Principles of Knowledge Representation and Reasoning

Qualitative Representation and Reasoning II: Allen's Interval Calculus

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Allen's Interval Calculus – Outline

Qualitative Temporal Representation and Reasoning

Often we do not want to talk about precise times:

- NLP – we do not have precise time points
- Planning – we do not want to commit to time points too early
- Scenario descriptions – we do not have the exact times or do not want to state them

What are the primitives in our representation system?

- Time points: actions and events are instantaneous, or we consider their beginning and ending
- Time intervals: actions and events have duration
- Reducibility? Expressiveness? Computational costs for reasoning?

Motivation: Example

Consider a planning scenario for multimedia generation:

P1: *Display* Picture1

P2: *Say* "Put the plug in."

P3: *Say* "The device should be shut off."

P4: *Point to* Plug-in-Picture1.

Temporal relations between events:

P2	should happen during	P1
P3	should happen during	P1
P2	should happen before or directly precede	P3
P4	should happen during or end together with	P2

- $\rightsquigarrow$ P4 happens before or directly precedes P3
- $\rightsquigarrow$ We could add the statement "P4 does not overlap with P3" without creating an inconsistency.

Allen's Interval Calculus

- Allen's interval calculus: time intervals and binary relations over them
- Time intervals: $X = (X^-, X^+)$, where X^- and X^+ are interpreted over the reals and $X^- < X^+$ ($\rightsquigarrow$ naïve approach)
- Relations between concrete intervals, e. g.:

  (1.0,2.0) strictly before (3.0,5.5)
  (1.0,3.0) meets (3.0,5.5)
  (1.0,4.0) overlaps (3.0,5.5)
  ...

$\rightsquigarrow$ Which relations are conceivable?

The Base Relations

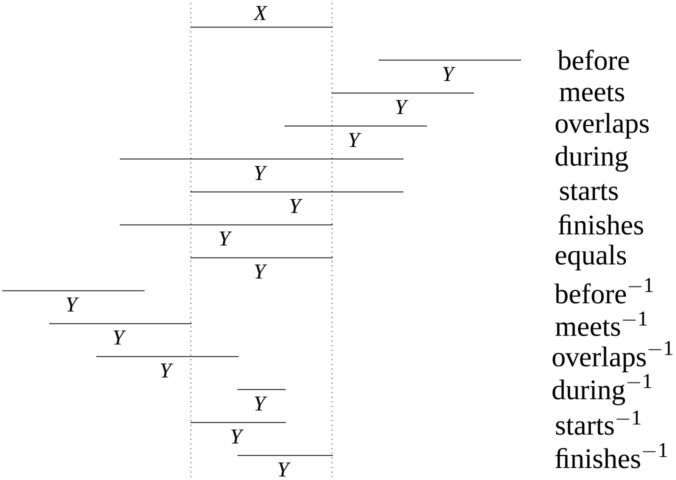
How many ways are there to order the four points of two intervals?

Relation	Symbol	Name
$\{(X,Y) : X^- < X^+ < Y^- < Y^+\}$	$\prec$	before
$\{(X,Y) : X^- < X^+ = Y^- < Y^+\}$	$\mathtt{m}$	meets
$\{(X,Y) : X^- < Y^- < X^+ < Y^+\}$	$\mathtt{o}$	overlaps
$\{(X,Y) : X^- = Y^- < X^+ < Y^+\}$	$\mathtt{s}$	starts
$\{(X,Y) : Y^- < X^- < X^+ = Y^+\}$	$\mathtt{f}$	finishes
$\{(X,Y) : Y^- < X^- < X^+ < Y^+\}$	$\mathtt{d}$	during
$\{(X,Y) : Y^- = X^- < X^+ = Y^+\}$	$\equiv$	equal

and the **converse** relations (obtained by exchanging X and Y)

$\leadsto$ These relations are JEPD.

