Foundations of AI

7. Propositional Logic

Rational Thinking, Logic, Resolution

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Contents

- Agents that think rationally
- The wumpus world
- Propositional logic: syntax and semantics
- Logical entailment
- Logical derivation (resolution)

Agents that Think Rationally

- Until now, the focus has been on agents that act rationally.
- Often, however, rational action requires rational (logical) thought on the agent's part.
- To that purpose, portions of the world must be represented in a knowledge base, or KB.
  - A KB is composed of sentences in a language with a truth theory (logic), i.e. we (being external) can interpret sentences as statements about the world. (semantics)
  - Through their form, the sentences themselves have a causal influence on the agent's behaviour in a way that is correlated with the contents of the sentences. (syntax)
- Interaction with the KB through ASK and TELL (simplified):

ASK(KB,α) = yes	exactly when α follows from the KB
TELL(KB,α) = KB′	so that α follows from KB′
FORGET(KB,α) = KB′	non-monotonic (will not be discussed)

3 Levels

In the context of knowledge representation, we can distinguish three levels [Newell 1990]:

Knowledge level: Most abstract level. Concerns the total knowledge contained in the KB. For example, the automated DB information system knows that a trip from Freiburg to Basel costs 18€.

Logical level: Encoding of knowledge in a formal language. *Price(Freiburg, Basel, 18.00)*

Implementation level: The internal representation of the sentences, for example:

- As a string "`Price(Freiburg, Basel, 18.00)`"
- As a value in a matrix

When ASK and TELL are working correctly, it is possible to remain on the knowledge level. Advantage: very comfortable user interface. The user has his/her own mental model of the world (statements about the world) and communicates it to the agent (TELL).

A Knowledge-Based Agent

A knowledge-based agent uses its knowledge base to

- represent its background knowledge
- store its observations
- store its executed actions
- … derive actions

```
function KB-AGENT(percept) returns an action
   static: KB, a knowledge base
           t, a counter, initially 0, indicating time

   TELL(KB, MAKE-PERCEPT-SENTENCE(percept, t))
   action ← ASK(KB, MAKE-ACTION-QUERY(t))
   TELL(KB, MAKE-ACTION-SENTENCE(action, t))
   t ← t + 1
   return action
```

The Wumpus World (1)

- A 4 x 4 grid
- In the square containing the wumpus and in the directly adjacent squares, the agent perceives a stench.
- In the squares adjacent to a pit, the agent perceives a breeze.
- In the square where the gold is, the agent perceives a glitter.
- When the agent walks into a wall, it perceives a bump.
- When the wumpus is killed, its scream is heard everywhere.
- Percepts are represented as a 5-tuple, e.g.,

