## **Foundations of AI**

## 14. Statistical Machine Learning

Bayesian Learning and Why Learning Works
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## **Statistical Learning Methods**

- In MDPs probability and utility theory allow agents to deal with uncertainty.
- To apply these techniques, however, the agents must first learn their probabilistic theories of the world from experience.
- We will discuss statistical learning methods as robust ways to learn probabilistic models.

#### **Contents**

- Statistical learning
- Why learning works

10/2

#### **An Example for Statistical Learning**

- The key concepts are data (evidence) and hypotheses.
- A candy manufacturer sells five kinds of bags that are indistinguishable from the outside:

h₁: 100% cherry

h<sub>2</sub>: 75% cherry and 25% lime

h<sub>3</sub>: 50% cherry and 50% lime

h<sub>4</sub>: 25% cherry and 75% lime

h<sub>5</sub>: 100% lime

• Given a sequence d<sub>1</sub>, ..., d<sub>N</sub> of candies observed, what is the most likely flavor of the next piece of candy?

#### **Bayesian Learning**

- Calculates the probability of each hypothesis, given the data.
- It then makes predictions using all hypotheses weighted by their probabilities (instead of a single best hypothesis).
- Learning is reduced to probabilistic inference.

10/5

#### **How to Make Predictions?**

 Suppose we want to make predictions about an unknown quantity X given the data d.

$$P(X \mid \mathbf{d}) = \sum_{i} P(X \mid h_i, \mathbf{d}) P(h_i \mid \mathbf{d})$$
$$= \sum_{i} P(X \mid h_i) P(h_i \mid \mathbf{d})$$

- Predictions are weighted averages over the predictions of the individual hypotheses.
- The key quantities are the hypothesis prior  $P(h_i)$  and the likelihood  $P(d/h_i)$  of the data under each hypothesis.

#### **Application of Bayes Rule**

- Let **D** represent all the data with observed value **d**.
- The probability of each hypothesis is obtained by Bayes rule:

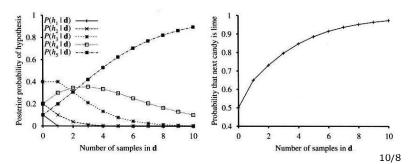
$$P(h_i \mid \mathbf{d}) = \alpha P(\mathbf{d} \mid h_i) P(h_i)$$

- The manufacturer tells us that the prior distribution over h<sub>1</sub>, ..., h<sub>5</sub> is given by
   <.1, .2, .4, .2, .1>
- We compute the likelihood of the data under the assumption that the observations are independently and identically distributed (i.i.d.):

$$P(\mathbf{d} \mid h_i) = \prod_j P(d_j \mid h_i)$$

#### **Example**

- Suppose the bag is an all-lime bag (h<sub>5</sub>)
- The first 10 candies are all lime.
- Then  $P(d/h_3)$  is  $0.5^{10}$  because half the candies in an  $h_3$  bag are lime.
- Evolution of the five hypotheses given 10 lime candies were observed (the values start at the prior!).



#### **Observations**

- The true hypothesis often dominates the Bayesian prediction.
- For any fixed prior that does not rule out the true hypothesis, the posterior of any false hypothesis will eventually vanish.
- The Bayesian prediction is optimal and, given the hypothesis prior, any other prediction will be correct less often.
- It comes at a price that the hypothesis space can be very large or infinite.

10/9

## **Maximum-Likelihood Hypothesis (ML)**

- A final simplification is to assume a uniform prior over the hypothesis space.
- In that case MAP-learning reduces to choosing the hypothesis that maximizes  $P(d/h_i)$ .
- This hypothesis is called the maximumlikelihood hypothesis (ML).
- ML-learning is a good approximation to MAP learning and Bayesian learning when there is a uniform prior and when the data set is large.

### Maximum a Posteriori (MAP)

- A common approximation is to make predictions based on a single most probable hypothesis.
- The maximum a posteriori (MAP) hypothesis is the one that maximizes  $P(h_i|d)$ .

$$\mathbf{P}(X \mid \mathbf{d}) \approx \mathbf{P}(X \mid h_{MAP})$$

- In the candy example,  $h_{MAP} = h_5$  after three lime candies in a row.
- The MAP learner the predicts that the fourth candy is lime with probability 1.0, whereas the Bayesian prediction is still 0.8.
- As more data arrive, MAP and Bayesian predictions become closer.
- Finding MAP hypotheses is often much easier than Bayesian learning.