The 13 Base Relations Graphically



Disjunctive Descriptions

- Assumption: We don't have precise information about the relation between X and Y, e. g.:

$$X \text{ o } Y \text{ or } X \text{ m } Y$$

- . . . modelled by sets of base relations (meaning the union of the relations):

$$X \{\mathtt{o}, \mathtt{m}\} Y$$

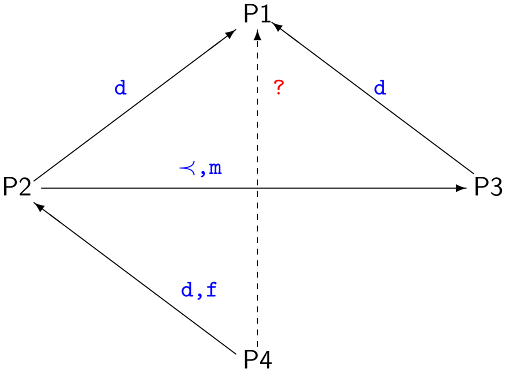
$\leadsto$ 2^{13} imprecise relations (incl. $\emptyset$ and **B**)

Example of an indefinite qualitative description:

$$\left\{ X \{\mathtt{o}, \mathtt{m}\} Y, \; Y \{\mathtt{m}\} Z, X \{\mathtt{o}, \mathtt{m}\} Z \right\}$$

Our Example . . . Formal

P1: *Display* Picture1
P2: *Say* "Put the plug in."
P3: *Say* "The device should be shut off."
P4: *Point to* Plug-in-Picture1.



Compose the constraints: $P4\,\{\mathtt{d},\mathtt{f}\}\,P2$ and $P2\,\{\mathtt{d}\}\,P1$: $P4\,\{\mathtt{d}\}\,P1$.

	$\prec$	$\succ$	d	d^{-1}	o	o^{-1}	m	m^{-1}	s	s^{-1}	f	f^{-1}
$\prec$	$\prec$	**B**	$\prec$ o m d s	$\prec$	$\prec$	$\prec$ o m d s	$\prec$	$\prec$ o m d s	$\prec$	$\prec$	$\prec$ o m d s	$\prec$
$\succ$	**B**	$\succ$	$\succ$ o^{-1} m^{-1} d f	$\succ$	$\succ$ o^{-1} m^{-1} d f	$\succ$	$\succ$ o^{-1} m^{-1} d f	$\succ$	$\succ$ o^{-1} m^{-1} d f	$\succ$	$\succ$	$\succ$
d	$\prec$	$\succ$	d	**B**	$\prec$ o m d s	$\succ$ o^{-1} m^{-1} d f	$\prec$	$\succ$	d	$\succ$ o^{-1} m^{-1} d f	d	$\prec$ o m d s
d^{-1}	$\prec$ o m d^{-1} f^{-1}	$\succ$ o^{-1} m^{-1} d^{-1} s^{-1}	**B**− $\prec\succ$ m m^{-1}	d^{-1}	o d^{-1} f^{-1}	o^{-1} d^{-1} s^{-1}	o d^{-1} f^{-1}	o^{-1} d^{-1} s^{-1}	o d^{-1} f^{-1}	d^{-1}	o^{-1} d^{-1} s^{-1}	d^{-1}
o	$\prec$	$\succ$ o^{-1} m^{-1} d^{-1} s^{-1}	o d s	$\prec$ o m d^{-1} f^{-1}	$\prec$ o m	**B**− $\prec\succ$ m m^{-1}	$\prec$	o^{-1} d^{-1} s^{-1}	o	d^{-1} f^{-1} o	d s o	$\prec$ o m
o^{-1}	$\prec$ o m, d^{-1} f^{-1}	$\succ$	o^{-1} d f	$\succ$, o^{-1} m^{-1} d^{-1} s^{-1}	**B**− $\prec\succ$ m m^{-1}	$\succ$ o^{-1} m^{-1}	o d^{-1} f^{-1}	$\succ$	o^{-1} d f	o^{-1} $\succ$ m^{-1}	o^{-1}	o^{-1} d^{-1} s^{-1}
m	$\prec$	$\succ$ o^{-1} m^{-1} d^{-1} s^{-1}	o d s	$\prec$	$\prec$	o d s	$\prec$	f f^{-1} $\equiv$	m	m	d s o	$\prec$
m^{-1}	$\prec$ o m d^{-1} f^{-1}	$\succ$	o^{-1} d f	$\succ$	o^{-1} d f	$\succ$	s s^{-1} $\equiv$	$\succ$	d f o^{-1}	$\succ$	m^{-1}	m^{-1}
s	$\prec$	$\succ$	d	$\prec$ o m d^{-1} f^{-1}	$\prec$ o m	o^{-1} d f	$\prec$	m^{-1}	s	s s^{-1} $\equiv$	d	$\prec$ m o
s^{-1}	$\prec$ o m d^{-1} f^{-1}	$\succ$	o^{-1} .d f	d^{-1}	o d^{-1} f^{-1}	o^{-1}	o d^{-1} f^{-1}	m^{-1}	s s^{-1} $\equiv$	s^{-1}	o^{-1}	d^{-1}
f	$\prec$	$\succ$	d	$\succ$ o^{-1} m^{-1} d^{-1} s^{-1}	o d s	$\succ$ o^{-1} m^{-1}	m	$\succ$	d	$\succ$ o^{-1} m^{-1}	f	f f^{-1} $\equiv$
f^{-1}	$\prec$	$\succ$ o^{-1} m^{-1} d^{-1} s^{-1}	o d s	d^{-1}	o	o^{-1} d^{-1} s^{-1}	m	s^{-1} o^{-1} d^{-1}	o	d^{-1}	f f^{-1} $\equiv$	f^{-1}