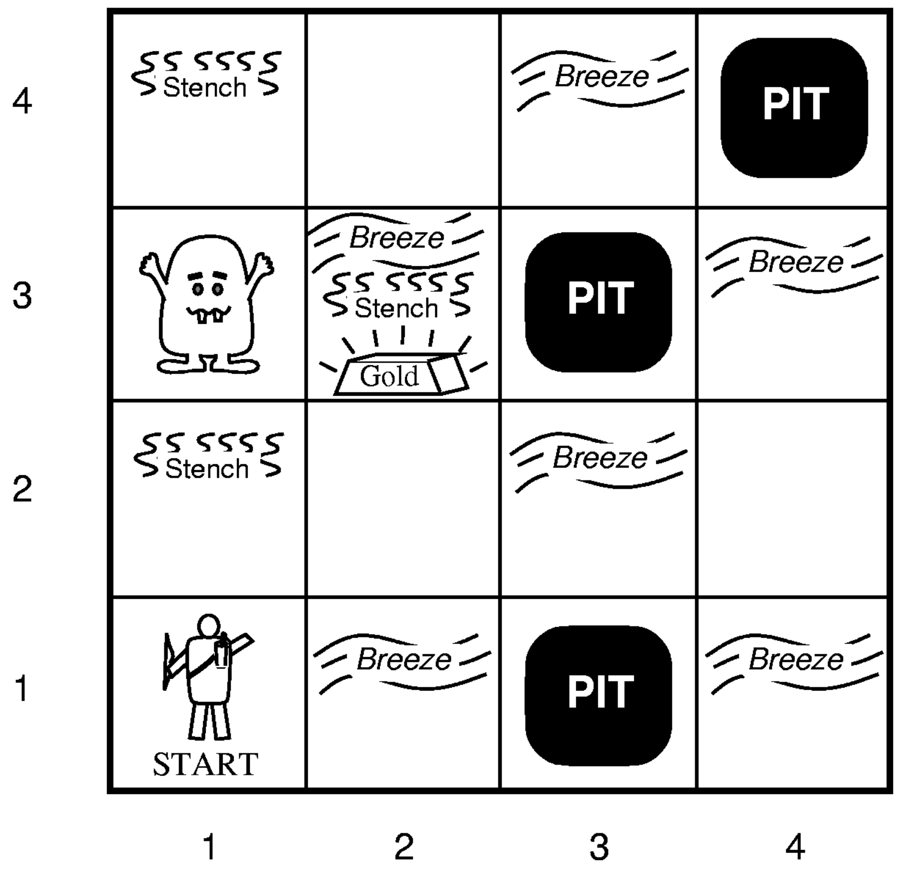
  [Stench, Breeze, Glitter, None, None]

  means that it stinks, there is a breeze and a glitter, but no bump and no scream. The agent *cannot* perceive its own location!

The Wumpus World (2)

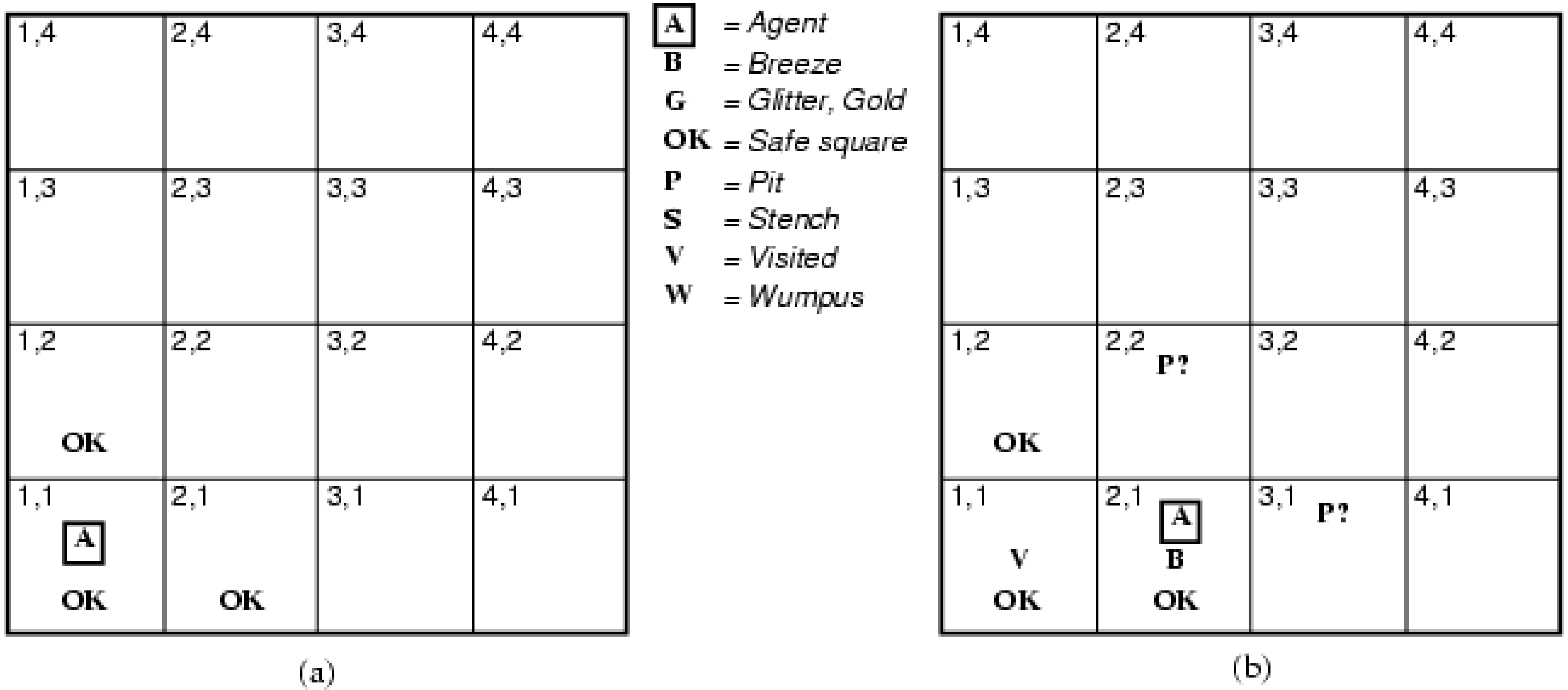
- Actions: Go forward, turn right by 90°, turn left by 90°, pick up an object in the same square (grab), shoot (there is only one arrow), leave the cave (only works in square [1,1]).
- The agent dies if it falls down a pit or meets a live wumpus.
- Initial situation: The agent is in square [1,1] facing east. Somewhere exists a wumpus, a pile of gold and 3 pits.
- Goal: Find the gold and leave the cave.

