Foundations of AI

9. Statistical Machine Learning

Bayesian Learning and Why Learning Works

Wolfram Burgard, Andreas Karwath, Bernhard Nebel, and Martin Riedmiller

Contents

Statistical Learning Methods

- In MDPs probability and utility theory allow agents to deal with uncertainty.
- To apply these techniques, however, the agents must first learn their probabilistic theories of the world from experience.
- We will discuss statistical learning methods as robust ways to learn probabilistic models.

An Example for Statistical Learning

- The key concepts are data (evidence) and hypotheses.
- A candy manufacturer sells five kinds of bags that are indistinguishable from the outside:

  h_1: 100% cherry

  h_2: 75% cherry and 25% lime

  h_3: 50% cherry and 50% lime

  h_4: 25% cherry and 75% lime

  h_5: 100% lime
- Given a sequence $d_1, \ldots, d_N$ of candies observed, what is the most likely flavor of the next piece of candy?

Bayesian Learning

- Calculates the probability of each hypothesis, given the data.
- It then makes predictions using all hypotheses weighted by their probabilities (instead of a single best hypothesis).
- Learning is reduced to probabilistic inference.

Application of Bayes Rule

- Let ***D*** represent all the data with observed value ***d***.
- The probability of each hypothesis is obtained by Bayes rule:

$$P(h_i \mid \mathbf{d}) = \alpha \, P(\mathbf{d} \mid h_i) \, P(h_i)$$

- The manufacturer tells us that the prior distribution over h_1, ..., h_5 is given by <.1, .2, .4, .2, .1>
- We compute the likelihood of the data under the assumption that the observations are independently and identically distributed (i.i.d.):

$$P(\mathbf{d} \mid h_i) = \prod_j P(d_j \mid h_i)$$

How to Make Predictions?

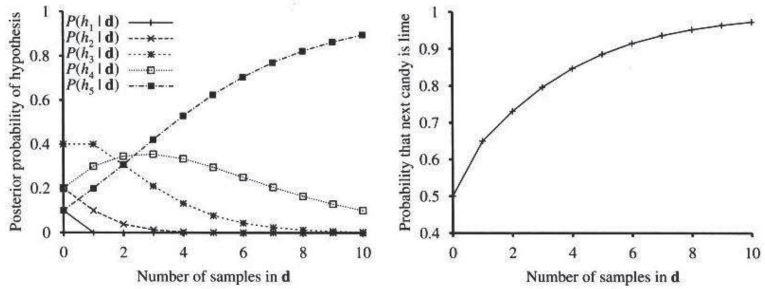
- Suppose we want to make predictions about an unknown quantity X given the data **d**.

$$\begin{aligned} \mathbf{P}(X \mid \mathbf{d}) &= \sum_i \mathbf{P}(X \mid h_i, \mathbf{d}) \mathbf{P}(h_i \mid \mathbf{d}) \\ &= \sum_i \mathbf{P}(X \mid h_i) \mathbf{P}(h_i \mid \mathbf{d}) \end{aligned}$$

- Predictions are weighted averages over the predictions of the individual hypotheses.
- The key quantities are the hypothesis prior $P(h_i)$ and the likelihood $P(d|h_i)$ of the data under each hypothesis.

Example

- Suppose the bag is an all-lime bag (h_5)
- The first 10 candies are all lime.
- Then $P(d|h_3)$ is 0.5^{10} because half the candies in an h_3 bag are lime.
- Evolution of the five hypotheses given 10 lime candies were observed (the values start at the prior!).



Observations

- The true hypothesis often dominates the Bayesian prediction.
- For any fixed prior that does not rule out the true hypothesis, the posterior of any false hypothesis will eventually vanish.
- The Bayesian prediction is optimal and, given the hypothesis prior, any other prediction will be correct less often.
- It comes at a price that the hypothesis space can be very large or infinite.

Maximum a Posteriori (MAP)

- A common approximation is to make predictions based on a single most probable hypothesis.
- The maximum a posteriori (MAP) hypothesis is the one that maximizes $P(h_i/d)$.

$$P(X \mid d) \approx P(X \mid h_{MAP})$$

- In the candy example, $h_{MAP} = h_5$ after three lime candies in a row.
- The MAP learner the predicts that the fourth candy is lime with probability 1.0, whereas the Bayesian prediction is still 0.8.
- As more data arrive, MAP and Bayesian predictions become closer.
- Finding MAP hypotheses is often much easier than Bayesian learning.

Maximum-Likelihood Hypothesis (ML)

- A final simplification is to assume a uniform prior over the hypothesis space.
- In that case MAP-learning reduces to choosing the hypothesis that maximizes $P(d/h_i)$.
- This hypothesis is called the maximum-likelihood hypothesis (ML).
- ML-learning is a good approximation to MAP learning and Bayesian learning when there is a uniform prior and when the data set is large.

Why Learning Works

How can we decide that h is close to f when f is unknown?
→ Probably approximately correct

Idea: Any wrong hypothesis will be found out after a reasonable number of examples, since it makes a wrong prediction.

