics that they can be embedded in the predicate logic.

$\Rightarrow$ Modal logics can be understood as a sublanguage of FOL.

Embedding Modal Logics in the Predicate Logic (1)

1. $\tau(p, x) = p(x)$ for propositional variables p
2. $\tau(\neg\phi, x) = \neg\tau(\phi, x)$
3. $\tau(\phi \vee \psi, x) = \tau(\phi, x) \vee \tau(\psi, x)$
4. $\tau(\phi \wedge \psi, x) = \tau(\phi, x) \wedge \tau(\psi, x)$
5. $\tau(\Box\phi, x) = \forall y(R(x, y) \rightarrow \tau(\phi, y))$ for some new y
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Embedding Modal Logics in the Predicate Logic (2)

Theorem

ϕ is K-valid if and only if $\forall x\, \tau(\phi, x)$ is valid in the predicate logic.

Theorem

ϕ is T-valid if and only if in the predicate logic the logical consequence $\{\forall x R(x,x)\} \models \forall x \tau(\phi, x)$ holds.

Example

$\Box p \wedge \Diamond(p \rightarrow q) \rightarrow \Diamond q$ is K-valid, because

$$\forall x(\forall x'(R(x,x') \rightarrow p(x')) \wedge \exists x'(R(x,x') \wedge (p(x') \rightarrow q(x'))) \rightarrow \exists x'(R(x,x') \wedge q(x')))$$

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Outlook

We only looked at some basic propositional modal logics.
There are also:

- modal first order logics (with quantification $\forall$ and $\exists$ and predicates)
- multi-modal logics: more than one modality, e.g. knowledge/belief operators for several agents
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