Principles of Knowledge Representation and Reasoning

Description Logics – Decidability and Complexity

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Decidability & Undecidability

Polynomial Cases

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The Complexity of Subsumption in TBoxes

Outlook

Decidability

L_2 is the fragment of first-order predicate logic using only two different variable names (*note*: variable names can be reused!).
$L_2^=$ the same including equality.

Theorem

. *$L_2^=$ is decidable.*

Corollary

Subsumption and satisfiability of concept descriptions is decidable in description logics using only the following concept and role forming operators: $C \sqcap D$, $C \sqcup D$, $\neg C$, $\forall r.C$, $\exists r.C$, $r \sqsubseteq s$, $r \sqcap s$, $r \sqcup s$, $\neg r$, r^{-1}.

Potential problems: Role composition and cardinality restrictions for role fillers. Cardinality restrictions, however, are not a real problem.

Undecidability

- $r \circ s$, $r \sqcap s$, $\neg r$, 1 [Schild 88]
- not relevant; Tarski had shown that already! – for relation algebras
- $r \circ s$, $r \doteq s$, $C \sqcap D$, $\forall r.C$ [Schmidt-Schauß 89]
- This is in fact a fragment of the early description logic *KL-ONE*, where people had hoped to come up with a complete subsumption algorithm

Decidable, Polynomial-Time Cases

- $\mathcal{FL}^-$ has obviously a polynomial subsumption problem (in the empty TBox) – the SUB algorithm needs only quadratic time.
- Donini *et al* [IJCAI 91] have shown that in the following languages subsumption can be decided using only polynomial time (and they are maximal wrt. this property):

$C \rightarrow A \,|\, \neg A \,|\, C \sqcap C' \,|\, \forall r.C \,|\, (\geq n\, r) \,|\, (\leq n\, r), \;\; r \rightarrow t \,|\, r^{-1}$
and
$C \rightarrow A \,|\, C \sqcap C' \,|\, \forall r.C \,|\, \exists r, \;\; r \rightarrow t \,|\, r^{-1} \,|\, r \sqcap r' \,|\, r \circ r'$
Open:
$C \rightarrow A \,|\, C \sqcap C' \,|\, \forall r.C \,|\, (\geq n\, r) \,|\, (\leq n\, r), \;\; r \rightarrow t \,|\, r \circ r'.$

How Hard is $\mathcal{ALC}$ Subsumption?

Proposition

$\mathcal{ALC}$ subsumption and unsatisfiability are co-NP-hard.

Proof.

Unsatisfiability and subsumption are reducible to each other. We give a reduction from UNSAT. A propositional formula φ over the atoms a_i is mapped to $\pi(\varphi)$:

$$\begin{array}{rcl} a_i & \mapsto & a_i \\ \psi \wedge \psi' & \mapsto & \pi(\psi) \sqcap \pi(\psi') \\ \psi' \vee \psi & \mapsto & \pi(\psi) \sqcup \pi(\psi') \\ \neg\psi & \mapsto & \neg\pi(\psi) \end{array}$$

Obviously, φ is satisfiable iff $\pi(\varphi)$ is satisfiable (use structural induction). If φ has a model, construct a model for $\pi(\varphi)$ with just one element t standing for the truth of the atoms and the formula. Conversely, if $\pi(\varphi)$ satisfiable, pick one element $d \in \pi(\varphi)^{\mathcal{I}}$ and set the truth value of atom a_i according to the fact that $d \in \pi(a_i)^{\mathcal{I}}$. □

How Hard Does It Get?

- Is $\mathcal{ALC}$ unsatisfiability and subsumption also complete for co-NP?
- Unlikely – since models of a single concept description can already become exponentially large!
- We will show PSPACE-completeness, whereby hardness is proved using a complexity result for (un)satisifiability in the modal logic K
- Satisifiability and unsatisfiability in K is PSPACE-complete

Reduction from K-Satisfiability

Lemma (Lower bound for $\mathcal{ALC}$)

$\mathcal{ALC}$ subsumption, unsatisfiability and satisfiability are all PSPACE-hard.