Add: “with high probability”

Stationarity as the basic assumption of PAC-Learning: training and test sets are selected from the same population of examples with the same probability distribution.

Some Notation

Key question: how many examples do we need?

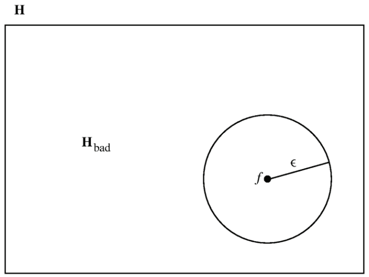
X	Set of examples
D	Distribution from which the examples are drawn
H	Hypothesis space ($f \in H$)
m	Number of examples in the training set

$$error(h) = P(h(x) \neq f(x) \mid x \text{ drawn from D}) \leq \epsilon$$

PAC-Learning

A hypothesis h is approximately correct if $error(h) \leq \epsilon$.

To show: After the training period with m examples, with high probability, all consistent hypotheses are approximately correct.



How high is the probability that a wrong hypothesis $h_b \in H_{bad}$ is consistent with the first m examples?

Sample Complexity

Assumption: $error(h) > \epsilon \Rightarrow$

P(h_b is consistent with 1 example) $\leq (1-\epsilon)$

P(h_b is consistent with N examples) $\leq (1-\epsilon)^N$

P(H_{bad} contains a consistent h) $\leq |H_{\text{bad}}|(1-\epsilon)^N$

Since $|H_{bad}| \leq |H|$

P(H_{bad} contains a consistent h) $\leq |H|(1-\epsilon)^N$

We want to limit this probability by some small number δ:

$$|H|(1-\epsilon)^N < \delta$$

Since $(1-\epsilon) \leq e^{-\epsilon}$, we derive

$$N \geq \frac{1}{\epsilon}\left(\log\left(\frac{1}{\delta}\right) + \log|H|\right)$$

Sample Complexity: Number of required examples, as a function of ϵ and δ.

Sample Complexity (2)

Example: Boolean functions

The number of Boolean functions over n attributes is $|H| = 2^{2^n}$.

The sample complexity therefore grows as 2^n.

Since the number of possible examples is also 2^n, any learning algorithm for the space of all Boolean functions will do no better than a lookup table, if it merely returns a hypothesis that is consistent with all known examples.

PAC-Learnable

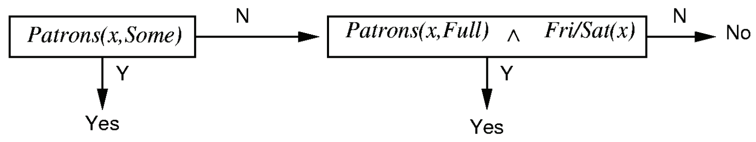
Definition: Consider a concept class F over a set of instances X and a Learner using hypothesis space H. F is PAC-learnable by L using H, if for all f in F, distributions D over X, ε and δ, learner L will with probability at least (1- δ) output a hypothesis h in H such that error_D(h) ≤ ε, in time that is polynomial in (1\ε), (1\δ), n, and size(h).

From T. Mitchell, Machine Learning

Learning from Decision Lists

In comparison to decision trees:

- The overall structure is simpler
- The individual tests are more complex



This represents the hypothesis

$$H_4 : \forall x\, WillWait(x) \Leftrightarrow Patrons(x, some) \vee [Patrons(x, full) \wedge Fri/Sat(x)]$$

If we allow tests of arbitrary size, then any Boolean function can be represented.

k-DL: Language with tests of length $\leq k$.

Note: k-DL includes decision trees of depth at most k.

Learnability of k-DL

```
function DECISION-LIST-LEARNING(examples) returns a decision list, No or failure

  if examples is empty then return the value No
  t ← a test that matches a nonempty subset examples_t of examples
       such that the members of examples_t are all positive or all negative
  if there is no such t then return failure
  if the examples in examples_t are positive then o ← Yes
  else o ← No
  return a decision list with initial test t and outcome o
         and remaining elements given by DECISION-LIST-LEARNING(examples − examples_t)
```

$|$ k-DL(n) $| \leq 3^{|Conj(n,k)|}$ $|$ $Conj(n,k)!$ $|$ (Yes,No,no-Test,all permutations)

$| Conj(n,k) | = \sum_{i=0}^{k} \binom{2n}{i} = O(n^k)$

(Combination without repeating pos/neg attributes)

$|$ k-DL(n) $| = 2^{O(n^k log(n^k))}$ (with Euler's summation formula)

$m \geq \frac{1}{\epsilon}(ln(\frac{1}{\delta}) + O(n^k log(n^k)))$

Summary (Statistical Learning Methods)

- Bayesian learning techniques formulate learning as a form of probabilistic inference.
- Maximum a posteriori (MAP) learning selects the most likely hypothesis given the data.
- Maximum likelihood learning selects the hypothesis that maximizes the likelihood of the data.

Summary
(Statistical Learning Theory)

Inductive learning as learning the representation of a function from example input/output pairs.

- Decision trees learn deterministic Boolean functions.
- PAC learning deals with the complexity of learning.
- Decision lists as functions that are easy to learn.