

# Foundations of AI

## 9. Predicate Logic

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Syntax and Semantics, Normal Forms,  
Herbrand Expansion, Resolution

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## Motivation

We can already do a lot with propositional logic. It is, however, annoying that there is no structure in the atomic propositions.

Example:

“All blocks are red”

“There is a block A”

It should follow that “A is red”

But propositional logic cannot handle this.

Idea: We introduce individual variables, predicates, functions, ... .

→ First-Order Predicate Logic (PL1)

# The Alphabet of First-Order Predicate Logic

## Symbols:

- Operators:  $\neg, \vee, \wedge, \forall, \exists, =$
- Variables:  $x, x_1, x_2, \dots, x', x'', \dots, y, \dots, z, \dots$
- Brackets:  $() , [] , \{ \}$
- Function symbols (e.g.,  $weight(), color()$ )
- Predicate symbols (e.g.,  $block(), red()$ )
- Predicate and function symbols have an arity (number of arguments).
  - 0-ary predicate: propositional logic atoms
  - 0-ary function: constant
- We suppose a countable set of predicates and functions of any arity.
- “=” is usually not considered a predicate, but a logical symbol

# The Grammar of First-Order Predicate Logic (1)

Terms (represent objects):

1. Every variable is a term.
2. If  $t_1, t_2, \dots, t_n$  are terms and  $f$  is an  $n$ -ary function, then

$$f(t_1, t_2, \dots, t_n)$$

is also a term.

Terms without variables: **ground terms**.

Atomic Formulae (represent statements about objects)

1. If  $t_1, t_2, \dots, t_n$  are terms and  $P$  is an  $n$ -ary predicate, then  $P(t_1, t_2, \dots, t_n)$  is an atomic formula.
2. If  $t_1$  and  $t_2$  are terms, then  $t_1 = t_2$  is an atomic formula.

Atomic formulae without variables: **ground atoms**. 09/5

# The Grammar of First-Order Predicate Logic (2)

Formulae:

1. Every atomic formula is a formula.
2. If  $\varphi$  and  $\psi$  are formulae and  $x$  is a variable, then

$\neg\varphi, \varphi \wedge \psi, \varphi \vee \psi, \varphi \Rightarrow \psi, \varphi \Leftrightarrow \psi, \exists x\varphi$  and  $\forall x\varphi$   
are also formulae.

$\forall, \exists$  are as strongly binding as  $\neg$ .

Propositional logic is part of the PL1 language:

1. Atomic formulae: only 0-ary predicates
2. Neither variables nor quantifiers.

# Alternative Notation

Here	Elsewhere
$\neg\varphi$	$\sim\varphi \quad \bar{\varphi}$
$\varphi \wedge \psi$	$\varphi \& \psi \quad \varphi \bullet \psi \quad \varphi, \psi$
$\varphi \vee \psi$	$\varphi   \psi \quad \varphi ; \psi \quad \varphi + \psi$
$\varphi \Rightarrow \psi$	$\varphi \rightarrow \psi \quad \varphi \supset \psi$
$\varphi \Leftrightarrow \psi$	$\varphi \leftrightarrow \psi \quad \varphi \equiv \psi$
$\forall x \varphi$	$(\forall x) \varphi \quad \wedge x \varphi$
$\exists x \varphi$	$(\exists x) \varphi \quad \vee x \varphi$

# Meaning of PL1-Formulae

Our example:  $\forall x [Block(x) \Rightarrow Red(x)], Block(a)$

For all objects  $x$  : If  $x$  is a block, then  $x$  is red and  $a$  is a block.

Generally:

- Terms are interpreted as objects.
- Universally-quantified variables denote all objects in the universe.
- Existentially-quantified variables represent one of the objects in the universe (made true by the quantified expression).
- Predicates represent subsets of the universe.

Similar to propositional logic, we define **interpretations**, **satisfiability**, **models**, **validity**, ...



# Semantics of PL1-Logic

**Interpretation:**  $I = \langle D, \bullet^I \rangle$  where  $D$  is an arbitrary, non-empty set and  $\bullet^I$  is a function that

- maps  $n$ -ary function symbols to functions over  $D$  :

$$f^I \in [D^n \rightarrow D]$$

- maps individual constants to elements of  $D$  :

$$a^I \in D$$

- maps  $n$ -ary predicate symbols to relations over  $D$  :

$$P^I \subseteq D^n$$

**Interpretation** of ground terms:

$$(f(t_1, \dots, t_n))^I = f^I(t_1^I, \dots, t_n^I)$$

**Satisfaction** of ground atoms  $P(t_1, \dots, t_n)$ :

$$I \models P(t_1, \dots, t_n) \text{ iff } \langle t_1^I, \dots, t_n^I \rangle \in P^I$$

## Example (1)

$$D = \{d_1, \dots, d_n \mid n > 1\}$$

$$a^I = d_1$$

$$b^I = d_2$$

$$c^I = \dots$$

$$\text{Block}^I = \{d_1\}$$

$$\text{Red}^I = D$$

$$I \models \text{Red}(b)$$

$$I \not\models \text{Block}(b)$$

## Example (2)

$$D = \{1, 2, 3, \dots\}$$

$$1^I = 1$$

$$2^I = 2$$

...

$$Even^I = \{2, 4, 6, \dots\}$$

$$succ^I = \{(1 \rightarrow 2), (2 \rightarrow 3), \dots\}$$

$$I \models Even(2)$$

$$I \not\models Even(succ(2))$$

# Semantics of PL1: Variable Assignment

Set of all variables  $V$ . Function  $\alpha: V \rightarrow D$

Notation:  $\alpha[x/d]$  is the same as  $\alpha$  apart from point  $x$ .