The Wumpus World (3):
A Sample Configuration



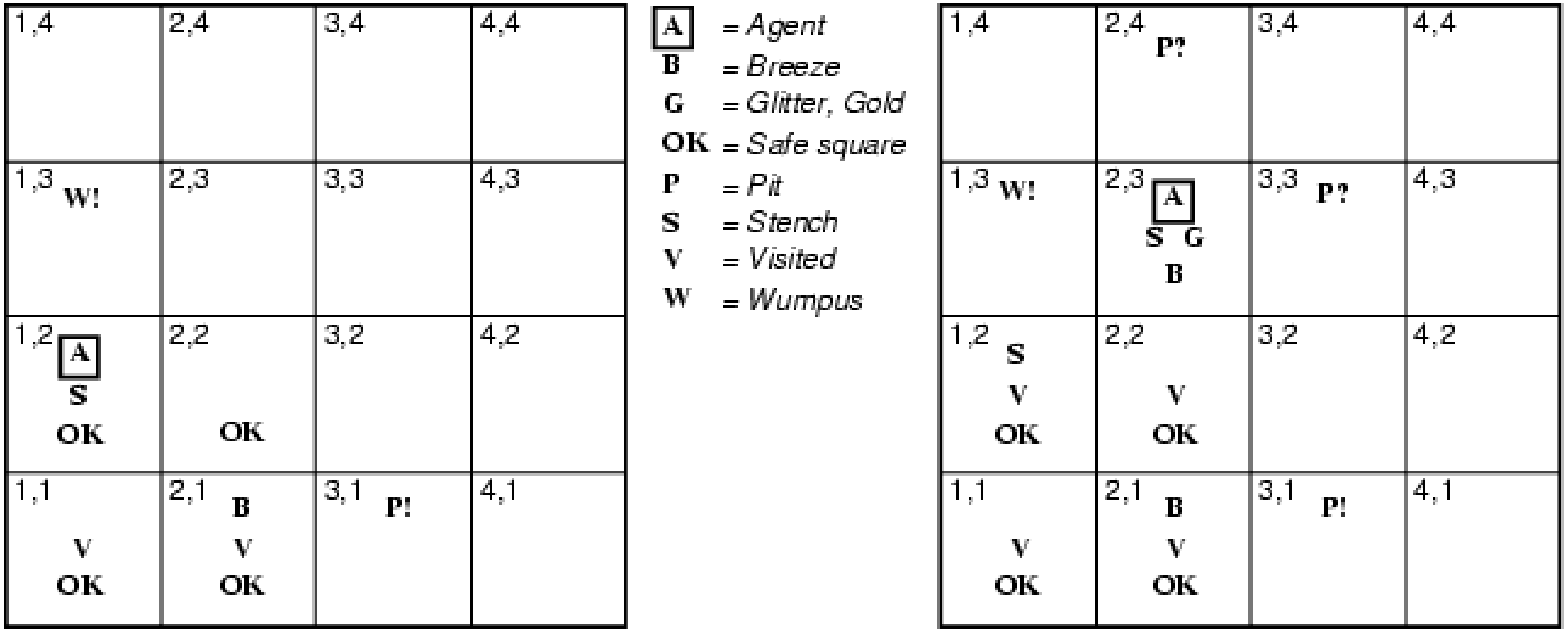
The Wumpus World (4)

