Principles of Knowledge Representation and Reasoning

Qualitative Representation and Reasoning II: Allen's Interval Calculus

Bernhard Nebel, Malte Helmert and Stefan Wölfl

Albert-Ludwigs-Universität Freiburg

June 24, 2008

Allen's Interval Calculus – Outline

1. Allen's Interval Calculus
    - Motivation
    - Intervals and Relations Between Them
    - Composing Interval Relations
2. Reasoning in Allen's Interval Calculus
3. A Maximal Tractable Sub-Algebra
4. Literature

Qualitative Temporal Representation and Reasoning

Often we do not want to talk about precise times:

- NLP – we do not have precise time points
- Planning – we do not want to commit to time points too early
- Scenario descriptions – we do not have the exact times or do not want to state them

What are the primitives in our representation system?

- Time points: actions and events are instantaneous, or we consider their beginning and ending
- Time intervals: actions and events have duration
- Reducibility? Expressiveness? Computational costs for reasoning?

Motivation: Example

Consider a planning scenario for multimedia generation:

P1: *Display* Picture1

P2: *Say* "Put the plug in."

P3: *Say* "The device should be shut off."

P4: *Point to* Plug-in-Picture1.

Temporal relations between events:

P2	should happen during	P1
P3	should happen during	P1
P2	should happen before or directly precede	P3
P4	should happen during or end together with	P2

- ⇝ P4 happens before or directly precedes P3
- ⇝ We could add the statement "P4 does not overlap with P3" without creating an inconsistency.

Allen's Interval Calculus

- Allen's interval calculus: time intervals and binary relations over them
- Time intervals: $X = (X^-, X^+)$, where X^- and X^+ are interpreted over the reals and $X^- < X^+$ ($\rightsquigarrow$ naïve approach)
- Relations between concrete intervals, e. g.:

  (1.0,2.0) strictly before (3.0,5.5)
  (1.0,3.0) meets (3.0,5.5)
  (1.0,4.0) overlaps (3.0,5.5)
  . . .

$\rightsquigarrow$ Which relations are conceivable?

The Base Relations

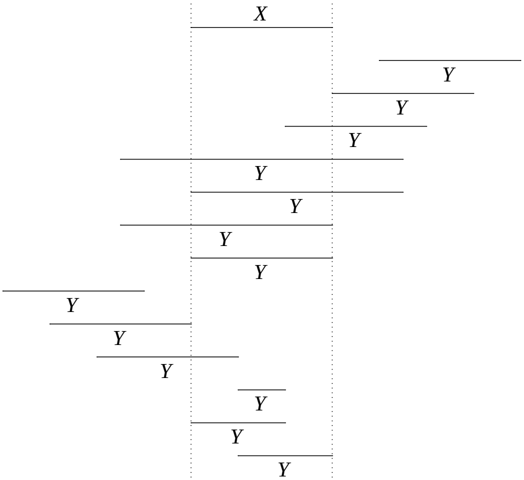
How many ways are there to order the four points of two intervals?

Relation	Symbol	Name
$\{(X,Y) : X^- < X^+ < Y^- < Y^+\}$	$\prec$	before
$\{(X,Y) : X^- < X^+ = Y^- < Y^+\}$	`m`	meets
$\{(X,Y) : X^- < Y^- < X^+ < Y^+\}$	`o`	overlaps
$\{(X,Y) : X^- = Y^- < X^+ < Y^+\}$	`s`	starts
$\{(X,Y) : Y^- < X^- < X^+ = Y^+\}$	`f`	finishes
$\{(X,Y) : Y^- < X^- < X^+ < Y^+\}$	`d`	during
$\{(X,Y) : Y^- = X^- < X^+ = Y^+\}$	$\equiv$	equal

and the **converse** relations (obtained by exchanging X and Y)

⇝ These relations are JEPD.

The 13 Base Relations Graphically



before
meets
overlaps
during
starts
finishes
equals
before^{-1}
meets^{-1}
overlaps^{-1}
during^{-1}
starts^{-1}
finishes^{-1}

Disjunctive Descriptions

- Assumption: We don't have precise information about the relation between X and Y, e. g.:

$$X \mathtt{o} Y \text{ or } X \mathtt{m} Y$$

- …modelled by sets of base relations (meaning the union of the relations):

$$X \{\mathtt{o}, \mathtt{m}\} Y$$

⇝ 2^{13} imprecise relations (incl. $\emptyset$ and **B**)

Example of an indefinite qualitative description:

$$\left\{ X \{\mathtt{o}, \mathtt{m}\} Y, Y \{\mathtt{m}\} Z, X \{\mathtt{o}, \mathtt{m}\} Z \right\}$$

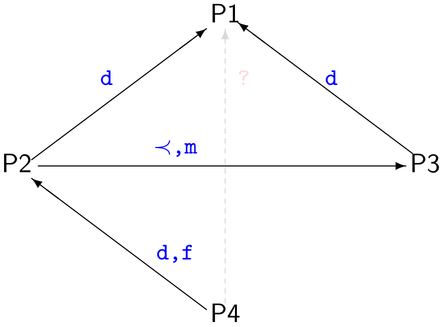
Our Example. . . Formally

P1: *Display* Picture1

P2: *Say* "Put the plug in."

P3: *Say* "The device should be shut off."

P4: *Point to* Plug-in-Picture1.



Compose the constraints: $P4\,\{\mathtt{d},\mathtt{f}\}\,P2$ and $P2\,\{\mathtt{d}\}\,P1$

$\rightsquigarrow P4\,\{\mathtt{d}\}\,P1$.