For  $x$  :  $\alpha[x/d](x) = d$ .

Interpretation of terms under  $I, \alpha$ :

$$\begin{aligned}x^{I, \alpha} &= \alpha(x) \\ a^{I, \alpha} &= a^I \\ (f(t_1, \dots, t_n))^{I, \alpha} &= f^I(t_1^{I, \alpha}, \dots, t_n^{I, \alpha})\end{aligned}$$

Satisfaction of atomic formulae:

$$I, \alpha \models P(t_1, \dots, t_n) \text{ iff } \langle t_1^{I, \alpha}, \dots, t_n^{I, \alpha} \rangle \in P^I$$

## Example

$$\begin{aligned}\alpha &= \{(x \rightarrow d_1), (y \rightarrow d_2)\} \\ I, \alpha &\models Red(x) \\ I, \alpha[y/d_1] &\models Block(y)\end{aligned}$$

# Semantics of PL1: Satisfiability

A formula  $\varphi$  is satisfied by an interpretation  $I$  and a variable assignment  $\alpha$ , i.e.,  $I, \alpha \models \varphi$ :

$$I, \alpha \models \top$$

$$I, \alpha \not\models \perp$$

$$I, \alpha \models \neg\varphi \text{ iff } I, \alpha \not\models \varphi$$

...

and all other propositional rules as well as

$$I, \alpha \models P(t_1, \dots, t_n) \quad \text{iff} \quad \langle t_1^{I, \alpha}, \dots, t_n^{I, \alpha} \rangle \in P^{I, \alpha}$$

$$I, \alpha \models \forall x\varphi \quad \text{iff} \quad \text{for all } d \in D, I, \alpha[x/d] \models \varphi$$

$$I, \alpha \models \exists x\varphi \quad \text{iff} \quad \text{there exists a } d \in D \text{ with } I, \alpha[x/d] \models \varphi$$

## Example

$$T = \{Block(a), Block(b), \forall x(Block(x) \Rightarrow Red(x))\}$$

$$D = \{d_1, \dots, d_n \mid n > 1\}$$

$$a^I = d_1$$

$$b^I = d_2$$

$$Block^I = \{d_1\}$$

$$Red^I = D$$

$$\alpha = \{(x \rightarrow d_1), (y \rightarrow d_2)\}$$

### Questions:

1.  $I, \alpha \models Block(b) \vee \neg Block(b)$ ?
2.  $I, \alpha \models Block(x) \Rightarrow (Block(x) \vee \neg Block(y))$ ?
3.  $I, \alpha \models Block(a) \wedge Block(b)$ ?
4.  $I, \alpha \models \forall x(Block(x) \Rightarrow Red(x))$ ?
5.  $I, \alpha \models \top$ ?

# Free and Bound Variables

$$\forall x [R(\boxed{y}, \boxed{z}) \wedge \exists y \{ \neg P(y, x) \vee R(y, \boxed{z}) \}]$$

The boxed appearances of  $y$  and  $z$  are **free**. All other appearances of  $x, y, z$  are **bound**.

Formulae with no free variables are called **closed** formulae or **sentences**. We form theories from closed formulae.

**Note:** With closed formulae, the concepts *logical equivalence, satisfiability, and implication, etc.* are not dependent on the variable assignment  $\alpha$  (i.e., we can always ignore all variable assignments).

With closed formulae,  $\alpha$  can be left out on the left side of the model relationship symbol:

$$I \models \varphi$$



# Terminology

An interpretation  $I$  is called a **model** of  $\varphi$  under  $\alpha$  if

$$I, \alpha \models \varphi$$

A PL1 formula  $\varphi$  can, as in propositional logic, be **satisfiable**, **unsatisfiable**, **falsifiable**, or **valid**.

Analogously, two formulae are **logically equivalent** ( $\varphi \equiv \psi$ ) if for all  $I, \alpha$  :

$$I, \alpha \models \varphi \text{ iff } I, \alpha \models \psi$$

**Note:**  $P(x) \not\equiv P(y)$ !

**Logical Implication** is also analogous to propositional logic.

Question: How can we define **derivation**?

# Prenex Normal Form

Because of the quantifiers, we cannot produce the CNF form of a formula directly.

First step: Produce the prenex normal form

quantifier prefix + (quantifier-free) matrix

$$Qx_1 Qx_2 Qx_3 \dots Qx_n \varphi$$

# Equivalences for the Production of Prenex Normal Form

$$\begin{aligned}(\forall x\varphi) \wedge \psi &\equiv \forall x(\varphi \wedge \psi) \text{ if } x \text{ not free in } \psi \\(\forall x\varphi) \vee \psi &\equiv \forall x(\varphi \vee \psi) \text{ if } x \text{ not free in } \psi \\(\exists x\varphi) \wedge \psi &\equiv \exists x(\varphi \wedge \psi) \text{ if } x \text{ not free in } \psi \\(\exists x\varphi) \vee \psi &\equiv \exists x(\varphi \vee \psi) \text{ if } x \text{ not free in } \psi \\ \forall x\varphi \wedge \forall x\psi &\equiv \forall x(\varphi \wedge \psi) \\ \exists x\varphi \vee \exists x\psi &\equiv \exists x(\varphi \vee \psi)\end{aligned}$$

$$\begin{aligned}\neg\forall x\varphi &\equiv \exists x\neg\varphi \\ \neg\exists x\varphi &\equiv \forall x\neg\varphi\end{aligned}$$

... and propositional logic equivalents

# Production of Prenex Normal Form

1. Eliminate  $\Rightarrow$  and  $\Leftrightarrow$
2. Move  $\neg$  inwards
3. Move quantifiers outwards

Example:

$$\begin{aligned} & \neg \forall x [(\forall x P(x)) \Rightarrow Q(x)] \\ \rightarrow & \neg \forall x [\neg(\forall x P(x)) \vee Q(x)] \\ \rightarrow & \exists x [(\forall x P(x)) \wedge \neg Q(x)] \end{aligned}$$

And now?

