Principles of Knowledge Representation and Reasoning

Description Logics – Reasoning Services and Reductions

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Semantic Networks and Description Logics III: Description Logics – Reasoning Services and Reductions

Example TBox & ABox

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\begin{aligned}
\texttt{Male} &\doteq \neg\texttt{Female}\\
\texttt{Human} &\sqsubseteq \texttt{Living\_entity}\\
\texttt{Woman} &\doteq \texttt{Human} \sqcap \texttt{Female}\\
\texttt{Man} &\doteq \texttt{Human} \sqcap \texttt{Male}\\
\texttt{Mother} &\doteq \texttt{Woman} \sqcap \exists\texttt{has-child.Human}\\
\texttt{Father} &\doteq \texttt{Man} \sqcap \exists\texttt{has-child.Human}\\
\texttt{Parent} &\doteq \texttt{Father} \sqcup \texttt{Mother}\\
\texttt{Grandmother} &\\
&\doteq \texttt{Woman} \sqcap \exists\texttt{has-child.Parent}\\
\texttt{Mother-without-daughter} &\\
&\doteq \texttt{Mother} \sqcap \forall\texttt{has-child.Male}\\
\texttt{Mother-with-many-children} &\\
&\doteq \texttt{Mother} \sqcap (\geq 3\, \texttt{has-child})
\end{aligned}
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DIANA:	Woman	
ELIZABETH:	Woman	
CHARLES:	Man	
EDWARD:	Man	
ANDREW:	Man	
DIANA:	Mother-without-daughter	
(ELIZABETH,	CHARLES):	has-child
(ELIZABETH,	EDWARD):	has-child
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Motivation: Reasoning Services

- What do we want to know?
- We want to check whether the *knowledge base* is reasonable:
  - Is each defined concept in a TBox satisfiable?
  - Is a given TBox satisfiable?
  - Is a given ABox satisfiable?
- What can we **conclude** from the represented knowledge?
  - Is concept X **subsumed** by concept Y?
  - Is an object a **instance** of a concept X?
- These problems can be **reduced** to logical satisfiability or implication – using the logical semantics.
- We take a different route: We will try to simplify these problems and then we specify *direct inference methods*.

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Satisfiability of Concept Descriptions in a TBox

- **Motivation**: Given a TBox $\mathcal{T}$ and a concept description C, does C make sense, i.e., is C **satisfiable**?
- **Test**:
  - Does there exist a *model* $\mathcal{I}$ of $\mathcal{T}$ such that $C^{\mathcal{I}} \neq \emptyset$?
  - Is the formula $\exists x\colon C(x)$ together with the formulas resulting from the translation of $\mathcal{T}$ satisfiable?
- **Example**: `Mother-without-daughter` $\sqcap$ $\forall$`has-child.Female` is unsatisfiable.

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- **Motivation**: Given a concept description C in "isolation", i.e., in an *empty TBox*, does C make sense, i.e., is C **satisfiable**?
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Reduction: Getting Rid of the TBox

- We can **reduce** satisfiability in a TBox to simple satisfiability.
- **Idea**:
  - Since TBoxes are *cycle-free*, one can understand a concept definition as a kind of “macro”
  - For a given TBox $\mathcal{T}$ and a given concept description C, all defined concept symbols appearing in C can be *expanded* until C contains only undefined concept symbols
  - An *expanded* concept description is then satisfiable iff C is satisfiable in $\mathcal{T}$
  - *Problem*: What do we do with partial definitions (using $\sqsubseteq$)?

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Normalized Terminologies

- A terminology is called **normalized** when it does not contain definitions using $\sqsubseteq$.
- In order to *normalize* a terminology, replace

$$A \sqsubseteq C$$

by

$$A \doteq A^* \sqcap C,$$

where A^* is a **fresh** concept symbol (not appearing elsewhere in $\mathcal{T}$).
- If $\mathcal{T}$ is a terminology, the normalized terminology is denoted by $\widetilde{\mathcal{T}}$.