10/10

## **Why Learning Works**

How can we decide that h is close to f when f is unknown?

→ Probably approximately correct

Stationarity as the basic assumption of PAC-Learning: training and test sets are selected from the same population of examples with the same probability distribution.

Key question: how many examples do we need?

*X* Set of examples

D Distribution from which the examples are drawn

*H* Hypothesis space  $(f \in H)$ 

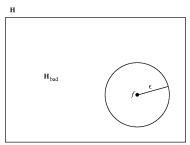
*m* Number of examples in the training set

$$error(h) = P(h(x) \neq f(x) \mid x \text{ drawn from D}) < \in$$

#### **PAC-Learning**

A hypothesis h is approximately correct if  $error(h) \leq \epsilon$ .

To show: After the training period with m examples, with high probability, all consistent hypotheses are approximately correct.



How high is the probability that a wrong hypothesis  $h_b \in H_{bad}$  is consistent with the first m examples?

10/13

## Sample Complexity (2)

Example: Boolean functions

The number of Boolean functions over n attributes is  $|H| = 2^{2^n}$ .

The sample complexity therefore grows as  $2^n$ .

Since the number of possible examples is also  $2^n$ , any learning algorithm for the space of all Boolean functions will do no better than a lookup table, if it merely returns a hypothesis that is consistent with all known examples.

## **Sample Complexity**

Assumption:  $error(h_h) > \in . \Rightarrow$ 

 $P(h_b \text{ is consistent with 1 example}) \leq (1 - \epsilon)$ 

 $P(h_h \text{ is consistent with } N \text{ examples}) \leq (1 - \epsilon)^N$ 

 $P(H_{bad} \text{ contains a consistent } h) \leq |H_{bad}| (1 - \epsilon)^{N}$ 

Since  $|H_{bad}| \leq |H|$ 

 $P(H_{had} \text{ contains a consistent } h) \leq |H| (1 - \epsilon)^N$ 

We want to limit this probability by some small number  $\delta$ :

$$|H| (1 - \epsilon)^N < \delta$$

Since  $(1 - \epsilon) \le e^{-\epsilon}$ , we derive

$$N \ge 1/\epsilon \left( \ln(1/\delta) + \ln|H| \right)$$

Sample Complexity: Number of required examples, as a function of  $\varepsilon$  and  $\delta.$ 

#### **Learning from Decision Lists**

In comparison to decision trees:

- The overall structure is simpler
- The individual tests are more complex



This represents the hypothesis

 $H_4: \forall x \, Will Wait(x) \Leftrightarrow Patrons(x, some) \vee [Patrons(x, full) \wedge Fri/Sat(x)]$ 

If we allow tests of arbitrary size, then any Boolean function can be represented.

k-DL: Language with tests of length  $\leq k$ .



## **Learnability of k-DL**

function DECISION-LIST-LEARNING(examples) returns a decision list, No or failure

if examples is empty then return the value No

 $t \leftarrow a$  test that matches a nonempty subset examples, of examples

such that the members of examples, are all positive or all negative

if there is no such t then return failure

 $\textbf{if the examples in } \textit{examples}_{t} \textit{ are positive } \textbf{then } o \leftarrow \textit{Yes}$ 

else  $o \leftarrow No$ 

return a decision list with initial test t and outcome o

and remaining elements given by DECISION-LIST-LEARNING(examples - examples,)

$$| \text{ k-DL}(n) | \le 3^{|Conj(n,k)|} | Conj(n,k)! |$$
 (Yes,No,no-Test,all permutations)  $| Conj(n,k) | = \sum_{i=0}^k {2n \choose i} = O(n^k)$ 

(Combination without repeating pos/neg attributes)

$$\mid$$
 k-DL $(n)\mid =2^{O(n^{k}log(n^{k}))}$  (with Euler's summation formula)

 $m \ge \frac{1}{\epsilon} (ln(\frac{1}{\delta}) + O(n^k log(n^k)))$ 

10/17

# **Summary** (Statistical Learning Theory)

Inductive learning as learning the representation of a function from example input/output pairs.

- Decision trees learn deterministic Boolean functions.
- PAC learning deals with the complexity of learning.
- Decision lists as functions that are easy to learn.

## **Summary** (Statistical Learning Methods)

- Bayesian learning techniques formulate learning as a form of probabilistic inference.
- Maximum a posteriori (MAP) learning selects the most likely hypothesis given the data.
- Maximum likelihood learning selects the hypothesis that maximizes the likelihood of the data.

10/18