Proof.

Extend the reduction given in the last proof by the following two rules – assuming that b is a fixed role name:

$$\begin{array}{rcl} \Box\psi & \mapsto & \forall b.\pi(\psi) \\ \Diamond\psi & \mapsto & \exists b.\pi(\psi) \end{array}$$

Again, obviously, φ is satisfiable iff $\pi(\varphi)$ is satisfiable (again using structural induction). If φ has a Kripke model, interpret each world w as an object in the universe of discourse that is an instances of the primitive concept $\pi(a_i)$ iff a_i is true in w. For the converse direction use the interpretation the other way around. □

Computational Complexity of $\mathcal{ALC}$ Subsumption

Lemma (Upper Bound for $\mathcal{ALC}$)

$\mathcal{ALC}$ subsumption, unsatisfiability and satisfiability are all in PSPACE.

Proof.

This follows from the tableau algorithm for $\mathcal{ALC}$. Although there may be exponentially many closed constraint systems, we can visit them step by step generating only one at a time. When closing a system, we have to consider only one role at a time – resulting in an only polynomial space requirement, i.e., satisfiability can be decided in PSPACE. □

Theorem (Complexity of $\mathcal{ALC}$)

$\mathcal{ALC}$ subsumption, unsatisfiability and satisfiability are all PSPACE-complete.

Further Consequences of the Reducibility of K to $\mathcal{ALC}$

- In the reduction we used only *one* role symbol. Are there modal logics that would require more than one such role symbol?
- ⇝ The **multi-modal logic** $K_{(n)}$ has n different Box operators $\Box_i$ (for n different agents)
- ⇝ $\mathcal{ALC}$ is a *notational variant* of $K_{(n)}$ [Schild, IJCAI-91]
- Are there perhaps other modal logics that correspond to other descriptions logics?
- ⇝ **propositional dynamic logic** (PDL), e.g., transitive closure, composition, role inverse, ...
- ⇝ DL can be thought as fragments of *first-order predicate logic*. However, they are much more similar to *modal logics*
- ⇝ Algorithms and complexity results can be borrowed. Works also the other way around

Expressive Power vs. Complexity

- Of course, one wants to have a description logic with high *expressive power*. However, high expressive power implies usually that the **computational complexity** of the reasoning problems might also be high, e.g., $\mathcal{FL}^-$ vs. $\mathcal{ALC}$
- Does it make sense to use a language such as $\mathcal{ALC}$ or even extensions (corresponding to PDL) with higher complexity?
- There are three approaches to this problem:
  1. Use only *small* description logics with *complete* inference algorithms
  2. Use *expressive* description logics, but employ *incomplete* inference algorithms
  3. Use *expressive* description logics with *complete* inference algorithms
- For a long time, only options 1 and 2 were studied. Meanwhile, most researcher concentrate on *option 3*!

Is Subsumption in the Empty TBox Enough?

- We have shown that we can *reduce* concept subsumption in a given TBox to concept subsumption in the empty TBox.
- However, it is not obvious that this can be done in *polynomial time*
- In particular, in the following example *unfolding* leads to an exponential blowup:

$$\begin{aligned} C_1 &\doteq \forall r.C_0 \sqcap \forall s.C_0 \\ C_2 &\doteq \forall r.C_1 \sqcap \forall s.C_1 \\ &\vdots \\ C_n &\doteq \forall r.C_{n-1} \sqcap \forall s.C_{n-1} \end{aligned}$$

- Unfolding C_n leads to a concept description with a size $\Omega(2^n)$
- Is it possible to **avoid** this blowup?
- Can we avoid exponential preprocessing?