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Normalizing is Reasonable

Theorem (Normalization Invariance)

If $\mathcal{I}$ is a model of the terminology $\mathcal{T}$, then there exists a model $\mathcal{I}'$ of $\widetilde{\mathcal{T}}$ (and vice versa*) such that for all concept symbols A appearing in $\mathcal{T}$ we have:*

$$A^{\mathcal{I}} = A^{\mathcal{I}'}.$$

Proof.

"⇒": Let $\mathcal{I}$ be a model of $\mathcal{T}$. This model should be *extended* to $\mathcal{I}'$ so that the freshly introduced concept symbols also get interpretations. Assume $(A \sqsubseteq C) \in \mathcal{T}$, i.e., we have $(A \doteq A^* \sqcap C) \in \widetilde{\mathcal{T}}$. Then set $A^{*\mathcal{I}'} = A^{\mathcal{I}}$. $\mathcal{I}'$ obviously satisfies $\widetilde{\mathcal{T}}$ and has the same interpretation for all symbols in $\mathcal{T}$.
$\Leftarrow$ Given a model $\mathcal{I}'$ of $\widetilde{\mathcal{T}}$, its restriction to symbols of $\mathcal{T}$ is the interpretation we looked for. □

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TBox Unfolding

- We say that a *normalized TBox* is **unfolded by one step** when all defined concept symbols on the right sides are replaced by their defining terms.
- **Example**: $\texttt{Mother} \doteq \texttt{Woman} \sqcap \ldots$ is unfolded to $\texttt{Mother} \doteq (\texttt{Human} \sqcap \texttt{Female}) \sqcap \ldots$
- We write $U(\mathcal{T})$ to denote a one-step unfolding and $U^n(\mathcal{T})$ to denote an *n-step unfolding*.
- We say $\mathcal{T}$ is **unfolded** if $U(\mathcal{T}) = \mathcal{T}$.
- We say that $U^n(\mathcal{T})$ is the **unfolding** of $\mathcal{T}$ if $U^n(\mathcal{T}) = U^{n+1}(\mathcal{T})$. If such an unfolding exists, it is denoted by $\widehat{\mathcal{T}}$

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- We say that $U^n(\mathcal{T})$ is the **unfolding** of $\mathcal{T}$ if $U^n(\mathcal{T}) = U^{n+1}(\mathcal{T})$. If such an unfolding exists, it is denoted by $\widehat{\mathcal{T}}$

Properties of Unfoldings (1): Existence

Theorem (Existence of unfolded terminology)

For each normalized terminology $\mathcal{T}$, there exists its unfolding $\widehat{\mathcal{T}}$.

Proof idea.

The main reason is that terminologies have to be *cycle-free*. The proof can be done by induction of the *definition depth* of concepts. □

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- All concept and role names *not appearing on the left hand side* in a terminology $\mathcal{T}$ are called **primitive components**.
- Interpretations restricted to primitive components are called **initial interpretations**.

Theorem (Model extension)

For each initial interpretation $\mathcal{J}$ of a normalized TBox, there exists a unique interpretation $\mathcal{I}$ extending $\mathcal{J}$ and satisfying $\mathcal{T}$.

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Use $\widehat{\mathcal{T}}$ and compute an interpretation for all defined symbols. □

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Each TBox has at least one model.

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- Similar to the unfolding of TBoxes, we can define **unfolding of concept descriptions**.
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An concept description C is satisfiable in a terminology $\mathcal{T}$ iff $\widehat{C}$ satisfiable in an empty terminology.

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- **Motivation**: Given a terminology $\mathcal{T}$ and two concept descriptions C and D, is C *subsumed by* (or a *sub-concept* of) D in $\mathcal{T}$ ($C \sqsubseteq_{\mathcal{T}} D$)?
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  - it is a *generalized sorting* problem!