Outlook

- Using the composition table and the rules about operations on relations, we can deduce new relations between time intervals.
- What would be a systematic approach?
- How costly is that?
- Is that complete?
- If not, could it be complete on a subset of the relation system?

Reasoning in Allen's Interval Calculus

Constraint Propagation – The Naive Algorithm

Enforcing path-consistency using the straight-forward method:
Let $Table[i,j]$ be an array of size $|n| \times |n|$ (n: number of intervals), in which we have recorded the constraints between the intervals.

EnforcePathConsistency1 ($\mathcal{C}$):
Input: a (binary) CSP $\mathcal{C} = \langle V, D, C \rangle$
Output: an equivalent, but path consistent CSP $\mathcal{C}'$

repeat
 for each pair (i,j), $1 \leq i,j \leq n$
  for each k with $1 \leq k \leq n$
   $Table[i,j] := Table[i,j] \cap (Table[i,k] \circ Table[k,j])$
  endfor
 endfor
until no entry in $Table$ is changed

- $\rightsquigarrow$ terminates;
- $\rightsquigarrow$ needs $O(n^5)$ intersections and compositions.

An $O(n^3)$ Algorithm

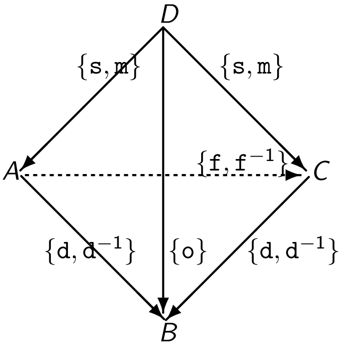
EnforcePathConsistency2 $(\mathcal{C})$:
Input: a (binary) CSP $\mathcal{C} = \langle V, D, C \rangle$
Output: an equivalent, but path consistent CSP $\mathcal{C}'$

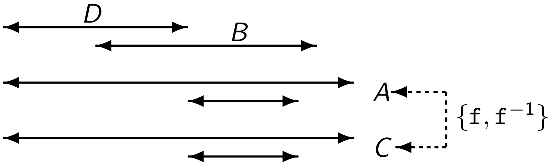
$Paths(i,j) = \{(i,j,k) : 1 \leq k \leq n\} \cup \{(k,i,j) : 1 \leq k \leq n\}$