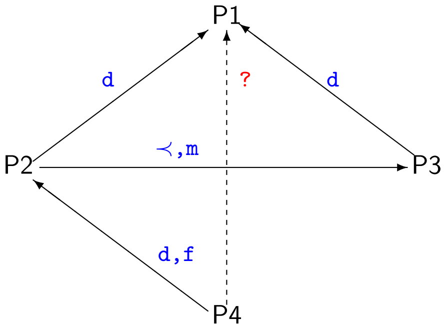
Our Example. . . Formally

P1: *Display* Picture1

P3: *Say* “The device should be shut off.”

P2: *Say* “Put the plug in.”

P4: *Point to* Plug-in-Picture1.



Compose the constraints: $P4\,\{\mathtt{d},\mathtt{f}\}\,P2$ and $P2\,\{\mathtt{d}\}\,P1$ $\rightsquigarrow$ $P4\,\{\mathtt{d}\}\,P1$.

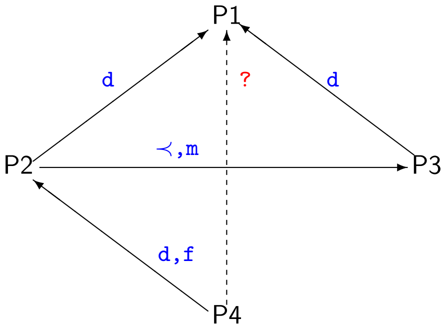
Our Example... Formally

P1: *Display* Picture1

P2: *Say* "Put the plug in."

P3: *Say* "The device should be shut off."

P4: *Point to* Plug-in-Picture1.



Compose the constraints: $P4\,\{\mathtt{d},\mathtt{f}\}\,P2$ and $P2\,\{\mathtt{d}\}\,P1$

$\rightsquigarrow P4\,\{\mathtt{d}\}\,P1$.

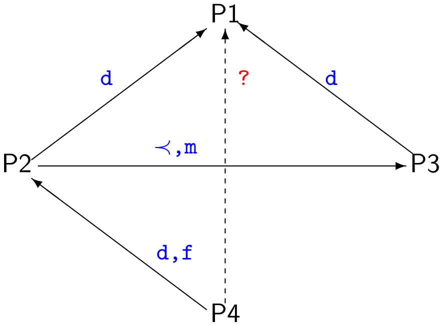
Our Example…Formally

P1: *Display* Picture1

P2: *Say* "Put the plug in."

P3: *Say* "The device should be shut off."

P4: *Point to* Plug-in-Picture1.



Compose the constraints: $P4\,\{\mathtt{d},\mathtt{f}\}\,P2$ and $P2\,\{\mathtt{d}\}\,P1$ $\rightsquigarrow$ $P4\,\{\mathtt{d}\}\,P1$.

Composition of Base Relations

	$\prec$	$\succ$	d	d^{-1}	o	o^{-1}	m	m^{-1}	s	s^{-1}	f	f^{-1}
$\prec$	$\prec$	**B**	$\prec$ o m d s	$\prec$	$\prec$	$\prec$ o m d s	$\prec$	$\prec$ o m d s	$\prec$	$\prec$	$\prec$ o m d s	$\prec$
$\succ$	**B**	$\succ$	$\succ$ o^{-1} m^{-1} d f	$\succ$	$\succ$ o^{-1} m^{-1} d f	$\succ$	$\succ$ o^{-1} m^{-1} d f	$\succ$	$\succ$ o^{-1} m^{-1} d f	$\succ$	$\succ$	$\succ$
d	$\prec$	$\succ$	d	**B**	$\prec$ o m d s	$\succ$ o^{-1} m^{-1} d f	$\prec$	$\succ$	d	$\succ$ o^{-1} m^{-1} d f	d	$\prec$ o m d s
d^{-1}	$\prec$ o m d^{-1} f^{-1}	$\succ$ o^{-1} m^{-1} d^{-1} s^{-1}	**B**$-$ $\prec\succ$ m m^{-1}	d^{-1}	o d^{-1} f^{-1}	o^{-1} d^{-1} s^{-1}	o d^{-1} f^{-1}	o^{-1} d^{-1} s^{-1}	o d^{-1} f^{-1}	d^{-1}	o^{-1} d^{-1} s^{-1}	d^{-1}
o	$\prec$	$\succ$ o^{-1} m^{-1} d^{-1} s^{-1}	o d s	$\prec$ o m d^{-1} f^{-1}	$\prec$ o m	**B**$-$ $\prec\succ$ m m^{-1}	$\prec$	o^{-1} d^{-1} s^{-1}	o	d^{-1} f^{-1} o	d s o	$\prec$ o m
o^{-1}	$\prec$ o m, d^{-1} f^{-1}	$\succ$	o^{-1} d f	$\succ, o^{-1}$ $m^{-1} d^{-1}$ s^{-1}	**B**$-$ $\prec\succ$ m m^{-1}	$\succ$ o^{-1} m^{-1}	o d^{-1} f^{-1}	$\succ$	o^{-1} d f	o^{-1} $\succ$ m^{-1}	o^{-1}	o^{-1} d^{-1} s^{-1}
m	$\prec$	$\succ$ o^{-1} $m^{-1} d^{-1}$ s^{-1}	o d s	$\prec$	$\prec$	o d s	$\prec$	f f^{-1} $\equiv$	m	m	d s o	$\prec$
m^{-1}	$\prec$ o m d^{-1} f^{-1}	$\succ$	o^{-1} d f	$\succ$	o^{-1} d f	$\succ$	s s^{-1} $\equiv$	$\succ$	d f o^{-1}	$\succ$	m^{-1}	m^{-1}
s	$\prec$	$\succ$	d	$\prec$ o m d^{-1} f^{-1}	$\prec$ o m	o^{-1} d f	$\prec$	m^{-1}	s	s s^{-1} $\equiv$	d	$\prec$ m o
s^{-1}	$\prec$ o m d^{-1} f^{-1}	$\succ$	o^{-1} $.d$ f	d^{-1}	o d^{-1} f^{-1}	o^{-1}	o d^{-1} f^{-1}	m^{-1}	s s^{-1} $\equiv$	s^{-1}	o^{-1}	d^{-1}
f	$\prec$	$\succ$	d	$\succ$ o^{-1} $m^{-1} d^{-1}$ s^{-1}	o d s	$\succ$ o^{-1} m^{-1}	m	$\succ$	d	$\succ$ o^{-1} m^{-1}	f	f f^{-1} $\equiv$
f^{-1}	$\prec$	$\succ$ o^{-1} $m^{-1} d^{-1}$ s^{-1}	o d s	d^{-1}	o	o^{-1} d^{-1} s^{-1}	m	s^{-1} o^{-1} d^{-1}	o	d^{-1}	f f^{-1} $\equiv$	f^{-1}

