

Principles of Knowledge Representation and Reasoning

Description Logics – Decidability and Complexity

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Decidability & Undecidability

Polynomial Cases

Complexity of \mathcal{ALC} Subsumption

Expressive Power vs. Complexity

The Complexity of Subsumption in TBoxes

Outlook

Decidability

L_2 is the fragment of first-order predicate logic using only two different variable names (*note*: variable names can be reused!).

$L_2^=$ the same including equality.

Theorem

. $L_2^=$ is decidable.

Corollary

Subsumption and satisfiability of concept descriptions is decidable in description logics using only the following concept and role forming operators: $C \sqcap D$, $C \sqcup D$, $\neg C$, $\forall r.C$, $\exists r.C$, $r \sqsubseteq s$, $r \sqcap s$, $r \sqcup s$, $\neg r$, r^{-1} .

Potential problems: Role composition and cardinality restrictions for role fillers. Cardinality restrictions, however, are not a real problem.

Undecidability

- ▶ $r \circ s, r \sqcap s, \neg r, 1$ [Schild 88]
- ▶ not relevant; Tarski had shown that already! – for relation algebras
- ▶ $r \circ s, r \dot{=} s, C \sqcap D, \forall r.C$ [Schmidt-Schauß 89]
- ▶ This is in fact a fragment of the early description logic *KL-ONE*, where people had hoped to come up with a complete subsumption algorithm

Decidable, Polynomial-Time Cases

- ▶ \mathcal{FL}^- has obviously a polynomial subsumption problem (in the empty TBox) – the SUB algorithm needs only quadratic time.
- ▶ Donini *et al* [IJCAI 91] have shown that in the following languages subsumption can be decided using only polynomial time (and they are maximal wrt. this property):

$$C \rightarrow A \mid \neg A \mid C \sqcap C' \mid \forall r.C \mid (\geq nr) \mid (\leq nr), \quad r \rightarrow t \mid r^{-1}$$

and

$$C \rightarrow A \mid C \sqcap C' \mid \forall r.C \mid \exists r, \quad r \rightarrow t \mid r^{-1} \mid r \sqcap r' \mid r \circ r'$$

Open:

$$C \rightarrow A \mid C \sqcap C' \mid \forall r.C \mid (\geq nr) \mid (\leq nr), \quad r \rightarrow t \mid r \circ r'.$$

How Hard is \mathcal{ALC} Subsumption?

Proposition

\mathcal{ALC} subsumption and unsatisfiability are co-NP-hard.

Proof.

Unsatisfiability and subsumption are reducible to each other. We give a reduction from UNSAT. A propositional formula φ over the atoms a_i is mapped to $\pi(\varphi)$:

$$\begin{aligned} a_i &\mapsto a_i \\ \psi \wedge \psi' &\mapsto \pi(\psi) \sqcap \pi(\psi') \\ \psi' \vee \psi &\mapsto \pi(\psi) \sqcup \pi(\psi') \\ \neg\psi &\mapsto \neg\pi(\psi) \end{aligned}$$

Obviously, φ is satisfiable iff $\pi(\varphi)$ is satisfiable (use structural induction). If φ has a model, construct a model for $\pi(\varphi)$ with just one element t standing for the truth of the atoms and the formula. Conversely, if $\pi(\varphi)$ is satisfiable, pick one element $d \in \pi(\varphi)^{\mathcal{I}}$ and set the truth value of atom a_i according to the fact that $d \in \pi(a_i)^{\mathcal{I}}$. \square

How Hard Does It Get?

- ▶ Is \mathcal{ALC} unsatisfiability and subsumption also **complete** for co-NP?
- ▶ Unlikely – since models of a single concept description can already become exponentially large!
- ▶ We will show **PSPACE-completeness**, whereby hardness is proved using a complexity result for (un)satisfiability in the modal logic K
- ▶ Satisfiability and unsatisfiability in K is PSPACE-complete

Reduction from K -Satisfiability

Lemma (Lower bound for \mathcal{ALC})

\mathcal{ALC} subsumption, unsatisfiability and satisfiability are all PSPACE-hard.