$Queue := \bigcup_{i,j} Paths(i,j)$

While $Q \neq \emptyset$
  select and delete (i,k,j) from Q
  $T := Table[i,j] \cap (Table[i,k] \circ Table[k,j])$
  if $T \neq Table[i,j]$
    $Table[i,j] := T$
    $Table[j,i] := T^{-1}$
    $Queue := Queue \cup Paths(i,j)$
  endif
endwhile

Example for Incompleteness





NP-Hardness

Theorem (Kautz & Vilain)

CSAT is NP-hard for Allen's interval calculus.

Proof.

Reduction from 3-colorability (original proof using 3Sat).

Let $G = (V, E)$, $V = \{v_1, \ldots, v_n\}$ be an instance of 3-colorability. Then we use the intervals $\{v_1, \ldots, v_n, 1, 2, 3\}$ with the following constraints:

$$
\begin{array}{cccl}
1 & \{\mathtt{m}\} & 2 & \\
2 & \{\mathtt{m}\} & 3 & \\
v_i & \{\mathtt{m}, \equiv, \mathtt{m}^{-1}\} & 2 & \forall v_i \in V \\
v_i & \{\mathtt{m}, \mathtt{m}^{-1}, \prec, \succ\} & v_j & \forall (v_i, v_j) \in E
\end{array}
$$

This constraint system is satisfiable *iff* G can be colored with 3 colors. □

Looking for Special Cases

- Idea: Let us look for polynomial special cases. In particular, let us look for sets of relations (subsets of the entire set of relations) that have an easy CSAT problem.
- Note: Interval formulae $X\ R\ Y$ can be expressed as clauses over atoms of the form *a op b*, where:
  - a and b are endpoints X^-, X^+, Y^- and Y^+ and
  - $op \in \{<, >, =, \leq, \geq\}$.
- Example: All base relations can be expressed as unit clauses.

Lemma

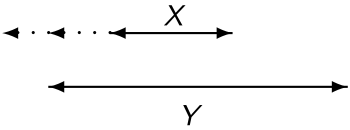
Let $\pi(\Theta)$ be the translation of Θ to clause form. Θ is satisfiable over intervals iff $\pi(\Theta)$ is satisfiable over the rational numbers.

The Continuous Endpoint Class

Continuous Endpoint Class $\mathcal{C}$: This is a subset of $\mathcal{A}$ such that there exists a clause form for each relation containing only unit clauses where $\neg(a = b)$ is forbidden.

Example: All basic relations and $\{\mathtt{d}, \mathtt{o}, \mathtt{s}\}$, because

$$\begin{aligned}\pi(X\ \{\mathtt{d}, \mathtt{o}, \mathtt{s}\}\ Y) \quad = \quad & \{X^- < X^+, Y^- < Y^+, \\ & X^- < Y^+, X^+ > Y^-, \\ & X^+ < Y^+\}\end{aligned}$$



Why Do We Have Completeness?

The set $\mathcal{C}$ is closed under intersection, composition, and converse (it is a sub-algebra wrt. these three operations on relations). This can be shown by using a computer program.

Lemma

Each 3-consistent interval CSP over $\mathcal{C}$ is globally consistent.

Theorem (van Beek)

Path consistency solves CMIN($\mathcal{C}$) and decides CSAT($\mathcal{C}$).

Proof.

Follows from the above lemma and the fact that a strongly n-consistent CSP is minimal. $\square$

Corollary

A path consistent interval CSP consisting of base relations only is satisfiable.

Helly's Theorem

Definition

A set $M \subseteq \mathbf{R}^n$ is convex iff for all pairs of points $a, b \in M$, all points on the line connecting a and b belong to M.