## Renaming of Variables

$\varphi[\frac{x}{t}]$  is obtained from  $\varphi$  by replacing all free appearances of  $x$  in  $\varphi$  by  $t$ .

**Lemma:** Let  $y$  be a variable that does not appear in  $\varphi$ . Then it holds that

$$\forall x\varphi \equiv \forall y\varphi[\frac{x}{y}] \quad \text{and} \quad \exists x\varphi \equiv \exists y\varphi[\frac{x}{y}]$$

**Theorem:** There exists an algorithm that calculates the prenex normal form of any formula.

# Derivation in PL1

Why is prenex normal form useful?

Unfortunately, there is no simple law as in propositional logic that allows us to determine satisfiability or general validity (by transformation into DNF or CNF).

But: We can reduce the satisfiability problem in predicate logic to the satisfiability problem in propositional logic. In general, however, this produces a very large number of propositional formulae (perhaps infinitely many)

Then: Apply resolution.

# Skolemization

Idea: **Elimination of existential quantifiers** by applying a function that produces the “right” element.

Theorem (**Skolem Normal Form**): Let  $\varphi$  be a closed formula in prenex normal form such that all quantified variables are pair-wise distinct and the function symbols  $g_1, g_2, \dots$  do not appear in  $\varphi$ . Let

$$\varphi = \forall x_1 \dots \forall x_i \exists y \psi,$$

then  $\varphi$  is satisfiable iff

$$\varphi' = \forall x_1 \dots \forall x_i \psi \left[ \frac{y}{g_i(x_1, \dots, x_i)} \right]$$

is satisfiable.

Example:  $\forall x \exists y [P(x) \Rightarrow Q(y)] \rightarrow \forall x [P(x) \Rightarrow Q(g(x))]$

# Skolem Normal Form

Skolem Normal Form: Prenex normal form without existential quantifiers. Notation:  $\varphi^*$  is the SNF of  $\varphi$ .

Theorem: It is possible to calculate the skolem normal form of every closed formula  $\varphi$ .

Example:  $\exists x((\forall x P(x)) \wedge \neg Q(x))$  develops as follows:

$$\exists y((\forall x P(x)) \wedge \neg Q(y))$$

$$\exists y(\forall x(P(x) \wedge \neg Q(y)))$$

$$\forall x(P(x) \wedge \neg Q(g_0))$$

Note: This transformation is **not an equivalence transformation**; it **only preserves satisfiability!**

Note: ... and is **not unique**.



# Ground Terms & Herbrand Expansion

The set of **ground terms** (or **Herbrand Universe**) over a set of SNF formulae  $\theta^*$  is the (infinite) set of all ground terms formed from the symbols of  $\theta^*$  (in case there is no constant symbol, one is added). This set is denoted by  $D(\theta^*)$ .

The **Herbrand expansion**  $E(\theta^*)$  is the instantiation of the Matrix  $\psi_i$  of all formulae in  $\theta^*$  through all terms  $t \in D(\theta^*)$ :

$$E(\theta^*) = \{ \psi_i[\frac{x_1}{t_1}, \dots, \frac{x_n}{t_n}] \mid (\forall x_1, \dots, x_n \psi_i) \in \theta^*, t_j \in D(\theta^*) \}$$

**Theorem (Herbrand):** Let  $\theta^*$  be a set of formulae in SNF. Then  $\theta^*$  is satisfiable iff  $E(\theta^*)$  is satisfiable.

**Note:** If  $D(\theta^*)$  and  $\theta^*$  are finite, then the Herbrand expansion is finite  $\rightarrow$  finite propositional logic theory.

**Note:** This is used heavily in AI and works well most of the time!

# Infinite Propositional Logic Theories

Can a finite proof exist when the set is infinite?

Theorem (compactness of propositional logic): A (countable) set of formulae of propositional logic is satisfiable if and only if every finite subset is satisfiable.

Corollary: A (countable) set of formulae in propositional logic is unsatisfiable if and only if a finite subset is unsatisfiable.

Corollary: (compactness of PL1): A (countable) set of formulae in predicate logic is satisfiable if and only if every finite subset is satisfiable.

# Recursive Enumeration and Decidability

We can construct a **semi-decision procedure** for **validity**, i.e., we can give a (rather inefficient) algorithm that *enumerates* all valid formulae step by step.

Theorem: The set of **valid** (and **unsatisfiable**) formulae in PL1 is **recursively enumerable**.

What about *satisfiable* formulae?

Theorem (**undecidability of PL1**): It is **undecidable**, whether a formula of PL1 is **valid**.

(Proof by reduction from PCP)

Corollary: The set of **satisfiable formulae** in PL1 is **not recursively enumerable**.

In other words: If a formula is valid, we can effectively confirm this fact. Otherwise, we can end up in an infinite loop.