Proof.

Extend the reduction given in the last proof by the following two rules – assuming that b is a fixed role name:

$$\Box\psi \mapsto \forall b.\pi(\psi)$$

$$\Diamond\psi \mapsto \exists b.\pi(\psi)$$

Again, **obviously**, φ is satisfiable iff $\pi(\varphi)$ is satisfiable (again using structural induction). If φ has a Kripke model, interpret each world w as an object in the universe of discourse that is an instances of the primitive concept $\pi(a_i)$ iff a_i is true in w . For the converse direction use the interpretation the other way around. \square

Computational Complexity of \mathcal{ALC} Subsumption

Lemma (Upper Bound for \mathcal{ALC})

\mathcal{ALC} subsumption, unsatisfiability and satisfiability are all in PSPACE.

Proof.

This follows from the tableau algorithm for \mathcal{ALC} . Although there may be exponentially many closed constraint systems, we can visit them step by step generating only one at a time. When closing a system, we have to consider only one role at a time – resulting in an only polynomial space requirement, i.e., satisfiability can be decided in PSPACE. \square

Theorem (Complexity of \mathcal{ALC})

\mathcal{ALC} subsumption, unsatisfiability and satisfiability are all PSPACE-complete.

Further Consequences of the Reducibility of K to \mathcal{ALC}

- ▶ In the reduction we used only *one* role symbol. Are there modal logics that would require more than one such role symbol?
- ↪ The **multi-modal logic** $K_{(n)}$ has n different Box operators \square_i (for n different agents)
- ↪ \mathcal{ALC} is a *notational variant* of $K_{(n)}$ [Schild, IJCAI-91]
 - ▶ Are there perhaps other modal logics that correspond to other descriptions logics?
- ↪ **propositional dynamic logic** (PDL), e.g., transitive closure, composition, role inverse, ...
- ↪ DL can be thought as fragments of *first-order predicate logic*. However, they are much more similar to *modal logics*
- ↪ Algorithms and complexity results can be borrowed. Works also the other way around

Expressive Power vs. Complexity

- ▶ Of course, one wants to have a description logic with high *expressive power*. However, high expressive power implies usually that the **computational complexity** of the reasoning problems might also be high, e.g., \mathcal{FL}^- vs. \mathcal{ALC}
- ▶ Does it make sense to use a language such as \mathcal{ALC} or even extensions (corresponding to PDL) with higher complexity?
- ▶ There are three approaches to this problem:
 1. Use only *small* description logics with *complete* inference algorithms
 2. Use *expressive* description logics, but employ *incomplete* inference algorithms
 3. Use *expressive* description logics with *complete* inference algorithms
- ▶ For a long time, only options 1 and 2 were studied. Meanwhile, most researcher concentrate on *option 3!*

Is Subsumption in the Empty TBox Enough?

- ▶ We have shown that we can *reduce* concept subsumption in a given TBox to concept subsumption in the empty TBox.
- ▶ However, it is not obvious that this can be done in *polynomial time*
- ▶ In particular, in the following example *unfolding* leads to an exponential blowup:

$$\begin{aligned}
 C_1 &\doteq \forall r.C_0 \sqcap \forall s.C_0 \\
 C_2 &\doteq \forall r.C_1 \sqcap \forall s.C_1 \\
 &\vdots \\
 C_n &\doteq \forall r.C_{n-1} \sqcap \forall s.C_{n-1}
 \end{aligned}$$

- ▶ Unfolding C_n leads to a concept description with a size $\Omega(2^n)$
- ▶ Is it possible to **avoid** this blowup?
- ▶ Can we avoid exponential preprocessing?