[1,2] and [2,1] are safe:



The Wumpus World (5)

The wumpus is in [1,3]!



(a) (b)

Declarative Languages

Before a system that is capable of learning, thinking, planning, explaining, … can be built, one must find a way to express knowledge.

We need a precise, declarative language.

- Declarative: System believes P iff it considers P to be true (one cannot believe P without an idea of what it means for the world to fulfill P).
- Precise: We must know,
  - which symbols represent sentences,
  - what it means for a sentence to be true, and
  - when a sentence follows from other sentences.

One possibility: Propositional Logic

Basics of Propositional Logic (1)

Propositions: The building blocks of propositional logic are indivisible, atomic statements (atomic propositions), e.g.,

- "The block is red"
- "The wumpus is in [1,3]"

and the logical connectives "and", "or" and "not", which we can use to build formulae.

Basics of Propositional Logic (2)

We are interested in knowing the following:

- When is a proposition true?
- When does a proposition follow from a knowledge base (KB)?
- Symbolically: $KB \models \varphi$
- Can we (syntactically) define the concept of *derivation*,
- Symbolically: $KB \vdash \varphi$
  such that it is equivalent to the concept of logical implication conclusion?

→ Meaning and implementation of ASK

Syntax of Propositional Logic

Countable alphabet Σ of atomic propositions: *P, Q, R*, ...

Logical formulae:

$P \in \Sigma$	atomic formula
$\bot$	falseness
$\top$	truth
$\neg\varphi$	negation
$\varphi \wedge \psi$	conjunction
$\varphi \vee \psi$	disjunction
$\varphi \Rightarrow \psi$	implication
$\varphi \Leftrightarrow \psi$	equivalence

Operator precedence: $\neg > \wedge > \vee > \Rightarrow = \Leftrightarrow$. (use brackets when necessary)

Atom: atomic formula

Literal: (possibly negated) atomic formula

Clause: disjunction of literals

Semantics: Intuition

Atomic propositions can be true (T) or false (F).

The truth of a formula follows from the truth of its atomic propositions (truth assignment or interpretation) and the connectives.

Example:

$$(P \vee Q) \wedge R$$

- If P and Q are *false* and R is *true*, the formula is *false*
- If P and R are *true*, the formula is *true* regardless of what Q is.