## Derivation in PL1

Clausal Form instead of Herbrand Expansion.  
Clauses are universally quantified disjunctions of literals; all variables are universally quantified

$$(\forall x_1, \dots, x_n)(l_1 \vee \dots \vee l_n)$$

$$l_1 \vee \dots \vee l_n$$

$$\{l_1, \dots, l_n\}$$

written as

or

# Production of Clausal Form from SNF

Skolem Normal Form

quantifier prefix + (quantifier-free) matrix

$$\forall x_1 \forall x_2 \forall x_3 \dots \forall x_n \varphi$$

1. Put Matrix into CNF using distribution rule
2. Eliminate universal quantifiers
3. Eliminate conjunction symbol
4. Rename variables so that no variable appears in more than one clause.

Theorem: It is possible to calculate the clausal form of every closed formula  $\varphi$ .

Note: Same remarks as for SNF

# Conversion to CNF (1)

Everyone who loves all animals is loved by someone:

$$\forall x[\forall y \text{Animal}(y) \Rightarrow \text{Loves}(x, y)] \Rightarrow [\exists y \text{Loves}(y, x)]$$

1. Eliminate biconditionals and implications

$$\forall x \neg[\forall y \neg \text{Animal}(y) \vee \text{Loves}(x, y)] \vee [\exists y \text{Loves}(y, x)]$$

2. Move  $\neg$  inwards:  $\neg \forall x p \equiv \exists x \neg p$  ,  $\neg \exists x p \equiv \forall x \neg p$

$$\forall x [\exists y \neg(\neg \text{Animal}(y) \vee \text{Loves}(x, y))] \vee [\exists y \text{Loves}(y, x)]$$

$$\forall x [\exists y \neg \neg \text{Animal}(y) \wedge \neg \text{Loves}(x, y)] \vee [\exists y \text{Loves}(y, x)]$$

$$\forall x [\exists y \text{Animal}(y) \wedge \neg \text{Loves}(x, y)] \vee [\exists y \text{Loves}(y, x)]$$

## Conversion to CNF (2)

3. Standardize variables: each quantifier should use a different one

$$\forall x[\exists y \textit{Animal}(y) \wedge \neg \textit{Loves}(x, y)] \vee [\exists z \textit{Loves}(z, x)]$$

4. Skolemize: a more general form of existential instantiation. Each existential variable is replaced by a Skolem function of the enclosing universally quantified variables:

$$\forall x[\textit{Animal}(F(x)) \wedge \neg \textit{Loves}(x, F(x))] \vee [\textit{Loves}(G(x), x)]$$

5. Drop universal quantifiers:

$$[\textit{Animal}(F(x)) \wedge \neg \textit{Loves}(x, F(x))] \vee [\textit{Loves}(G(x), x)]$$

6. Distribute  $\wedge$  over  $\vee$  :

$$[\textit{Animal}(F(x)) \vee \textit{Loves}(G(x), x)] \wedge [\neg \textit{Loves}(x, F(x)) \vee \textit{Loves}(G(x), x)]$$

# Clauses and Resolution

Assumption: All formulae in the KB are clauses.

Equivalently, we can assume that the KB is a *set of clauses*.

Due to commutativity, associativity, and idempotence of  $\vee$ , clauses can also be understood as *sets of literals*. The *empty set of literals* is denoted by  $\square$ .

Set of clauses:  $\Delta$

Set of literals: C, D

Literal:  $l$

Negation of a literal:  $\neg l$



# Propositional Resolution

$$\frac{C_1 \cup \{l\}, C_2 \cup \{\bar{l}\}}{C_1 \cup C_2}$$

$C_1 \cup C_2$  are called **resolvents** of the **parent clauses**  $C_1 \cup \{l\}$  and  $C_1 \cup \{\bar{l}\}$ .  $l$  and  $\bar{l}$  are the **resolution literals**.

**Example:**  $\{a, b, \neg c\}$  resolves with  $\{a, d, c\}$  to  $\{a, b, d\}$ .

**Note:** The resolvent is not equivalent to the parent clauses, but it follows from them!

**Notation:**  $R(\Delta) = \Delta \cup \{C \mid C \text{ is a resolvent of two clauses from } \Delta\}$

# What Changes?

## Examples

$$\{\{Nat(s(0)), \neg Nat(0)\}, \{Nat(0)\}\} \vdash \{Nat(s(0))\}$$

$$\{\{Nat(s(0)), \neg Nat(x)\}, \{Nat(0)\}\} \vdash \{Nat(s(0))\}$$

$$\{\{Nat(s(x)), \neg Nat(x)\}, \{Nat(0)\}\} \vdash \{Nat(s(0))\}$$

We need **unification**, a way to make literals identical.  
Based on the notion of **substitution**, e.g.,  $\{\frac{x}{0}\}$ .

# Substitutions

A substitution  $s = \left\{ \frac{v_1}{t_1}, \dots, \frac{v_n}{t_n} \right\}$  substitutes variables  $v_i$  for terms  $t_i$  ( $t_i$  does NOT contain  $v_i$ ).

Applying a substitution  $s$  to an expression  $\varphi$  yields the expression  $\varphi s$  which is  $\varphi$  with all occurrences of  $v_i$  replaced by  $t_i$  for all  $i$ .