Outlook

- Using the composition table and the rules about operations on relations, we can deduce new relations between time intervals.
- What would be a systematic approach?
- How costly is that?
- Is that complete?
- If not, could it be complete on a subset of the relation system?

Reasoning in Allen's Interval Calculus

1. Allen's Interval Calculus
2. Reasoning in Allen's Interval Calculus
   - Enforcing Path Consistency
   - NP-Hardness Example
   - The Continuous Endpoint Class
   - Completeness for the CEP Class
3. A Maximal Tractable Sub-Algebra
4. Literature

Constraint Propagation – The Naive Algorithm

Enforcing path consistency using the straight-forward method: Let *Table*$[i,j]$ be an array of size $n \times n$ (n: number of intervals) in which we record the constraints between the intervals.

EnforcePathConsistency1($\mathcal{C}$)

Input: a (binary) CSP $\mathcal{C} = \langle V, D, C \rangle$
Output: an equivalent, but path consistent CSP $\mathcal{C}'$

```
repeat
    for each pair $(i, j)$, $1 \leq i, j \leq n$
        for each $k$ with $1 \leq k \leq n$
            $Table[i,j] := Table[i,j] \cap (Table[i,k] \circ Table[k,j])$
until no entry in $Table$ is changed
```

- ⇝ terminates;
- ⇝ needs $O(n^5)$ intersections and compositions.

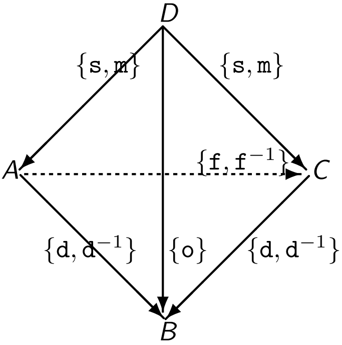
An $O(n^3)$ Algorithm

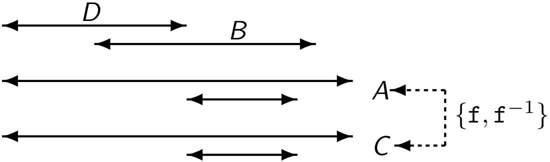
EnforcePathConsistency2($\mathcal{C}$)

Input: a (binary) CSP $\mathcal{C} = \langle V, D, C\rangle$
Output: an equivalent, but path consistent CSP $\mathcal{C}'$

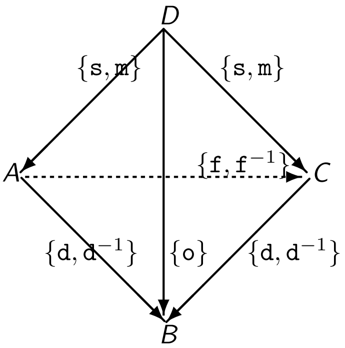
```
$\mathit{Paths}(i,j) = \{(i,j,k) : 1 \leq k \leq n\} \cup \{(k,i,j) : 1 \leq k \leq n\}$

$\mathit{Queue} := \bigcup_{i,j} \mathit{Paths}(i,j)$

while $Q \neq \emptyset$
    select and delete $(i,k,j)$ from $Q$
    $T := \mathit{Table}[i,j] \cap (\mathit{Table}[i,k] \circ \mathit{Table}[k,j])$
    if $T \neq \mathit{Table}[i,j]$
        $\mathit{Table}[i,j] := T$
        $\mathit{Table}[j,i] := T^{-1}$
        $\mathit{Queue} := \mathit{Queue} \cup \mathit{Paths}(i,j)$
```

