Principles of Knowledge Representation and Reasoning

Complexity Theory

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Motivation for Using Complexity Theory

- Complexity theory can answer questions on how easy or hard a problem is
- Gives hints on what algorithms could be appropriate, e.g.:
    - algorithms for polynomial-time problems are usually easy to design
    - for NP-complete problems, backtracking and local search work well
- Gives hints on what type of algorithm will (most probably) not work
    - for problems that are believed to be harder than NP-complete ones, simple backtracking will not work
- Gives hint on what sub-problems might be interesting

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Algorithms and Turing Machines

- We use Turing machines as formal models of algorithms
- This is justified, because:
  - we assume that Turing machines can compute all computable functions
  - the resource requirements (in term of time and memory) of a Turing machine are only polynomially worse than other models
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- Often, however, we use the notion of nondeterministic TMs: NDTM

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Problems, Solutions, and Complexity

- A problem is a set of pairs (I, A) of strings in $\{0,1\}^*$.
  I: Instance; A: Answer.
  If $A \in \{0,1\}$: decision problem
- A decision problem is the same as a formal language: namely the set of strings formed by the instances with answer 1
- An algorithm decides (or solves) a problem if it computes the right answer for all instances.
- The complexity of an algorithm is a function

$$T : \mathbf{N} \to \mathbf{N},$$

  measuring the number of basic steps (or memory requirement) the algorithm needs to compute an answer depending on the size of the instance.
- The complexity of a problem is the complexity of the most efficient algorithm that solves this problem.

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Complexity Classes P and NP

Problems are categorized into complexity classes according to the requirements of computational resources:

- The class of problems decidable on deterministic Turing machines in polynomial time: P
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- The class of problems decidable on non-deterministic Turing machines in polynomial time: NP
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Upper and Lower Bounds

- Upper bounds (membership in a class) are usually easy to prove:
  - provide an algorithm
  - show that the resource bounds are respected
- Lower bounds (hardness for a class) are usually difficult to show:
  - the technical tool here is the polynomial reduction (or any other appropriate reduction)
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Polynomial Reductions

- Given two languages L_1 and L_2, L_1 can be polynomially reduced to L_2, written $L_1 \leq_p L_2$, iff there exists a polynomially computable function f such that

$$x \in L_1 \text{ iff } f(x) \in L_2$$

- It cannot be harder to decide L_1 than L_2
- L is hard for a class C (C-hard) iff all languages of this class can be reduced to L.
- L is complete for C (C-complete) iff L is C-hard and $L \in C$.

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NP-complete Problems

- A problem is **NP-complete** iff it is NP-hard and in NP.
- Example: **SAT** – the satisfiability problem for propositional logic – is NP-complete (Cook/Karp)
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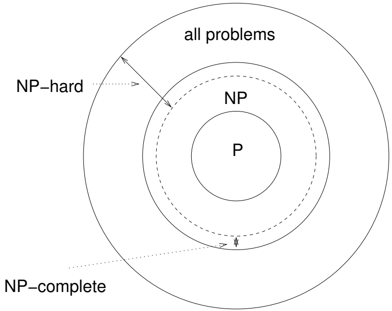

The Complexity Class co-NP

- Note that there is some asymmetry in the definition of NP:
  - It is clear that we can decide SAT by using a NDTM with polynomially bounded computation
  - There exists an accepting computation of polynomial length iff the formula is satisfiable
  - What if we want to solve UNSAT, the complementary problem?
  - It seems necessary to check all possible truth-assignments!
- Define co-$C = \{L | \Sigma^* - L \in C\}$, provided Σ is our alphabet
- co-NP $= \{L | \Sigma^* - L \in \text{NP}\}$
- For example UNSAT, TAUT $\in$ co-NP!
- Note: P is closed under complement, i.e.,

$$\text{P} \subseteq \text{NP} \cap \text{co-NP}$$

PSPACE

There are problems even more difficult than NP and co-NP.

Definition ((N)PSPACE)

PSPACE (NPSPACE) is the class of decision problems that can be decided on deterministic (non-deterministic) Turing machines using only polynomially many tape cells.