## Substitution Examples

$$P(x, f(y), B)$$

$$P(z, f(w), B) \quad s = \left\{ \frac{x}{z}, \frac{y}{w} \right\}$$

$$P(x, f(A), B) \quad s = \left\{ \frac{y}{A} \right\}$$

$$P(g(z), f(A), B) \quad s = \left\{ \frac{x}{g(z)}, \frac{y}{A} \right\}$$

$$P(C, f(A), A)$$

## Composing Substitutions

Composing substitutions  $s_1$  and  $s_2$  gives  $s_1s_2$  which is that substitution obtained by first applying  $s_2$  to the terms in  $s_1$  and adding remaining term/variable pairs (**not occurring** in  $s_1$ ) to  $s_1$ .

$$\text{Example: } \left\{ \frac{z}{g(x,y)} \right\} \left\{ \frac{x}{A}, \frac{y}{B}, \frac{w}{C}, \frac{z}{D} \right\} = \left\{ \frac{z}{g(A,B)}, \frac{x}{A}, \frac{y}{B}, \frac{w}{C} \right\}$$

Application example:  $P(x, y, z) \rightarrow P(A, B, g(A, B))$

# Properties of substitutions

For a formula  $\varphi$  and substitutions  $s_1, s_2$

$$(\varphi s_1) s_2 = \varphi(s_1 s_2)$$

$$(s_1 s_2) s_3 = s_1(s_2 s_3) \quad \text{associativity}$$

$$s_1 s_2 \neq s_2 s_1 \quad \text{no commutativity!}$$

# Unification

Unifying a set of expressions  $\{w_i\}$

Find substitution  $s$  such that  $w_i s = w_j s$  for all  $i, j$

Example

$$\{P(x, f(y), B), P(x, f(B), B)\}$$

$$s = \left\{ \frac{y}{B}, \frac{z}{A} \right\}$$

not the simplest unifier

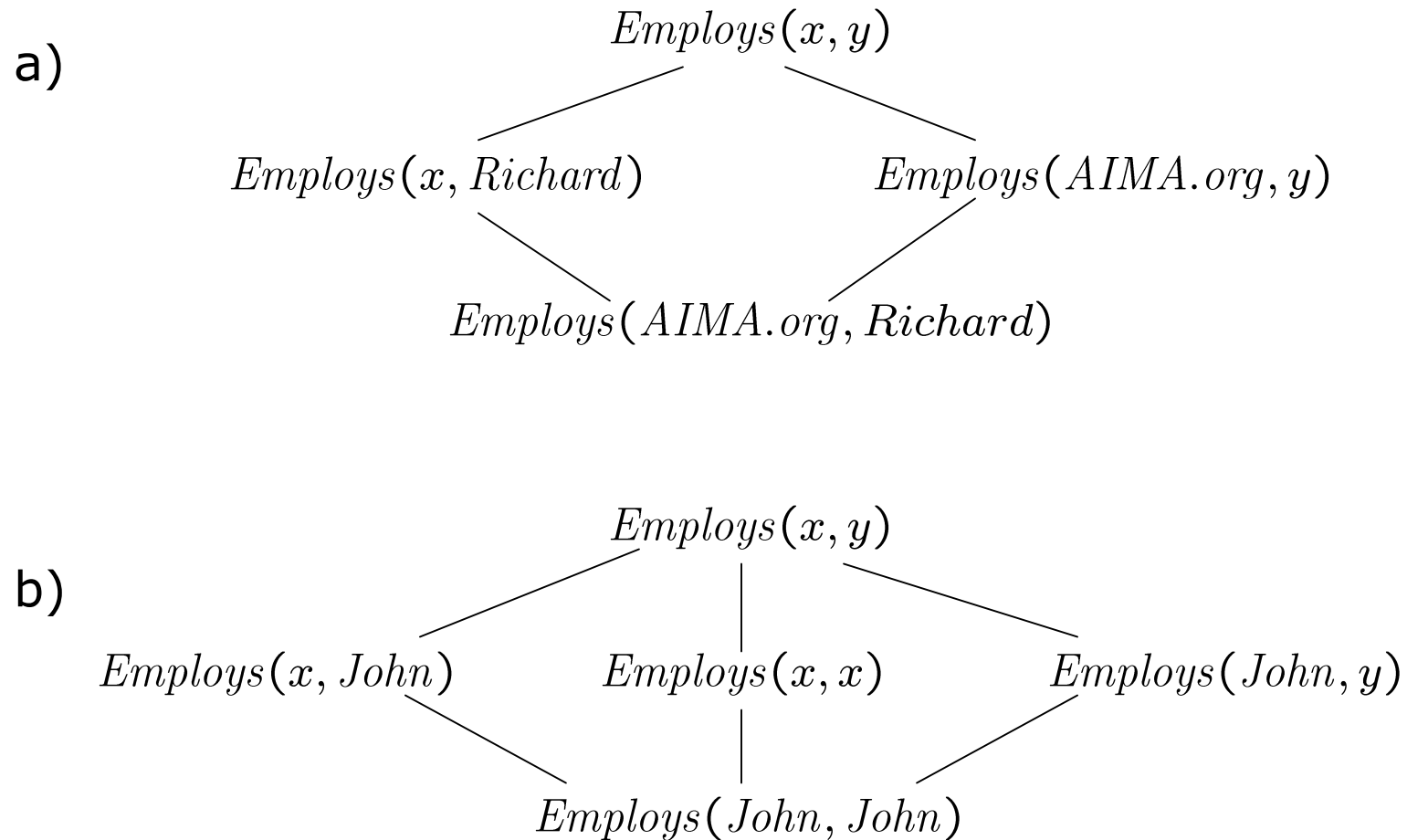
$$s = \left\{ \frac{y}{B} \right\}$$

most general unifier (mgu)

The **most general unifier**, the **mgu**,  $g$  of  $\{w_i\}$  has the property that if  $s$  is any unifier of  $\{w_i\}$  then there exists a substitution  $s'$  such that  $\{w_i\} s = \{w_i\} g s'$