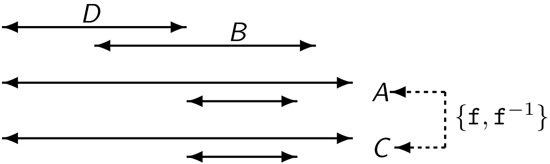
Example for Incompleteness



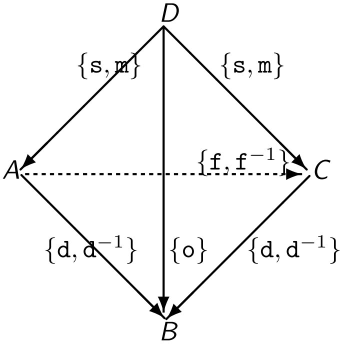


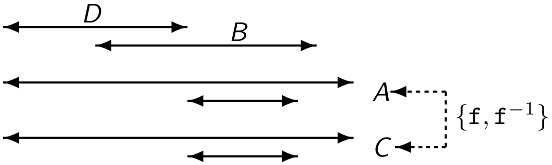
Example for Incompleteness



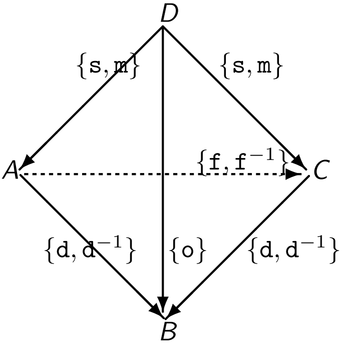


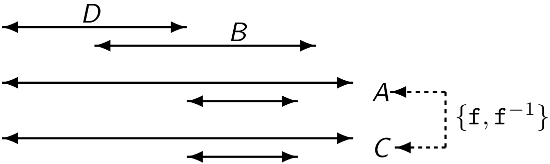
Example for Incompleteness





Example for Incompleteness





NP-Hardness

Theorem (Kautz & Vilain)

CSAT is NP-hard for Allen's interval calculus.

Proof.

Reduction from 3-colorability (original proof using 3Sat).

Let $G = (V, E)$, $V = \{v_1, \ldots, v_n\}$ be an instance of 3-colorability. Then we use the intervals $\{v_1, \ldots, v_n, 1, 2, 3\}$ with the following constraints:

$$\begin{array}{cccl} 1 & \{\mathtt{m}\} & 2 & \\ 2 & \{\mathtt{m}\} & 3 & \\ v_i & \{\mathtt{m}, \equiv, \mathtt{m}^{-1}\} & 2 & \forall v_i \in V \\ v_i & \{\mathtt{m}, \mathtt{m}^{-1}, \prec, \succ\} & v_j & \forall (v_i, v_j) \in E \end{array}$$

This constraint system is satisfiable *iff* G can be colored with 3 colors. ☐

Looking for Special Cases

- Idea: Let us look for polynomial special cases. In particular, let us look for sets of relations (subsets of the entire set of relations) that have an easy CSAT problem.
- Note: Interval formulae $X\ R\,Y$ can be expressed as clauses over atoms of the form $a\ op\ b$, where:
  - a and b are endpoints X^-, X^+, Y^- and Y^+ and
  - $op \in \{<, >, =, \leq, \geq\}$.
- Example: All base relations can be expressed as unit clauses.

Lemma

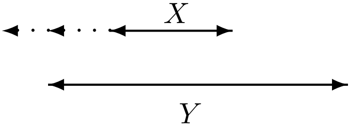
Let $\pi(\Theta)$ be the translation of Θ to clause form. Θ is satisfiable over intervals iff $\pi(\Theta)$ is satisfiable over the rational numbers.

The Continuous Endpoint Class

Continuous Endpoint Class $\mathcal{C}$: This is a subset of $\mathcal{A}$ such that there exists a clause form for each relation containing only unit clauses where $\neg(a = b)$ is forbidden.

Example: All basic relations and $\{\mathtt{d}, \mathtt{o}, \mathtt{s}\}$, because

$$\begin{aligned}\pi(X\,\{\mathtt{d},\mathtt{o},\mathtt{s}\}\,Y) \;=\; \{\, & X^- < X^+, Y^- < Y^+, \\ & X^- < Y^+, X^+ > Y^-, \\ & X^+ < Y^+\}\end{aligned}$$



Why Do We Have Completeness?

The set $\mathcal{C}$ is closed under intersection, composition, and converse (it is a sub-algebra wrt. these three operations on relations). This can be shown by using a computer program.

Lemma

Each 3-consistent interval CSP over $\mathcal{C}$ is globally consistent.

Theorem (van Beek)

Path consistency solves CMIN($\mathcal{C}$) and decides CSAT($\mathcal{C}$).

(Proof: Follows from the above lemma and the fact that a strongly n-consistent CSP is minimal.)

Corollary

A path consistent interval CSP consisting of base relations only is satisfiable.

Helly's Theorem

Definition

A set $M \subseteq \mathbf{R}^n$ is convex iff for all pairs of points $a, b \in M$, all points on the line connecting a and b belong to M.

Theorem (Helly)

Let F be a finite family of at least $n+1$ convex sets in $\mathbf{R}^n$. If all sub-families of F with $n+1$ sets have a non-empty intersection, then $\bigcap F \neq \emptyset$.

Strong n-Consistency (1)

Proof (Part 1).

We prove the claim by induction over k with $k \leq n$.

Base case: $k = 1, 2, 3 \quad \surd$

Induction assumption: Assume strong $(k-1)$-consistency (and non-emptiness of all relations)

Induction step: From the assumption, it follows that there is an instantiation of $k-1$ variables X_i to pairs (s_i, e_i) satisfying the constraints R_{ij} between the $k-1$ variables.

We have to show that we can extend the instantiation to any kth variable.

Strong n-Consistency (1)

Proof (Part 1).

We prove the claim by induction over k with $k \leq n$.

Base case: $k = 1, 2, 3$ $\quad\surd$

Induction assumption: Assume strong $(k-1)$-consistency (and non-emptiness of all relations)

Induction step: From the assumption, it follows that there is an instantiation of $k-1$ variables X_i to pairs (s_i, e_i) satisfying the constraints R_{ij} between the $k-1$ variables.

We have to show that we can extend the instantiation to any kth variable.

Strong n-Consistency (1)

Proof (Part 1).

We prove the claim by induction over k with $k \leq n$.

