

Principles of Knowledge Representation and Reasoning

Predicate Logic

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Motivation

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Semantics

- Interpretations

- Variable Maps

- Definition of Truth

- Terminology

- Free and Bound Variables

- Open and Closed Formulae

Literature

Why First-Order Logic (FOL)?

- ▶ In propositional logic, the only building blocks are atomic propositions.
- ▶ We cannot talk about the internal structures of these propositions.
- ▶ **Example:**
 - ▶ *All CS students know formal logic*
 - ▶ *Peter is a CS student*
 - ▶ Therefore, *Peter knows formal logic*
 - ▶ Not possible in propositional logic
- ▶ **Idea:** We introduce **predicates**, **functions**, **object variables** and **quantifiers**.

Syntax

- ▶ **variable** symbols: x, y, z, \dots
- ▶ n -ary **function** symbols: f, g, \dots
- ▶ **constant** symbols: a, b, c, \dots
- ▶ n -ary **predicate** symbols: P, Q, \dots
- ▶ logical symbols: $\forall, \exists, =, \neg, \wedge, \dots$

Terms	t	\longrightarrow	x	variable
			$f(t_1, \dots, t_n)$	function application
			a	constant

Formulae	φ	\longrightarrow	$P(t_1, \dots, t_n)$	atomic formulae
			$t = t'$	identity formulae
			\dots	propositional connectives
			$\forall x \varphi'$	universal quantification
			$\exists x \varphi'$	existential quantification

ground term, etc.: term, etc. without variable occurrences

Semantics: Idea

- ▶ In FOL, the **universe of discourse** consists of objects, functions over these objects, and relations over these objects.
- ▶ Function symbols are mapped to functions, predicate symbols are mapped to relations, and terms to objects.
- ▶ **Notation**: Instead of $\mathcal{I}(x)$ we write $x^{\mathcal{I}}$.
- ▶ **Note**: Usually one considers **all possible** non-empty universes. (However, sometimes the interpretations are restricted to particular domains, e.g. integers or real numbers.)
- ▶ Satisfiability and validity is then considered wrt all these universes.

Formal Semantics: Interpretations

Interpretations: $\mathcal{I} = \langle \mathcal{D}, \cdot^{\mathcal{I}} \rangle$ with \mathcal{D} being an arbitrary non-empty set and $\cdot^{\mathcal{I}}$ being a function which maps

- ▶ n -ary function symbols f to n -ary functions $f^{\mathcal{I}} \in [\mathcal{D}^n \rightarrow \mathcal{D}]$,
- ▶ constant symbols a to objects $a^{\mathcal{I}} \in \mathcal{D}$, and
- ▶ n -ary predicates P to n -ary relations $P^{\mathcal{I}} \subseteq \mathcal{D}^n$.

Interpretation of ground terms:

$$(f(t_1, \dots, t_n))^{\mathcal{I}} = f^{\mathcal{I}}(t_1^{\mathcal{I}}, \dots, t_n^{\mathcal{I}}) (\in \mathcal{D})$$

Truth of ground atoms:

$$\mathcal{I} \models P(t_1, \dots, t_n) \quad \text{iff} \quad \langle t_1^{\mathcal{I}}, \dots, t_n^{\mathcal{I}} \rangle \in P^{\mathcal{I}}$$

Examples

$$\mathcal{D} = \{d_1, \dots, d_n\}, n \geq 2$$

$$a^{\mathcal{I}} = d_1$$

$$b^{\mathcal{I}} = d_2$$

$$\text{eye}^{\mathcal{I}} = \{d_1\}$$

$$\text{red}^{\mathcal{I}} = \mathcal{D}$$

$$\mathcal{I} \models \text{red}(b)$$

$$\mathcal{I} \not\models \text{eye}(b)$$

$$\mathcal{D} = \{1, 2, 3, \dots\}$$

$$1^{\mathcal{I}} = 1$$

$$2^{\mathcal{I}} = 2$$

$$\vdots$$

$$\text{even}^{\mathcal{I}} = \{2, 4, 6, \dots\}$$

$$\text{succ}^{\mathcal{I}} = \{(1 \mapsto 2), (2 \mapsto 3), \dots\}$$

$$\mathcal{I} \not\models \text{even}(3)$$

$$\mathcal{I} \models \text{even}(\text{succ}(3))$$

Formal Semantics: Variable Maps

V is the set of variables. Function $\alpha: V \rightarrow \mathcal{D}$ is a **variable map**.