**Property:** The common instance produced is unique up to **alphabetic variants** (variable renaming)

# Subsumption Lattice





# Disagreement Set

The disagreement set of a set of expressions  $\{w_i\}$  is the set of sub-terms  $\{t_i\}$  of  $\{w_i\}$  at the first position in  $\{w_i\}$  for which the  $\{w_i\}$  disagree

## Examples

$\{P(x, A, f(y)), P(v, B, z)\}$  gives  $\{x, v\}$

$\{P(x, A, f(y)), P(x, B, z)\}$  gives  $\{A, B\}$

$\{P(x, y, f(y)), P(x, B, z)\}$  gives  $\{y, B\}$

# Unification Algorithm

Unify(Terms):

Initialize  $k \leftarrow 0$  ;

Initialize  $T_k = \text{Terms}$ ;

Initialize  $s_k = \{\}$  ;

\*If  $T_k$  is a singleton, then output  $s_k$  . Otherwise continue.

Let  $D_k$  be the disagreement set of  $T_k$  .

If there exists a var  $v_k$  and a term  $t_k$  in  $D_k$  such that  $v_k$  does not occur in  $t_k$  , continue. Otherwise, exit with failure.

$s_{k+1} \leftarrow s_k \left\{ \frac{v_k}{t_k} \right\}$  ;

$T_{k+1} \leftarrow T_k \left\{ \frac{v_k}{t_k} \right\}$  ;

$k \leftarrow k + 1$  ;

Goto \*.

## Example

$$\{P(x, f(y), y), P(z, f(B), B)\}$$

# Binary Resolution

$$\frac{C_1 \cup \{l_1\}, C_2 \cup \{\bar{l}_2\}}{[C_1 \cup C_2]_s}$$

where  $s = \text{mgu}(l_1, l_2)$ , the most general unifier  
 $[C_1 \cup C_2]_s$  is the **resolvent** of the **parent clauses**  $C_1 \cup \{l_1\}$  and  $C_2 \cup \{\bar{l}_2\}$ .

$C_1 \cup \{l_1\}$  and  $C_2 \cup \{\bar{l}_2\}$  do not share variables  
 $l_1$  and  $l_2$  are the **resolution literals**.

**Examples:**  $\{\{Nat(s(0)), \neg Nat(0)\}, \{Nat(0)\}\} \vdash \{Nat(s(0))\}$   
 $\{\{Nat(s(0)), \neg Nat(x)\}, \{Nat(0)\}\} \vdash \{Nat(s(0))\}$   
 $\{\{Nat(s(x)), \neg Nat(x)\}, \{Nat(0)\}\} \vdash \{Nat(s(0))\}$

## Some Further Examples

Resolve  $P(x) \vee Q(f(x))$  and  $R(g(x)) \vee \neg Q(f(A))$

Standardizing the variables apart gives  $P(x) \vee Q(f(x))$   
and  $R(g(y)) \vee \neg Q(f(A))$

Substitution  $s = \{\frac{x}{A}\}$       Resolvent  $P(A) \vee R(g(y))$

Resolve  $P(x) \vee Q(x, y)$  and  $\neg P(A) \vee \neg R(B, z)$

Standardizing the variables apart

Substitution  $s = \{\frac{x}{A}\}$  and Resolvent  $Q(A, y) \vee \neg R(B, z)$

# Factoring

$$\frac{C_1 \cup \{l_1\} \cup \{l_2\}}{[C \cup l_1]_s}$$

where  $s = \text{mgu}(l_1, l_2)$  is the most general unifier.

Needed because:

$$\{\{P(u), P(v)\}, \{\neg P(x), \neg P(y)\}\} \models \square$$

but  $\square$  cannot be derived by binary resolution

Factoring yields:

$\{P(u)\}$  and  $\{\neg P(x)\}$  whose resolvent is  $\square$ .

# Derivations

Notation:  $R(\Delta) = \Delta \cup \{C \mid C \text{ is a resolvent or a factor of two clauses from } \Delta\}$

We say  $D$  can be derived from  $\Delta$ , i.e.,

$$\Delta \vdash D,$$

If there exist  $C_1, C_2, C_3, \dots, C_n = D$  such that

$C_i \in R(\Delta \cup \{C_1, \dots, C_{i-1}\})$  for  $1 \leq i \leq n$ .

## Example

From Russell and Norvig :

The law says it is a crime for an American to sell weapons to hostile nations. The country Nono, an enemy of America, has some missiles, and all of its missiles were sold to it by Colonel West, who is American.

Prove that Col. West is a criminal.



## Example

... it is a crime for an American to sell weapons to hostile nations:

$American(x) \wedge weapon(y) \wedge Sells(x, y, z) \wedge Hostile(z) \Rightarrow Criminal(x)$

Nono ... has some missiles, i.e.,  $\exists x Owns(Nono, x) \wedge Missile(x)$  :

$Owns(Nono, M_1)$  and  $Missile(M_1)$

... all of its missiles were sold to it by Colonel West.