Theorem (Helly)

Let F be a family of at least $n+1$ convex sets in $\mathbf{R}^n$. If all sub-families of F with $n+1$ sets have a non-empty intersection, then $\bigcap F \neq \emptyset$.

Strong n-Consistency (1)

Proof.

We prove the claim by induction over k with $k \leq n$.

Base case: $k = 1, 2, 3 \quad \surd$

Induction assumption: Assume strong $k - 1$-consistency (and non-emptiness of all relations)

Induction step: From the assumption, it follows that there is an instantiation of $k - 1$ variables X_i to pairs (s_i, e_i) satisfying the constraints R_{ij} between the $k - 1$ variables.

We have to show that we can extend the instantiation to any kth variable.

Strong n-Consistency (2): Instantiating the kth Variable

Proof (Part 2).

The instantiation of the $k-1$ variables X_i to (s_i, e_i) restricts the instantiation of X_k.

Note: Since $R_{ij} \in \mathcal{C}$ by assumption, these restrictions can be expressed by inequalities of the form:

$$s_i < X_k^+ \ \wedge \ e_j \geq X_k^- \ \wedge \ \ldots$$

Such inequalities define convex subsets in $\mathbf{R}^2$.

$\rightsquigarrow$ Consider sets of 3 inequalities ($=$ 3 convex sets).

Strong n-Consistency (3): Using Helly's Theorem

Proof (Part 3).

Case 1: All 3 inequalities mention only X_k^- (or mention only X_k^+). Then it suffices to consider only 2 of these inequalities (the strongest). Because of 3-consistency, there exists at least 1 common point satisfying these 3 inequalities.

Case 2: The inequalities mention X_k^- and X_k^+, but it does not contain the inequality $X_k^- < X_k^+$. Then there are at most 2 inequalities with the same variable and we have the same situation as in Case 1.

Case 3: The set contains the inequality $X_k^- < X_k^+$. In this case, only three intervals (incl. X_k) can be involved and by the same argument as above there exists a common point.

- $\rightsquigarrow$ With Helly's Theorem, it follows that there exists a consistent instantiation for all subsets of variables.
- $\rightsquigarrow$ Strong k-consistency for all $k \leq n$.

□

Outlook

- CMIN($\mathcal{C}$) can be computed in $O(n^3)$ time (for n being the number of intervals) using the path consistency algorithm.
- $\mathcal{C}$ is a set of relations occurring "naturally" when observations are uncertain.
- $\mathcal{C}$ contains 83 relations (incl. the impossible and the universal relations).
- Are there larger sets such that path consistency computes minimal CSPs? Probably not
- Are there larger sets of relations that permit polynomial satisfiability testing? Yes

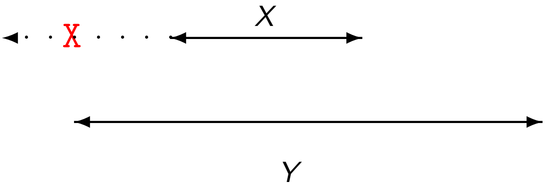
A Maximal Tractable Sub-Algebra

The EP-Subclass

End-Point Subclass: $\mathcal{P} \subseteq \mathcal{A}$ is the subclass that permits a clause form containing only unit clauses ($a \neq b$ is allowed).

Example: all basic relations and $\{\mathtt{d}, \mathtt{o}\}$ since

$$\begin{aligned}\pi(X \{\mathtt{d}, \mathtt{o}\} Y) \quad = \quad & \{X^- < X^+, Y^- < Y^+, \\ & X^- < Y^+, X^+ > Y^-, X^- \neq Y^-, \\ & X^+ < Y^+\}\end{aligned}$$



Theorem (Vilain & Kautz 86, Ladkin & Maddux 88)

The path-consistency method decides CSAT($\mathcal{P}$).

The ORD-Horn Subclass

ORD-Horn Subclass: $\mathcal{H} \subseteq \mathcal{A}$ is the subclass that permits a clause form containing only Horn clauses , where only the following literals are allowed:

$$a \leq b, a = b, a \neq b$$

$\neg a \leq b$ is not allowed!