Notation: $\alpha[x/d]$ is identical to α except for x where $\alpha[x/d](x) = d$.

Interpretation of terms under \mathcal{I}, α :

$$\begin{aligned} x^{\mathcal{I}, \alpha} &= \alpha(x) \\ a^{\mathcal{I}, \alpha} &= a^{\mathcal{I}} \\ (f(t_1, \dots, t_n))^{\mathcal{I}, \alpha} &= f^{\mathcal{I}}(t_1^{\mathcal{I}, \alpha}, \dots, t_n^{\mathcal{I}, \alpha}) \end{aligned}$$

Truth of atomic formulae:

$$\mathcal{I}, \alpha \models P(t_1, \dots, t_n) \quad \text{iff} \quad \langle t_1^{\mathcal{I}, \alpha}, \dots, t_n^{\mathcal{I}, \alpha} \rangle \in P^{\mathcal{I}}$$

Example (cont'd):

$$\alpha = \{x \mapsto d_1, y \mapsto d_2\} \quad \mathcal{I}, \alpha \models \text{red}(x) \quad \mathcal{I}, \alpha[y/d_1] \models \text{eye}(y)$$

Formal Semantics: Truth

Truth of φ by \mathcal{I} under α ($\mathcal{I}, \alpha \models \varphi$) is defined as follows.

$\mathcal{I}, \alpha \models P(t_1, \dots, t_n)$	iff $\langle t_1^{\mathcal{I}, \alpha}, \dots, t_n^{\mathcal{I}, \alpha} \rangle \in P^{\mathcal{I}}$
$\mathcal{I}, \alpha \models t_1 = t_2$	iff $t_1^{\mathcal{I}, \alpha} = t_2^{\mathcal{I}, \alpha}$
$\mathcal{I}, \alpha \models \neg \varphi$	iff $\mathcal{I}, \alpha \not\models \varphi$
$\mathcal{I}, \alpha \models \varphi \wedge \psi$	iff $\mathcal{I}, \alpha \models \varphi$ and $\mathcal{I}, \alpha \models \psi$
$\mathcal{I}, \alpha \models \varphi \vee \psi$	iff $\mathcal{I}, \alpha \models \varphi$ or $\mathcal{I}, \alpha \models \psi$
$\mathcal{I}, \alpha \models \varphi \rightarrow \psi$	iff if $\mathcal{I}, \alpha \models \varphi$, then $\mathcal{I}, \alpha \models \psi$
$\mathcal{I}, \alpha \models \varphi \leftrightarrow \psi$	iff $\mathcal{I}, \alpha \models \varphi$, iff $\mathcal{I}, \alpha \models \psi$
$\mathcal{I}, \alpha \models \forall x \varphi$	iff $\mathcal{I}, \alpha[x/d] \models \varphi$ for all $d \in \mathcal{D}$
$\mathcal{I}, \alpha \models \exists x \varphi$	iff $\mathcal{I}, \alpha[x/d] \models \varphi$ for some $d \in \mathcal{D}$

Examples

$$\Theta = \left\{ \begin{array}{l} \text{eye}(a), \text{eye}(b) \\ \forall x(\text{eye}(x) \rightarrow \text{red}(x)) \end{array} \right\}$$

$$\mathcal{D} = \{d_1, \dots, d_n\}, \quad n > 1$$

$$a^{\mathcal{I}} = d_1$$

$$b^{\mathcal{I}} = d_1$$

$$\text{eye}^{\mathcal{I}} = \{d_1\}$$

$$\text{red}^{\mathcal{I}} = \mathcal{D}$$

$$\alpha = \{(x \mapsto d_1), (y \mapsto d_2)\}$$

Questions:

$\mathcal{I}, \alpha \models \text{eye}(b) \vee \neg \text{eye}(b)$? **Yes**

$\mathcal{I}, \alpha \models \text{eye}(x) \rightarrow$
 $\text{eye}(x) \vee \text{eye}(y)$? **Yes**

$\mathcal{I}, \alpha \models \text{eye}(x) \rightarrow \text{eye}(y)$? **No**

$\mathcal{I}, \alpha \models \text{eye}(a) \wedge \text{eye}(b)$? **Yes**

$\mathcal{I}, \alpha \models \forall x(\text{eye}(x) \rightarrow \text{red}(x))$?
Yes

$\mathcal{I}, \alpha \models \Theta$? **Yes**

Terminology

\mathcal{I}, α is a **model** of φ iff

$$\mathcal{I}, \alpha \models \varphi.$$

A formula can be **satisfiable**, **unsatisfiable**, **falsifiable**, **valid**, ...

Two formulae φ and ψ are **logically equivalent** (symb.: $\varphi \equiv \psi$) iff for all \mathcal{I}, α :

$$\mathcal{I}, \alpha \models \varphi \text{ iff } \mathcal{I}, \alpha \models \psi.$$

Note: $P(x) \not\equiv P(y)$!

Logical implication is also analogous to propositional logic:

$$\Theta \models \varphi \text{ iff for all } \mathcal{I}, \alpha \text{ s.t. } \mathcal{I}, \alpha \models \Theta \text{ also } \mathcal{I}, \alpha \models \varphi.$$

Free and Bound Variables

Variables can be **free** or **bound** (by a quantifier) in a formula:

$$\text{free}(x) = \{x\}$$

$$\text{free}(f(t_1, \dots, t_n)) = \text{free}(t_1) \cup \dots \cup \text{free}(t_n)$$

$$\text{free}(t_1 = t_2) = \text{free}(t_1) \cup \text{free}(t_2)$$

$$\text{free}(P(t_1, \dots, t_n)) = \text{free}(t_1) \cup \dots \cup \text{free}(t_n)$$

$$\text{free}(\neg\varphi) = \text{free}(\varphi)$$

$$\text{free}(\varphi * \psi) = \text{free}(\varphi) \cup \text{free}(\psi), \text{ for } * = \vee, \wedge, \rightarrow, \leftrightarrow$$

$$\text{free}(\Xi x \varphi) = \text{free}(\varphi) \setminus \{x\}, \text{ for } \Xi = \forall, \exists$$

Example: $\forall x (\mathbf{R}(\mathbf{y}, \mathbf{z}) \wedge \exists y (\neg P(y, x) \vee \mathbf{R}(y, \mathbf{z})))$

Framed occurrences are free, all others are bound.

Open & Closed Formulae

- ▶ Formulae without free variables are called **closed formulae** or **sentences**. Formulae with free variables are called **open formulae**.
- ▶ Closed formulae are all we need when we want to state something about the world. Open formulae (and variable maps) are only necessary for technical reasons (semantics of \forall and \exists).
- ▶ Note that *logical equivalence*, *satisfiability*, and *entailment* are independent from variable maps if we consider only closed formulae.
- ▶ For closed formulae, we omit α in connection with \models :

$$\mathcal{I} \models \varphi.$$

Important Theorems

Theorem (Compactness)

Let $\Phi \cup \{\psi\}$ be a set of closed formulae.

- (a) $\Phi \models \psi$ iff there exists a finite subset $\Phi' \subseteq \Phi$ s. t. $\Phi' \models \psi$.
- (b) Φ is satisfiable iff each finite subset $\Phi' \subseteq \Phi$ is satisfiable.

Theorem (Löwenheim-Skolem)

Each countable set of closed formulae that is satisfiable is satisfiable on a countable domain.

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