$\forall x Missiles(x) \wedge Owns(Nono, x) \Rightarrow Sells(West, x, Nono)$

Missiles are weapons:

$Missile(x) \Rightarrow Weapon(x)$

An enemy of America counts as "hostile":

$Enemy(x, America) \Rightarrow Hostile(x)$

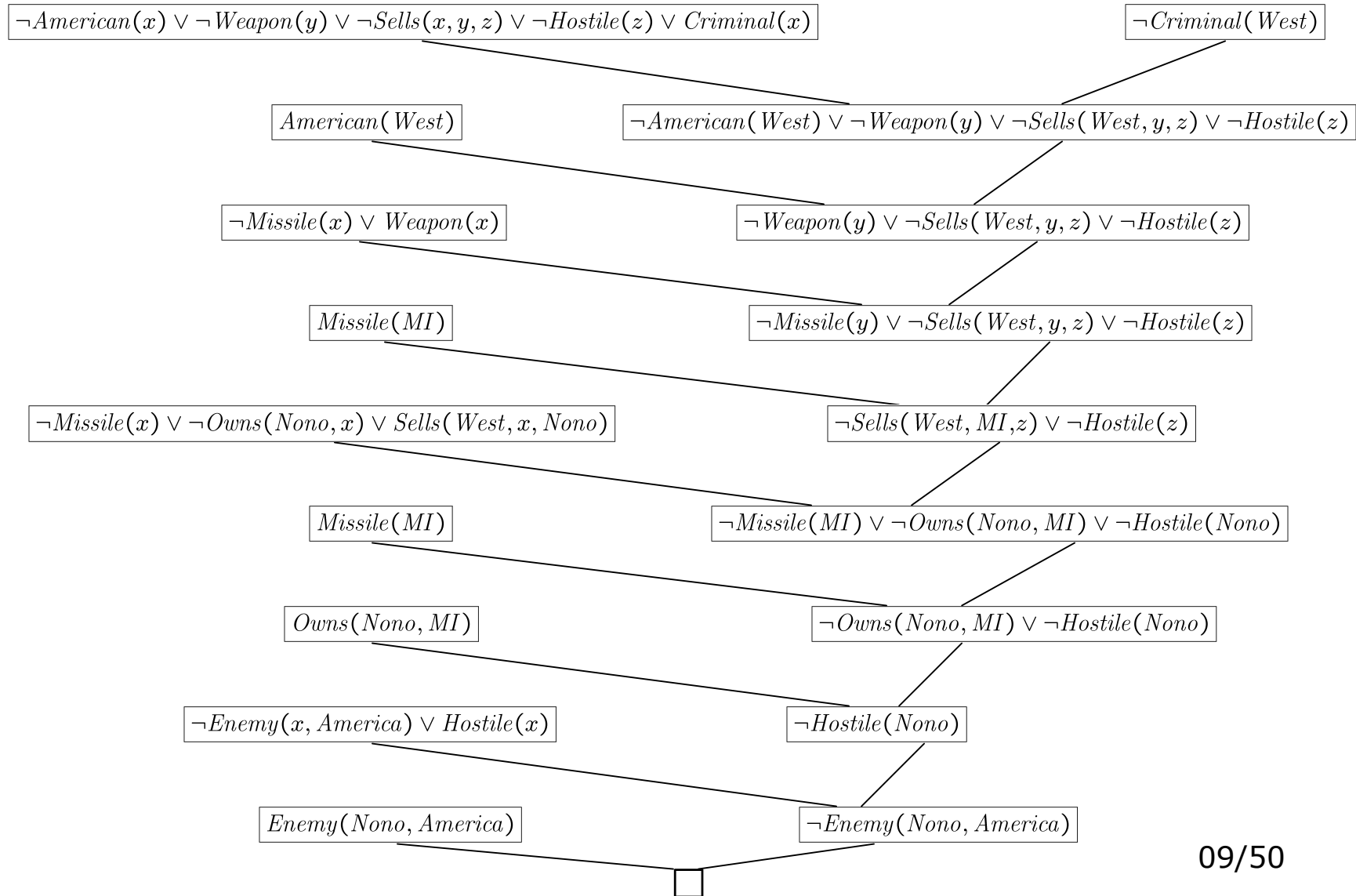
West, who is American ...