Example: all $R \in \mathcal{P}$ and $\{\mathtt{o}, \mathtt{s}, \mathtt{f}^{-1}\}$:

$$\pi(X\{\mathtt{o}, \mathtt{s}, \mathtt{f}^{-1}\}Y) = \Big\{ \begin{array}{l} X^- \leq X^+, X^- \neq X^+, \\ Y^- \leq Y^+, Y^- \neq Y^+, \\ X^- \leq Y^-, \\ X^- \leq Y^+, X^- \neq Y^+, \\ Y^- \leq X^+, X^+ \neq Y^-, \\ X^+ \leq Y^+, \\ X^- \neq Y^- \vee X^+ \neq Y^+ \Big\}. \end{array}$$

Partial Orders: The *ORD* Theory

Let *ORD* be the following theory:

$$
\begin{array}{lllll}
\forall x, y, z: & x \leq y \wedge y \leq z & \rightarrow & x \leq z & \textit{(transitivity)} \\
\forall x: & x \leq x & & & \textit{(reflexivity)} \\
\forall x, y: & x \leq y \wedge y \leq x & \rightarrow & x = y & \textit{(anti-symmetry)} \\
\forall x, y: & x = y & \rightarrow & x \leq y & \textit{(weakening of} = \textit{)} \\
\forall x, y: & x = y & \rightarrow & y \leq x & \textit{(weakening of} = \textit{)}.
\end{array}
$$

- *ORD* describes partially ordered sets, $\leq$ being the ordering relation.
- *ORD* is a Horn theory
- What is missing wrt to *dense* and *linear* orders?

Satisfiability over Partial Orders

Proposition

Let Θ be a CSP over $\mathcal{H}$. Θ is satisfiable over interval interpretations iff $\pi(\Theta) \cup ORD$ is satisfiable over arbitrary interpretations.

Proof.

$\Rightarrow$: Since the reals form a partially ordered set (i. e., satisfy *ORD*), this direction is trivial.
$\Leftarrow$: Each extension of a partial order to a linear order satisfies all formulae of the form $a \leq b$, $a = b$, and $a \neq b$ which have been satisfied over the original partial order. □

Complexity of CSAT($\mathcal{H}$)

Let $ORD_{\pi(\Theta)}$ be the propositional theory resulting from instantiating all axioms with the endpoints occurring in $\pi(\Theta)$.

Proposition

$ORD \cup \pi(\Theta)$ is satisfiable iff $ORD_{\pi(\Theta)} \cup \pi(\Theta)$ is so.

Proof idea: Herbrand expansion!

□

Theorem

CSAT($\mathcal{H}$) can be decided in polynomial time.

Proof.

CSAT($\mathcal{H}$) instances can be translated into a propositional Horn theory with blowup $O(n^3)$ according to the previous Prop., and such a theory is decidable in polynomial time. □

$$\mathcal{C} \subset \mathcal{P} \subset \mathcal{H} \quad \text{with} \quad |\mathcal{C}| = 83,\ |\mathcal{P}| = 188,\ |\mathcal{H}| = 868$$

Path-Consistency and the OH-Class

Lemma

Let Θ be a path-consistent set over $\mathcal{H}$. Then

$$(X\{\}Y) \notin \Theta \quad \textit{iff} \quad \Theta \textit{ is satisfiable}$$

Proof Idea.

One can show that $ORD_{\pi(\Theta)} \cup \pi(\Theta)$ is closed wrt positive unit resolution. Since this inference rule is refutation complete for Horn theories, the claim follows. □

Lemma

$\mathcal{H}$ is closed under intersection, composition, and conversion.

Theorem

The path-consistency method decides CSAT($\mathcal{H}$).

- ⇝ Maximality of $\mathcal{H}$?
- ⇝ Do we have to check all 8192 - 868 extensions?