Semantics: Formally

A truth assignment of the atoms in Σ, or an interpretation over Σ, is a function

$$I : \Sigma \to \{T, F\}$$

Interpretation $I(\varphi)$ or φ^I of a formula φ :

$$
\begin{array}{lll}
I \models \top & & \\
I \not\models \bot & & \\
I \models P & \text{iff} & P^I = T \\
I \not\models \neg\varphi & \text{iff} & I \models \varphi \\
I \models \varphi \wedge \psi & \text{iff} & I \models \varphi \text{ and } I \models \psi \\
I \models \varphi \vee \psi & \text{iff} & I \models \varphi \text{ or } I \models \psi \\
I \models \varphi \Rightarrow \psi & \text{iff} & \text{if } I \models \varphi\text{, then } I \models \psi \\
I \models \varphi \Leftrightarrow \psi & \text{iff} & \text{if } I \models \varphi \text{ if and only if } I \models \psi
\end{array}
$$

I satisfies φ ($I \models \varphi$) or φ is true under I, when $I(\varphi) = T$.

Example

$$I : \begin{cases} P \rightarrow T \\ Q \rightarrow T \\ R \rightarrow F \\ S \rightarrow T \\ \ldots \end{cases}$$

$$\varphi = ((\mathsf{P} \vee \mathsf{Q}) \Leftrightarrow (\mathsf{R} \vee \mathsf{S})) \wedge (\neg(\mathsf{P} \wedge \mathsf{Q}) \wedge (\mathsf{R} \wedge \neg \mathsf{S})).$$

Question: $I \models \varphi$?

Terminology

An interpretation I is called a model of φ if $I \models \varphi$.

An interpretation is a model of a set of formulae if it fulfils all formulae of the set.

A formula φ is

- satisfiable if there exists I that satisfies φ,
- unsatisfiable if φ is not satisfiable,
- falsifiable if there exists I that doesn't satisfy φ, and
- valid (a tautology) if $I \models \varphi$ holds for all I.

Two formulae are

- logically equivalent ($\varphi \equiv \psi$) if $I \models \varphi$ iff $I \models \psi$ holds for all I.

The Truth Table Method

How can we decide if a formula is satisfiable, valid, etc.?

→Generate a truth table

Example: Is $\varphi = ((P \lor H) \land \neg H) \Rightarrow P$ valid?

P	H	$P \lor H$	$(P \lor H) \land \neg H$	$(P \lor H) \land \neg H \Rightarrow P$
F	F	F	F	T
F	T	T	F	T
T	F	T	T	T
T	T	T	F	T

Since the formula is true for all possible combinations of truth values (satisfied under all interpretations), φ is valid.

Satisfiability, falsifiability, unsatisfiability likewise.

Normal Forms

- A formula is in **conjunctive normal form** (CNF) if it consists of a conjunction of disjunctions of literals $l_{i,j}$, i.e., if it has the following form: $\bigwedge_{i=1}^{n} (\bigvee_{j=1}^{m_i} l_{i,j})$
- A formula is in **disjunctive normal form** (DNF) if it consists of a disjunction of conjunctions of literals: $\bigvee_{i=1}^{n} (\bigwedge_{j=1}^{m_i} l_{i,j})$
- For every formula, there exists at least one equivalent formula in CNF and one in DNF.
- A formula in DNF is satisfiable iff one disjunct is satisfiable.
- A formula in CNF is valid iff every conjunct is valid.

Producing CNF

1. Eliminate $\Rightarrow$ and $\Leftrightarrow$: $\alpha \Rightarrow \beta \rightarrow (\neg\alpha \lor \beta)$ etc.
2. Move $\neg$ inwards: $\neg(\alpha \land \beta) \rightarrow (\neg\alpha \lor \neg\beta)$ etc.
3. Distribute $\lor$ over $\land$: $((\alpha \land \beta) \lor \gamma) \rightarrow ((\alpha \lor \gamma) \land (\beta \lor \gamma))$
4. Simplify: $\alpha \lor \alpha \rightarrow \alpha$ etc.

The result is a conjunction of disjunctions of literals

An analogous process converts any formula to an equivalent formula in DNF.

- During conversion, formulae can expand *exponentially*.
- Note: Conversion to CNF formula can be done *polynomially* if only satisfiability should be preserved

Logical Implication: Intuition

A set of formulae (a KB) usually provides an incomplete description of the world, i.e., leaves the truth values of a proposition open.