TBox Subsumption for Small Languages

- **Question**: Can we decide in polynomial time *TBox subsumption* for a description logic such as $\mathcal{FL}^-$, for which concept subsumption in the empty TBox can be decided in polynomial time?
- Let us consider $\mathcal{FL}_0$: $C \sqcap D, \forall r.C$ with *terminological axioms*.
- Subsumption without a TBox can be done easily, using a structural subsumption algorithm.
- Unfolding + strucural subsumption gives us an exponential algorithm.

Complexity of TBox Subsumption

Theorem (Complexity of TBox subsumption)

TBox subsumption for $\mathcal{FL}_0$ is NP-hard.

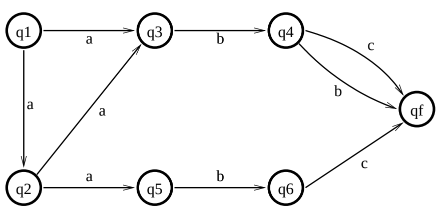
Proof sketch.

We use the **NDFA-equivalence problem**, which is NP-complete for *cycle-free* automatons and PSPACE-complete for general NDFAs. We transform a cycle-free NDFA to a $\mathcal{FL}_0$-terminology with the mapping π as follows:

$$\begin{array}{rcl} \text{automaton } A & \mapsto & \text{terminology } \mathcal{T}_A \\ \text{state } q & \mapsto & \text{concept name } q \\ \text{terminal state } q_f & \mapsto & \text{concept name } q_f \\ \text{input symbol } r & \mapsto & \text{role name } r \end{array}$$

r-transition from q to $q' \mapsto q \doteq \ldots \sqcap \forall r : q' \sqcap \ldots$ □

"Proof" by Example



$$\begin{array}{lcl} q_1 & \doteq & \forall a.q_3 \sqcap \forall a.q_2 \\ q_2 & \doteq & \forall a.q_3 \sqcap \forall a.q_5 \\ q_3 & \doteq & \forall b.q_4 \\ q_4 & \doteq & \forall b.q_f \sqcap \forall c.q_f \\ q_5 & \doteq & \forall b.q_6 \\ q_6 & \doteq & \forall b.q_f \\ q_1 & \equiv & \forall abc.q_f \sqcap \forall abb.q_f \sqcap \\ & & \forall aabc.q_f \sqcap \forall aabb.q_f \\ q_2 & \equiv & \forall abb.q_f \sqcap \forall abc.q_f \\ q_1 & \sqsubseteq_{\mathcal{T}} & q_2 \quad \text{and} \quad \mathcal{L}(q_2) \subseteq \mathcal{L}(g_1) \end{array}$$

In general, we have: $\mathcal{L}(q) \subseteq \mathcal{L}(q')$ iff $q' \sqsubseteq_{\mathcal{T}} q$, from which the *correctness of the reduction* and the *complexity result* follows.

What Does This Complexity Result Mean?

- Note that for expressive languages such as $\mathcal{ALC}$, we do not notice any difference!
- The TBox subsumption complexity result for less expressive languages does not play a large role *in practice*
- **Pathological situations** do not happen very often
- In fact, if the definition depth is logarithmic in the size of the TBox, the whole problem vanishes.
- However, in order to protect oneself against such problems, one often uses **lazy unfolding**
- Similarly, also for the $\mathcal{ALC}$ concept descriptions, one notices that they are usually very well behaved.

Outlook

- Description logics have a long history (Tarski's relation algebras and Brachman's KL-ONE)
- Early on, either small languages with provably easy reasoning problems (e.g., the system **CLASSIC**) or large languages with incomplete inference algorithms (e.g., the system **Loom**) were used.
- Meanwhile, one uses complete algorithms on very large descriptions logics (e.g., **SHIQ**), e.g. in the systems **FaCT** and **RACER**
- RACER can handle KBs with up to 160,000 concepts (example from *unified medical language system*) in reasonable time (less than one day computing time)
- Description logics are used as the semantic backbone for **OWL** (a Web-language extending RDF)

Literature

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