Base case: $k = 1, 2, 3 \quad \surd$

Induction assumption: Assume strong $(k-1)$-consistency (and non-emptiness of all relations)

Induction step: From the assumption, it follows that there is an instantiation of $k-1$ variables X_i to pairs (s_i, e_i) satisfying the constraints R_{ij} between the $k-1$ variables.

We have to show that we can extend the instantiation to any kth variable.

Strong n-Consistency (1)

Proof (Part 1).

We prove the claim by induction over k with $k \leq n$.

Base case: $k = 1, 2, 3 \quad \surd$

Induction assumption: Assume strong $(k-1)$-consistency (and non-emptiness of all relations)

Induction step: From the assumption, it follows that there is an instantiation of $k-1$ variables X_i to pairs (s_i, e_i) satisfying the constraints R_{ij} between the $k-1$ variables.

We have to show that we can extend the instantiation to any kth variable.

Strong n-Consistency (1)

Proof (Part 1).

We prove the claim by induction over k with $k \leq n$.

Base case: $k = 1, 2, 3 \quad \surd$

Induction assumption: Assume strong $(k-1)$-consistency (and non-emptiness of all relations)

Induction step: From the assumption, it follows that there is an instantiation of $k-1$ variables X_i to pairs (s_i, e_i) satisfying the constraints R_{ij} between the $k-1$ variables.

We have to show that we can extend the instantiation to any kth variable.

Strong n-Consistency (2): Instantiating the kth Variable

Proof (Part 2).

The instantiation of the $k - 1$ variables X_i to (s_i, e_i) restricts the instantiation of X_k.

Note: Since $R_{ij} \in \mathcal{C}$ by assumption, these restrictions can be expressed by inequalities of the form:

$$s_i < X_k^+ \wedge e_j \geq X_k^- \wedge \ldots$$

Such inequalities define convex subsets in $\mathbf{R}^2$.

$\rightsquigarrow$ Consider sets of 3 inequalities (= 3 convex sets).

Strong n-Consistency (2): Instantiating the kth Variable

Proof (Part 2).

The instantiation of the $k-1$ variables X_i to (s_i, e_i) restricts the instantiation of X_k.

Note: Since $R_{ij} \in \mathcal{C}$ by assumption, these restrictions can be expressed by inequalities of the form:

$$s_i < X_k^+ \ \wedge \ e_j \geq X_k^- \ \wedge \ \ldots$$

Such inequalities define convex subsets in $\mathbf{R}^2$.

$\leadsto$ Consider sets of 3 inequalities (= 3 convex sets).

Strong n-Consistency (2): Instantiating the kth Variable

Proof (Part 2).

The instantiation of the $k-1$ variables X_i to (s_i, e_i) restricts the instantiation of X_k.

Note: Since $R_{ij} \in \mathcal{C}$ by assumption, these restrictions can be expressed by inequalities of the form:

$$s_i < X_k^+ \ \wedge\ e_j \geq X_k^- \ \wedge\ \ldots$$

Such inequalities define convex subsets in $\mathbf{R}^2$.

$\rightsquigarrow$ Consider sets of 3 inequalities (= 3 convex sets).

Strong n-Consistency (2): Instantiating the kth Variable

Proof (Part 2).

The instantiation of the $k-1$ variables X_i to (s_i, e_i) restricts the instantiation of X_k.

Note: Since $R_{ij} \in \mathcal{C}$ by assumption, these restrictions can be expressed by inequalities of the form:

$$s_i < X_k^+ \wedge e_j \geq X_k^- \wedge \ldots$$

Such inequalities define convex subsets in $\mathbf{R}^2$.

$\rightsquigarrow$ Consider sets of 3 inequalities (= 3 convex sets).

Strong n-Consistency (3): Using Helly's Theorem

Proof (Part 3).

Case 1: All 3 inequalities mention only X_k^- (or mention only X_k^+). Then it suffices to consider only 2 of these inequalities (the strongest). Because of 3-consistency, there exists at least 1 common point satisfying these 2 inequalities.

Case 2: The inequalities mention X_k^- and X_k^+, but do not contain the inequality $X_k^- < X_k^+$. Then there are at most 2 inequalities with the same variable and we have the same situation as in Case 1.

Case 3: The set contains the inequality $X_k^- < X_k^+$. In this case, only three intervals (incl. X_k) can be involved and by 3-consistency there exists a common point.

$\rightsquigarrow$ With Helly's Theorem, there exists an instantiation consistent with all inequalities.

$\rightsquigarrow$ Strong k-consistency for all $k \leq n$. □

Strong n-Consistency (3): Using Helly's Theorem

Proof (Part 3).

Case 1: All 3 inequalities mention only X_k^- (or mention only X_k^+). Then it suffices to consider only 2 of these inequalities (the strongest). Because of 3-consistency, there exists at least 1 common point satisfying these 2 inequalities.

Case 2: The inequalities mention X_k^- and X_k^+, but do not contain the inequality $X_k^- < X_k^+$. Then there are at most 2 inequalities with the same variable and we have the same situation as in Case 1.

Case 3: The set contains the inequality $X_k^- < X_k^+$. In this case, only three intervals (incl. X_k) can be involved and by 3-consistency there exists a common point.

$\rightsquigarrow$ With Helly's Theorem, there exists an instantiation consistent with all inequalities.

$\rightsquigarrow$ Strong k-consistency for all $k \leq n$. □

Strong n-Consistency (3): Using Helly's Theorem

Proof (Part 3).

Case 1: All 3 inequalities mention only X_k^- (or mention only X_k^+). Then it suffices to consider only 2 of these inequalities (the strongest). Because of 3-consistency, there exists at least 1 common point satisfying these 2 inequalities.