Some facts about PSPACE:

- PSPACE is closed under complements (as all other deterministic classes)
- PSPACE is identical to NPSPACE (because non-deterministic Turing machines can be simulated on deterministic TMs using only quadratic space)
- NP⊆PSPACE (because in polynomial time one can “visit” only polynomial space, i.e., NP⊆NPSPACE)
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A decision problem (or language) is PSPACE-complete, if it is in PSPACE and all other problems in PSPACE can be polynomially reduced to it.

Intuitively, PSPACE-complete problems are the "hardest" problems in PSPACE (similar to NP-completeness). They appear to be "harder" than NP-complete problems from a *practical point of view*.

An example for a PSPACE-complete problem is the NDFA equivalence problem:

> **Instance**: *Two non-deterministic finite state automata* A_1 *and* A_2.
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- There are complexity classes above PSPACE (EXPTIME, EXPSPACE, NEXPTIME, DEXPTIME . . .)
- there are (infinitely many) classes between NP and PSPACE (the polynomial hierarchy defined by oracle machines)
- there are (infinitely many) classes inside P (circuit classes with different depths)
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Oracle Turing Machines

- An Oracle Turing machine ((N)OTM) is a Turing machine (DTM, NDTM) with the possibility to query an oracle (i. e., a different Turing machine without resource restrictions) whether it accepts or rejects a given string.
- Computation by the oracle does not cost anything!
- Formalization:
  - a tape onto which strings for the oracle are written,
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- Usage of OTMs answers what-if questions: What if we could solve the oracle-problem efficiently?

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- Idea: The "classical" reduction can be seen as calling a subroutine once.
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- Polynomial reducibility implies Turing reducibility, but not *vice versa*!
- NP-hardness and co-NP-hardness with respect to Turing reducibility are equivalent!
- Turing reducibility can also be applied to general search problems!

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Complexity Classes Based on Oracle TMs

1. P^{NP} = decision problems solved by poly-time DTMs with an oracle for a decision problem in NP.
2. NP^{NP} = decision problems solved by poly-time NDTMs with an oracle for a decision problem in NP.
3. $\text{co-}NP^{NP}$ = complements of decision problems solved by poly-time NDTMs with an oracle for a decision problem in NP.
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Example

- Consider the Minimum Equivalent Expression (MEE) problem:

  Instance: *A well-formed Boolean formula ϕ using the standard connectives (not $\leftrightarrow$) and a nonnegative integer K.*
  Question: *Is there a well-formed Boolean formula ϕ' that contains K or fewer literal occurrences and that is logical equivalent to ϕ?*

- This problem is NP-hard (wrt. to Turing reductions).
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The Polynomial Hierarchy

The complexity classes based on OTMs form an infinite hierarchy.

The polynomial hierarchy PH

$$\begin{array}{lll} \Sigma_0^p = \mathrm{P} & \Pi_0^p = \mathrm{P} & \Delta_0^p = \mathrm{P} \\ \Sigma_{i+1}^p = \mathrm{NP}^{\Sigma_i^p} & \Pi_{i+1}^p = \text{co-}\Sigma_{i+1}^p & \Delta_{i+1}^p = P^{\Sigma_i^p} \end{array}$$

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Quantified Boolean Formulae: Definition

- If ϕ is a propositional formula, P is the set of Boolean variables used in ϕ and σ is a sequence of $\exists p$ and $\forall p$, one for every $p \in P$, then $\sigma\phi$ is a QBF.
- A formula $\exists x \phi$ is true if and only if $\phi[\top/x] \vee \phi[\bot/x]$ is true. (Equivalently, $\phi[\top/x]$ *is true or* $\phi[\bot/x]$ *is true*.)
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The evaluation problem of QBF generalizes both the *satisfiability* and *validity/tautology problems* of propositional logic.

The latter are respectively NP-complete and co-NP-complete whereas the former is PSPACE-complete.

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The formulae $\forall x \exists y (x \leftrightarrow y)$ and $\exists x \exists y (x \wedge y)$ are true.

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Special cases corresponding to SAT and TAUT:
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