Complexity of Sub-Algebras

Let $\hat{S}$ be the closure of $S \subseteq \mathcal{A}$ under converse, intersection, and composition (i.e., the carrier of the least sub-algebra generated by S)

Theorem

CSAT($\hat{S}$) can be polynomially transformed to CSAT(S).

Proof Idea.

All relations in $\hat{S} - S$ can be modeled by a fixed number of compositions, intersections, and conversions of relations in S, introducing perhaps some fresh variables. □

- ⇝ Polynomiality of S extends to $\hat{S}$.
- ⇝ NP-hardness of $\hat{S}$ is inherited by all generating sets S.
- ⇝ Note: $\mathcal{H} = \hat{\mathcal{H}}$.

Minimal Extensions of the $\mathcal{H}$-Subclass

A *computer-aided* case analysis leads to the following result:

Lemma

There are only two minimal sub-algebras that strictly contain $\mathcal{H}$: $\mathcal{X}_1, \mathcal{X}_2$

$$N_1 = \{\mathtt{d}, \mathtt{d}^{-1}, \mathtt{o}^{-1}, \mathtt{s}^{-1}, \mathtt{f}\} \in \mathcal{X}_1$$
$$N_2 = \{\mathtt{d}^{-1}, \mathtt{o}, \mathtt{o}^{-1}, \mathtt{s}^{-1}, \mathtt{f}^{-1}\} \in \mathcal{X}_2$$

The clause form of these relations contain “proper” disjunctions!

Theorem

CSAT($\mathcal{H} \cup \{N_i\}$) is NP-complete.

Question: Are there other maximal tractable subclasses?

"Interesting" Subclasses

Interesting subclasses of $\mathcal{A}$ should contain all basic relations.

A *computer-aided* case analysis reveals: For $S \supseteq \{\{B\} : B \in \mathbf{B}\}$ it holds that

1. $\hat{S} \subseteq \mathcal{H}$, or
2. N_1 or N_2 is in $\hat{S}$.

In case 2, one can show: CSAT(S) is NP-complete.

$\rightsquigarrow$ $\mathcal{H}$ is the only maximal tractable subclass that is interesting.

Meanwhile, there is a complete classification of all sub-algebras containing at least one basic relation [IJCAI 2001] ... but the question for sub-algebras not containing a basic relation is open.

Relevance?

Theoretical:

We now know the boundary between polynomial and NP-hard reasoning problems along the dimension *expressiveness*. $\oplus$

Practical:

All known applications either need only $\mathcal{P}$ or they need more than $\mathcal{H}$! $\ominus$

Backtracking methods might profit from the result because the branching factor is lower. ?

- $\rightsquigarrow$ How difficult is CSAT($\mathcal{A}$) in practice?
- $\rightsquigarrow$ What are the relevant branching factors?

Solving General Allen CSPs

- Backtracking algorithm using path-consistency as a forward-checking method
- Relies on tractable fragments of Allen's calculus: split relations into relations of a tractable fragment, and backtrack over these.
- Refinements and evaluation of different heuristics

$\rightsquigarrow$ Which tractable fragment should one use?

Branching Factors

- If the labels are split into base relations, then on average a label is split into

  6.5 relations

- If the labels are split into pointizable relations ($\mathcal{P}$), then on average a label is split into

  2.955 relations

- If the labels are split into ORD-Horn relations ($\mathcal{H}$), then on average a label is split into

  2.533 relations

$\rightsquigarrow$ A difference of 0.422

$\rightsquigarrow$ This makes a difference for "hard" instances.

Summary

- Allen's interval calculus is often adequate for describing relative orders of events that have duration.
- The satisfiability problem for CSPs using the relations is NP-complete.
- For the continuous endpoint class, minimal CSPs can be computed using the path-consistency method.
- For the larger ORD-Horn class, CSAT is still decided by the path-consistency method.
- Can be used in practice for backtracking algorithms.

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