Case 2: The inequalities mention X_k^- and X_k^+, but do not contain the inequality $X_k^- < X_k^+$. Then there are at most 2 inequalities with the same variable and we have the same situation as in Case 1.

Case 3: The set contains the inequality $X_k^- < X_k^+$. In this case, only three intervals (incl. X_k) can be involved and by 3-consistency there exists a common point.

$\rightsquigarrow$ With Helly's Theorem, there exists an instantiation consistent with all inequalities.

$\rightsquigarrow$ Strong k-consistency for all $k \leq n$. ☐

Strong n-Consistency (3): Using Helly's Theorem

Proof (Part 3).

Case 1: All 3 inequalities mention only X_k^- (or mention only X_k^+). Then it suffices to consider only 2 of these inequalities (the strongest). Because of 3-consistency, there exists at least 1 common point satisfying these 2 inequalities.

Case 2: The inequalities mention X_k^- and X_k^+, but do not contain the inequality $X_k^- < X_k^+$. Then there are at most 2 inequalities with the same variable and we have the same situation as in Case 1.

Case 3: The set contains the inequality $X_k^- < X_k^+$. In this case, only three intervals (incl. X_k) can be involved and by 3-consistency there exists a common point.

$\rightsquigarrow$ With Helly's Theorem, there exists an instantiation consistent with all inequalities.

$\rightsquigarrow$ Strong k-consistency for all $k \leq n$. ☐

Outlook

- CMIN($\mathcal{C}$) can be computed in $O(n^3)$ time (for n being the number of intervals) using the path consistency algorithm.
- $\mathcal{C}$ is a set of relations occurring "naturally" when observations are uncertain.
- $\mathcal{C}$ contains 83 relations (incl. the impossible and the universal relations).
- Are there larger sets such that path consistency computes minimal CSPs? Probably not.
- Are there larger sets of relations that permit polynomial satisfiability testing? Yes.

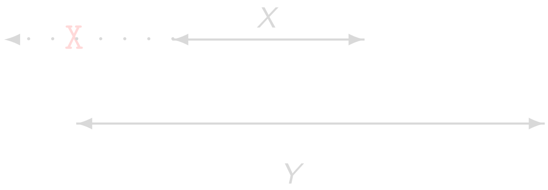
A Maximal Tractable Sub-Algebra

The EP-Subclass

End-Point Subclass: $\mathcal{P} \subseteq \mathcal{A}$ is the subclass that permits a clause form containing only unit clauses ($a \neq b$ is allowed).

Example: all basic relations and $\{\mathtt{d}, \mathtt{o}\}$ since

$$\begin{aligned}\pi(X\ \{\mathtt{d}, \mathtt{o}\}\ Y) \ = \ \{ & X^- < X^+, Y^- < Y^+, \\ & X^- < Y^+, X^+ > Y^-, X^- \neq Y^-, \\ & X^+ < Y^+\}\end{aligned}$$



Theorem (Vilain & Kautz 86, Ladkin & Maddux 88)

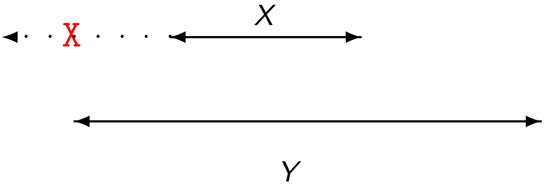
Enforcing path consistency decides CSAT($\mathcal{P}$).

The EP-Subclass

End-Point Subclass: $\mathcal{P} \subseteq \mathcal{A}$ is the subclass that permits a clause form containing only unit clauses ($a \neq b$ is allowed).

Example: all basic relations and $\{\mathtt{d}, \mathtt{o}\}$ since

$$\begin{aligned}\pi(X\,\{\mathtt{d},\mathtt{o}\}\,Y) \quad = \quad & \{X^- < X^+, Y^- < Y^+, \\ & X^- < Y^+, X^+ > Y^-, X^- \neq Y^-, \\ & X^+ < Y^+\}\end{aligned}$$



Theorem (Vilain & Kautz 86, Ladkin & Maddux 88)

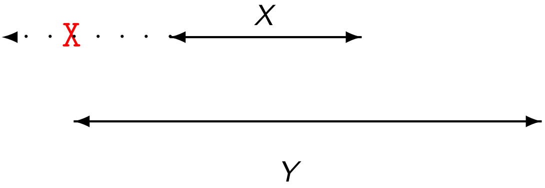
Enforcing path consistency decides CSAT($\mathcal{P}$).

The EP-Subclass

End-Point Subclass: $\mathcal{P} \subseteq \mathcal{A}$ is the subclass that permits a clause form containing only unit clauses ($a \neq b$ is allowed).

Example: all basic relations and $\{\mathtt{d}, \mathtt{o}\}$ since

$$\begin{aligned}\pi(X\,\{\mathtt{d},\mathtt{o}\}\,Y) \quad = \quad & \{X^- < X^+, Y^- < Y^+, \\ & X^- < Y^+, X^+ > Y^-, X^- \neq Y^-, \\ & X^+ < Y^+\}\end{aligned}$$



Theorem (Vilain & Kautz 86, Ladkin & Maddux 88)

Enforcing path consistency decides CSAT($\mathcal{P}$).

The ORD-Horn Subclass

ORD-Horn Subclass: $\mathcal{H} \subseteq \mathcal{A}$ is the subclass that permits a clause form containing only Horn clauses where only the following literals are allowed:

$$a \leq b, a = b, a \neq b$$

$\neg a \leq b$ is not allowed!