$American(West)$

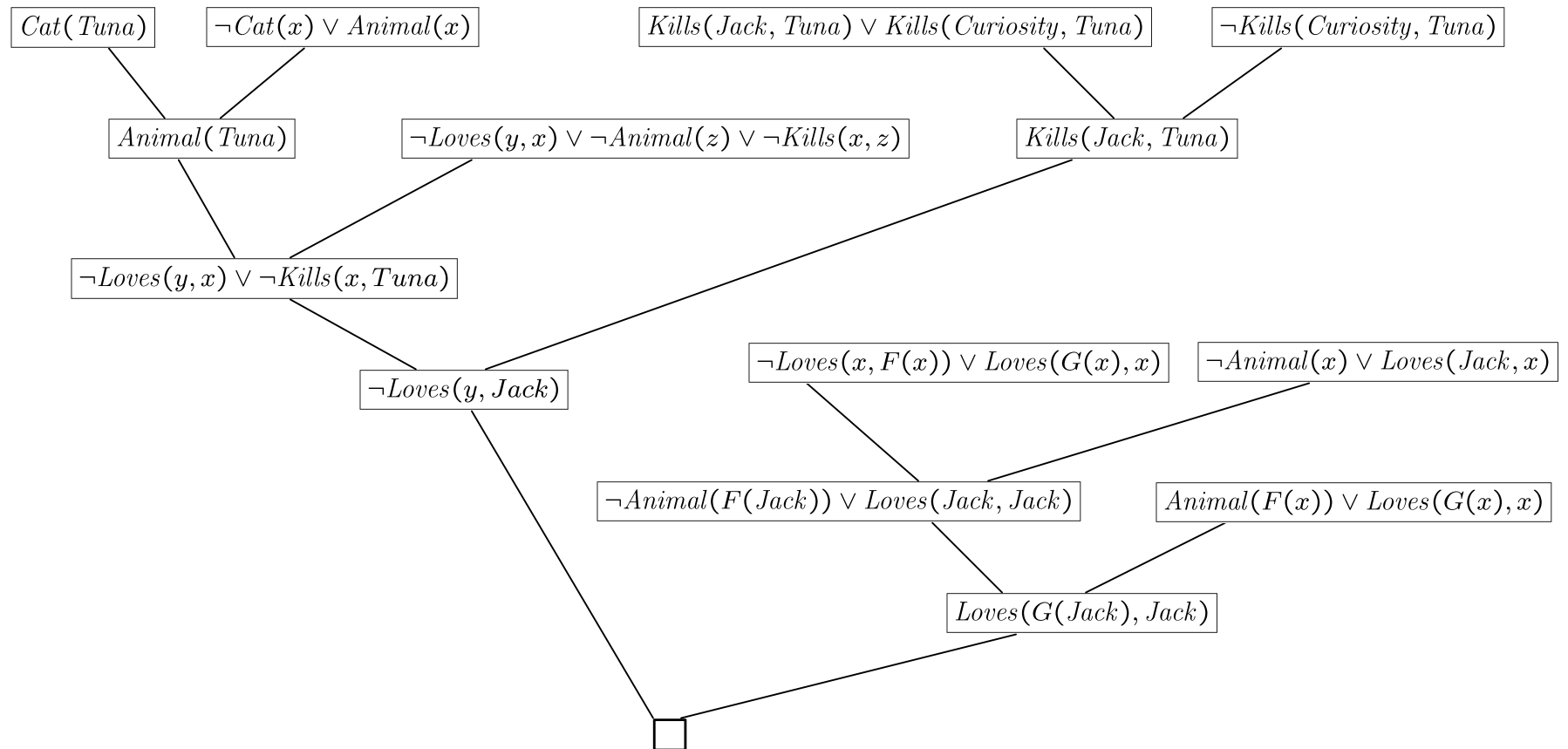
The country Nono, an enemy of America

$Enemy(Nono, America)$

# An Example



# Another Example



# Properties of Resolution

**Lemma:** (soundness) If  $\Delta \vdash D$  , then  $\Delta \models D$  .

**Lemma:** resolution is refutation-complete:

$\Delta$  is unsatisfiable implies  $\Delta \vdash \square$  .

**Theorem:**  $\Delta$  is unsatisfiable iff  $\Delta \vdash \square$  .

**Technique:**

to prove that  $\Delta \models C$

negate  $C$  and prove that  $\Delta \cup \{\neg C\} \vdash \square$  .

# The Lifting Lemma

## Lemma:

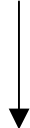
Let  $C_1$  and  $C_2$  be two clauses with no shared variables, and let  $C_1'$  and  $C_2'$  be ground instances of  $C_1$  and  $C_2$ . If  $C'$  is a resolvent of  $C_1'$  and  $C_2'$ , then there exists a clause  $C$  such that

- (1)  $C$  is a resolvent of  $C_1$  and  $C_2$
- (2)  $C'$  is a ground instance of  $C$

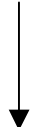
Can be easily generalized to derivations

# The general picture

Any set of sentences  $S$  is representable in clausal form

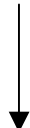


Assume  $S$  is unsatisfiable, and in clausal form



← Herbrand's theorem

Some set  $S'$  of ground instances is unsatisfiable



← Ground resolution theorem

Resolution can find a contradiction in  $S'$



← Lifting lemma

There is a resolution proof for the contradiction in  $S$

# Closing Remarks: Processing

- **PL1-Resolution**: forms the basis of
  - most state of the art theorem provers for PL1
  - the programming language **Prolog**
    - only Horn clauses
    - considerably more efficient methods.
  - not dealt with : search/resolution strategies
- **Finite theories**: In applications, we often have to deal with a fixed set of objects. **Domain closure axiom**:
  - $\forall x[x = c_1 \vee x = c_2 \vee \dots \vee x = c_n]$
  - Translation into finite propositional theory is possible.

## Closing Remarks: Possible Extensions

- PL1 is definitely very expressive, but in some circumstances we would like more ...
- **Second-Order Logic:** Also over predicate quantifiers
  - $\forall x, y[(x = y) \Leftrightarrow \{\forall p[p(x) \Leftrightarrow p(y)]\}]$
- Validity is no longer semi-decidable (we have lost compactness)
- **Lambda Calculus:** Definition of predicates, e.g.,  $\lambda x, y[\exists z P(x, z) \wedge Q(z, y)]$  defines a new predicate of arity 2
- Reducible to PL1 through Lambda-Reduction
- **Uniqueness quantifier:**  $\exists! x \varphi(x)$  – there is exactly one  $x$  ...
- Reduction to PL1:
  - $\exists x[\varphi(x) \wedge \forall y\{\varphi(y) \Rightarrow x = y\}]$



# Summary

- PL1 makes it possible to structure statements, thereby giving us considerably **more expressive power than propositional logic**.
- Formulae consist of **terms** and **atomic formulae**, which, together with **connectors** and **quantifiers**, can be put together to produce formulae.
- Interpretations in PL1 consist of a **universe** and an **interpretation function**.
- The **Herbrand Theory** shows that satisfiability in PL1 can be reduced to satisfiability in propositional logic (although infinite sets of formulae can arise under certain circumstances).
- **Resolution** is **refutation complete**
- **Validity** in PL1 is **not decidable** (it is only semi-decidable)