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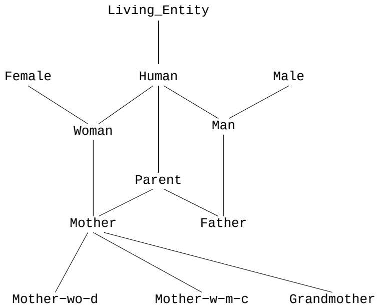

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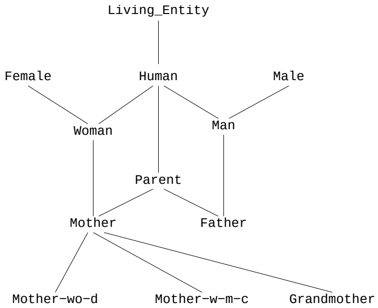

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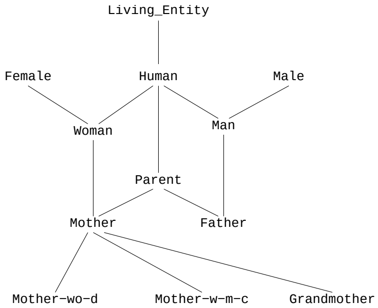

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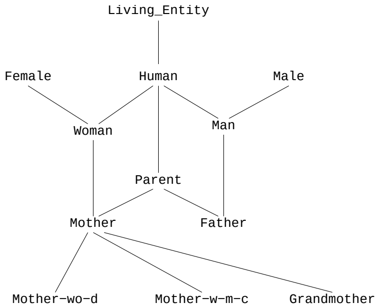

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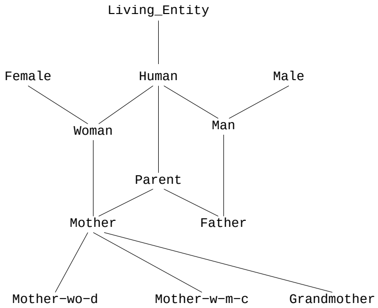

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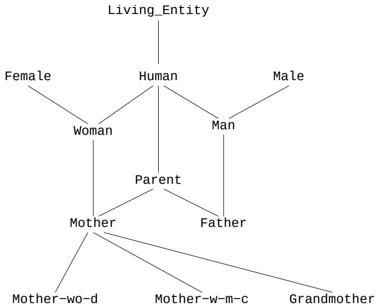

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- **Test**: Is $\mathcal{T} \cup \mathcal{A}$ satisfiable?
- **Example**: If we extend our example with
  `MARGRET: Woman`
  `(DIANA,MARGRET): has-child`,

  then the ABox becomes unsatisfiable in the given TBox.
- **Reduction**:
  - to satisfiability of an ABox
  - *Normalize* terminology, then *unfold* all concept and role descriptions in the ABox

Instance Relations

- **Motivation**: Which additional ABox formulas of the form $a\colon C$ follow logically from a given ABox and TBox?
- **Test**:
  - Is $a^{\mathcal{I}} \in C^{\mathcal{I}}$ true in all models of $\mathcal{I}$ of $\mathcal{T} \cup \mathcal{A}$?
  - Does the formula $C(a)$ logically follow from the translation of $\mathcal{A}$ and $\mathcal{T}$ to predicate logic?
- **Reductions**:
  - Instance relations wrt. an ABox and a TBox can be reduced to instance relations wrt. ABox.
  - Use *normalization* and *unfolding*
  - Instance relations in an ABox can be reduced to ABox unsatisfiability:

$$a\colon C \text{ holds in } \mathcal{A} \text{ iff } \mathcal{A} \cup \{a\colon \neg C\} \text{ is unsatisfiable}$$

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- WILLIAM: ¬ Female?
- ELIZABETH: Mother-without-daughter?
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- no (only male, but not necessarily human!)

Realization

- **Idea**: For a given object a, determine the **most specialized concept symbols** such that a is an instance of these concepts
- **Motivation**:
  - Similar to *classification*
  - Is the minimal representation of the instance relations (in the set of concept symbols)
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- **Reduction**: Can be reduced to (a sequence of) instance relation tests.

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