Example: all $R \in \mathcal{P}$ and $\{\mathtt{o}, \mathtt{s}, \mathtt{f}^{-1}\}$:

$$\pi(X\{\mathtt{o}, \mathtt{s}, \mathtt{f}^{-1}\}Y) = \Big\{ \begin{array}{l} X^- \leq X^+, X^- \neq X^+, \\ Y^- \leq Y^+, Y^- \neq Y^+, \\ X^- \leq Y^-, \\ X^- \leq Y^+, X^- \neq Y^+, \\ Y^- \leq X^+, X^+ \neq Y^-, \\ X^+ \leq Y^+, \\ X^- \neq Y^- \vee X^+ \neq Y^+ \end{array} \Big\}.$$

The ORD-Horn Subclass

ORD-Horn Subclass: $\mathcal{H} \subseteq \mathcal{A}$ is the subclass that permits a clause form containing only Horn clauses where only the following literals are allowed:

$$a \leq b, a = b, a \neq b$$

$\neg a \leq b$ is not allowed!

Example: all $R \in \mathcal{P}$ and $\{\mathtt{o}, \mathtt{s}, \mathtt{f}^{-1}\}$:

$$\pi(X\{\mathtt{o}, \mathtt{s}, \mathtt{f}^{-1}\}Y) = \Big\{ \begin{array}{l} X^- \leq X^+, X^- \neq X^+, \\ Y^- \leq Y^+, Y^- \neq Y^+, \\ X^- \leq Y^-, \\ X^- \leq Y^+, X^- \neq Y^+, \\ Y^- \leq X^+, X^+ \neq Y^-, \\ X^+ \leq Y^+, \\ X^- \neq Y^- \vee X^+ \neq Y^+ \Big\}. \end{array}$$

Partial Orders: The ORD Theory

Let ORD be the following theory:

$$\begin{array}{llcll}
\forall x, y, z\colon & x \le y \wedge y \le z & \rightarrow & x \le z & \textit{(transitivity)} \\
\forall x\colon & x \le x & & & \textit{(reflexivity)} \\
\forall x, y\colon & x \le y \wedge y \le x & \rightarrow & x = y & \textit{(anti-symmetry)} \\
\forall x, y\colon & x = y & \rightarrow & x \le y & \textit{(weakening of} =\textit{)} \\
\forall x, y\colon & x = y & \rightarrow & y \le x & \textit{(weakening of} =\textit{).}
\end{array}$$

- ORD describes partially ordered sets, $\le$ being the ordering relation.
- ORD is a Horn theory
- What is missing wrt. *dense* and *linear* orders?

Satisfiability over Partial Orders

Proposition

Let Θ be a CSP over $\mathcal{H}$. Θ is satisfiable over interval interpretations iff $\pi(\Theta) \cup ORD$ is satisfiable over arbitrary interpretations.

Proof.

$\Rightarrow$: Since the reals form a partially ordered set (i. e., satisfy ORD), this direction is trivial.

$\Leftarrow$: Each extension of a partial order to a linear order satisfies all formulae of the form $a \leq b$, $a = b$, and $a \neq b$ which have been satisfied over the original partial order. ☐

Satisfiability over Partial Orders

Proposition

Let Θ be a CSP over $\mathcal{H}$. Θ is satisfiable over interval interpretations iff $\pi(\Theta) \cup ORD$ is satisfiable over arbitrary interpretations.

Proof.

$\Rightarrow$: Since the reals form a partially ordered set (i. e., satisfy ORD), this direction is trivial.

$\Leftarrow$: Each extension of a partial order to a linear order satisfies all formulae of the form $a \leq b$, $a = b$, and $a \neq b$ which have been satisfied over the original partial order. ☐

Complexity of CSAT($\mathcal{H}$)

Let $ORD_{\pi(\Theta)}$ be the propositional theory resulting from instantiating all axioms with the endpoints occurring in $\pi(\Theta)$.

Proposition

$ORD \cup \pi(\Theta)$ is satisfiable iff $ORD_{\pi(\Theta)} \cup \pi(\Theta)$ is so.

Proof idea: Herbrand expansion!

Theorem

CSAT($\mathcal{H}$) can be decided in polynomial time.

Proof.

CSAT($\mathcal{H}$) instances can be translated into a propositional Horn theory with blowup $O(n^3)$ according to the previous Prop., and such a theory is decidable in polynomial time. ☐

$\mathcal{C} \subset \mathcal{P} \subset \mathcal{H} \quad \text{with} \quad |\mathcal{C}| = 83, \; |\mathcal{P}| = 188, \; |\mathcal{H}| = 868$

Complexity of CSAT($\mathcal{H}$)

Let $ORD_{\pi(\Theta)}$ be the propositional theory resulting from instantiating all axioms with the endpoints occurring in $\pi(\Theta)$.

Proposition

$ORD \cup \pi(\Theta)$ is satisfiable iff $ORD_{\pi(\Theta)} \cup \pi(\Theta)$ is so.

Proof idea: Herbrand expansion!

Theorem

CSAT($\mathcal{H}$) can be decided in polynomial time.

Proof.

CSAT($\mathcal{H}$) instances can be translated into a propositional Horn theory with blowup $O(n^3)$ according to the previous Prop., and such a theory is decidable in polynomial time. ☐

$\mathcal{C} \subset \mathcal{P} \subset \mathcal{H}$ with $|\mathcal{C}| = 83, \ |\mathcal{P}| = 188, \ |\mathcal{H}| = 868$

Path Consistency and the OH-Class

Lemma

Let Θ be a path-consistent set over $\mathcal{H}$. Then

$$(X\{\}Y) \notin \Theta \quad \textit{iff} \quad \Theta \textit{ is satisfiable}$$

Proof idea: One can show that $ORD_{\pi(\Theta)} \cup \pi(\Theta)$ is closed wrt. positive unit resolution. Since this inference rule is refutation complete for Horn theories, the claim follows.