TBox Subsumption for Small Languages

- ▶ **Question:** Can we decide in polynomial time *TBox subsumption* for a description logic such as \mathcal{FL}^- , for which concept subsumption in the empty TBox can be decided in polynomial time?
- ▶ Let us consider $\mathcal{FL}_0 : C \sqcap D, \forall r.C$ with *terminological axioms*.
- ▶ Subsumption without a TBox can be done easily, using a structural subsumption algorithm.
- ▶ Unfolding + structural subsumption gives us an **exponential** algorithm.

Complexity of TBox Subsumption

Theorem (Complexity of TBox subsumption)

TBox subsumption for \mathcal{FL}_0 is NP-hard.

Proof sketch.

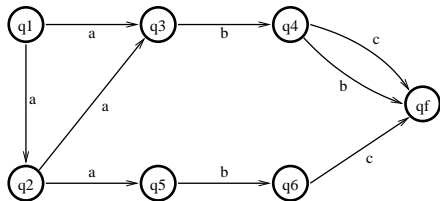
We use the **NFA-equivalence problem**, which is NP-complete for *cycle-free* automata and PSPACE-complete for general NFAs. We transform a cycle-free NFA to a \mathcal{FL}_0 -terminology with the mapping π as follows:

automaton A	\mapsto	terminology \mathcal{T}_A
state q	\mapsto	concept name q
terminal state q_f	\mapsto	concept name q_f
input symbol r	\mapsto	role name r

r -transition from q to $q' \mapsto q \dot{=} \dots \sqcap \forall r : q' \sqcap \dots$

□

“Proof” by Example



$$q_1 \stackrel{\cdot}{=} \forall a. q_3 \sqcap \forall a. q_2$$

$$q_2 \stackrel{\cdot}{=} \forall a. q_3 \sqcap \forall a. q_5$$

$$q_3 \stackrel{\cdot}{=} \forall b. q_4$$

$$q_4 \stackrel{\cdot}{=} \forall b. q_f \sqcap \forall c. q_f$$

$$q_5 \stackrel{\cdot}{=} \forall b. q_6$$

$$q_6 \stackrel{\cdot}{=} \forall b. q_f$$

$$q_1 \equiv \forall abc. q_f \sqcap \forall abb. q_f \sqcap \forall aabc. q_f \sqcap \forall aabb. q_f$$

$$q_2 \equiv \forall abb. q_f \sqcap \forall abc. q_f$$

$$q_1 \sqsubseteq_{\mathcal{T}} q_2 \text{ and } \mathcal{L}(q_2) \subseteq \mathcal{L}(g_1)$$

In general, we have: $\mathcal{L}(q) \subseteq \mathcal{L}(q')$ iff $q' \sqsubseteq_{\mathcal{T}} q$, from which the *correctness of the reduction* and the *complexity result* follows.

What Does This Complexity Result Mean?

- ▶ Note that for expressive languages such as ALC , we do not notice any difference!
- ▶ The TBox subsumption complexity result for less expressive languages does not play a large role *in practice*
- ▶ **Pathological situations** do not happen very often
- ▶ In fact, if the definition depth is logarithmic in the size of the TBox, the whole problem vanishes.
- ▶ However, in order to protect oneself against such problems, one often uses **lazy unfolding**
- ▶ Similarly, also for the ALC concept descriptions, one notices that they are usually very well behaved.

Outlook

- ▶ Description logics have a long history (Tarski's relation algebras and Brachman's KL-ONE)
- ▶ Early on, either small languages with provably easy reasoning problems (e.g., the system **CLASSIC**) or large languages with incomplete inference algorithms (e.g., the system **Loom**) were used.
- ▶ Meanwhile, one uses complete algorithms on very large descriptions logics (e.g., **SHIQ**), e.g. in the systems **FaCT** and **RACER**
- ▶ RACER can handle KBs with up to 160,000 concepts (example from *unified medical language system*) in reasonable time (less than one day computing time)
- ▶ Description logics are used as the semantic backbone for **OWL** (a Web-language extending RDF)

Literature



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