Theorem

Enforcing path consistency decides CSAT($\mathcal{H}$).

- ⇝ Maximality of $\mathcal{H}$?
- ⇝ Do we have to check all $8192 - 868$ extensions?

Complexity of Sub-Algebras

Let $\hat{S}$ be the closure of $S \subseteq \mathcal{A}$ under converse, intersection, and composition (i.e., the carrier of the least sub-algebra generated by S).

Theorem

CSAT($\hat{S}$) can be polynomially transformed to CSAT(S).

Proof Idea.

All relations in $\hat{S} - S$ can be modeled by a fixed number of compositions, intersections, and conversions of relations in S, introducing perhaps some fresh variables. ☐

$\rightsquigarrow$ Polynomiality of S extends to $\hat{S}$.

$\rightsquigarrow$ NP-hardness of $\hat{S}$ is inherited by all generating sets S.

$\rightsquigarrow$ Note: $\mathcal{H} = \hat{\mathcal{H}}$.

Minimal Extensions of the $\mathcal{H}$-Subclass

A computer-aided case analysis leads to the following result:

Lemma

There are only two minimal sub-algebras that strictly contain $\mathcal{H}$*:* $\mathcal{X}_1, \mathcal{X}_2$

$$N_1 = \{\mathtt{d}, \mathtt{d}^{-1}, \mathtt{o}^{-1}, \mathtt{s}^{-1}, \mathtt{f}\} \in \mathcal{X}_1$$
$$N_2 = \{\mathtt{d}^{-1}, \mathtt{o}, \mathtt{o}^{-1}, \mathtt{s}^{-1}, \mathtt{f}^{-1}\} \in \mathcal{X}_2$$

The clause form of these relations contain "proper" disjunctions!

Theorem

CSAT$(\mathcal{H} \cup \{N_i\})$ *is NP-complete.*

Question: Are there other maximal tractable subclasses?

"Interesting" Subclasses

Interesting subclasses of $\mathcal{A}$ should contain all basic relations.

A *computer-aided* case analysis reveals:
For $S \supseteq \{\{B\} : B \in \mathbf{B}\}$ it holds that

1. $\hat{S} \subseteq \mathcal{H}$, or
2. N_1 or N_2 is in $\hat{S}$.

In case 2, one can show: CSAT(S) is NP-complete.

$\rightsquigarrow$ $\mathcal{H}$ is the only interesting maximal tractable subclass.

If we include non-interesting subalgebras, there exist exactly 18 tractable classes.

Relevance?

Theory: $\oplus$ We now know the boundary between polynomial and NP-hard reasoning problems along the dimension *expressiveness*.

Practice: $\ominus$ All known applications either need only $\mathcal{P}$ or they need more than $\mathcal{H}$!

Backtracking methods might profit from the result by reducing the branching factor.

- $\rightsquigarrow$ How difficult is CSAT($\mathcal{A}$) in practice?
- $\rightsquigarrow$ What are the relevant branching factors?

Solving General Allen CSPs

- Backtracking algorithm using path consistency as a forward-checking method
- Relies on tractable fragments of Allen's calculus: split relations into relations of a tractable fragment, and backtrack over these.
- Refinements and evaluation of different heuristics

⇝ Which tractable fragment should one use?

Branching Factors

- If the labels are split into base relations, then on average a label is split into

  6.5 relations

- If the labels are split into pointizable relations ($\mathcal{P}$), then on average a label is split into

  2.955 relations

- If the labels are split into ORD-Horn relations ($\mathcal{H}$), then on average a label is split into

  2.533 relations

$\leadsto$ A difference of 0.422

$\leadsto$ This makes a difference for "hard" instances.

Summary

- Allen's interval calculus is often adequate for describing relative orders of events that have duration.
- The satisfiability problem for CSPs using the relations is NP-complete.
- For the continuous endpoint class, minimal CSPs can be computed using the path-consistency method.
- For the larger ORD-Horn class, CSAT is still decided by the path-consistency method.
- Can be used in practice for backtracking algorithms.

Literature I

J. F. Allen.
Maintaining knowledge about temporal intervals.
Communications of the ACM 26(11):832–843, 1983.
Also in *Readings in Knowledge Representation*.

P. van Beek and R. Cohen.
Exact and approximate reasoning about temporal relations.
Computational Intelligence, 6:132–144, 1990.

B. Nebel and H.-J. Bürckert.
Reasoning about temporal relations: A maximal tractable subclass of Allen's interval algebra.
Journal of the ACM 42(1):43–66, 1995.

B. Nebel.
Solving hard qualitative temporal reasoning problems: Evaluating the efficiency of using the ORD-Horn class.
CONSTRAINTS 1(3):175–190, 1997.

Literature II

A. Krokhin, P. Jeavons and P. Jonsson.
A complete classification of complexity in Allen's algebra in the presence of a non-trivial basic relation.
Proceedings of the 17th Internation Joint Conferenc on Artificial Intelligence (IJCAI 2001), pp. 83–88, Seattle, WA, 2001.

A. Krokhin, P. Jeavons and P. Jonsson.
Reasoning about Temporal Relations: The Tractable Subalgebras of Allen's Interval Algebra.
Journal of the ACM 50(5):591–640, 2003.