Example: $\mathrm{KB} = \{P \lor Q,\ R \lor \neg P,\ S\}$

is definitive with respect to S, but leaves P, Q, R open (although they cannot take on arbitrary values).

Models of the KB:

P	Q	R	S
F	T	F	T
F	T	T	T
T	F	T	T
T	T	T	T

In all models of the KB, $Q \lor R$ is true, i.e., $Q \lor R$ follows logically from KB.

Logical Implication: Formal

The formula φ follows logically from the KB if φ is true in all models of the KB (symbolically $KB \models \varphi$):

$$KB \models \varphi \text{ iff } I \models \varphi \text{ for all models I of KB}$$

Note: The $\models$ symbol is a *meta-symbol*

Some properties of logical implication relationships:

- Deduction theorem: KB $\cup \{\varphi\} \models \psi$ iff KB $\models \varphi \Rightarrow \psi$
- Contraposition theorem: KB $\cup \{\varphi\} \models \neg\psi$ iff KB $\cup \{\psi\} \models \neg\varphi$
- Contradiction theorem: KB $\cup \{\varphi\}$ is unsatisfiable iff KB $\models \neg\varphi$

Question: Can we determine KB $\models \varphi$ without considering all interpretations (the truth table method)?

Proof of the Deduction Theorem

"$\Rightarrow$" Assumption: $KB \cup \{\varphi\} \models \psi$, i.e., every model of $KB \cup \{\varphi\}$ is also a model of ψ.

Let I be any model of KB. If I is also a model of φ, then it follows that I is also a model of ψ.

This means that I is also a model of $\varphi \Rightarrow \psi$, i.e., $KB \models \varphi \Rightarrow \psi$.

"$\Leftarrow$" Assumption: $KB \models \varphi \Rightarrow \psi$. Let I be any model of KB that is also a model of φ, i.e., $I \models KB \cup \{\varphi\}$.

From the assumption, I is also a model of $\varphi \Rightarrow \psi$ and thereby also of ψ, i.e., $KB \cup \{\varphi\} \models \psi$.

Proof of the Contraposition Theorem

$$KB \cup \{\varphi\} \models \neg\psi$$

$$\text{iff } KB \models \varphi \Rightarrow \neg\psi \qquad (1)$$

$$\text{iff } KB \models (\neg\varphi \vee \neg\psi)$$

$$\text{iff } KB \models (\neg\psi \vee \neg\varphi)$$

$$\text{iff } KB \models \psi \Rightarrow \neg\varphi$$

$$\text{iff } KB \cup \{\psi\} \models \neg\varphi \qquad (2)$$

Note:

(1) and (2) are applications of the deduction theorem.

Inference Rules, Calculi and Proofs

We can often derive new formulae from formulae in the KB. These new formulae should follow logically from the syntactical structure of the KB formulae.

Example: If the KB is $\{\ldots, (\varphi \Rightarrow \psi), \ldots, \varphi, \ldots\}$, then ψ is a logical consequence of KB

→ Inference rules, e.g., $\frac{\varphi, \varphi \Rightarrow \psi}{\psi}$

Calculus: Set of inference rules (potentially including so-called logical axioms)

Proof step: Application of an inference rule on a set of formulae.

Proof: Sequence of proof steps where every newly-derived formula is added, and in the last step, the goal formula is produced.

Soundness and Completeness

In the case where in the calculus C there is a proof for a formula φ, we write

$$KB \vdash_C \varphi$$

(optionally without subscript C).

A calculus C is sound (or correct) if all formulae that are derivable from a KB actually follow logically.

$$KB \vdash_C \varphi \text{ implies } \mathrm{KB} \models \varphi$$

This normally follows from the soundness of the inference rules and the logical axioms.

A calculus is complete if every formula that follows logically from the KB is also derivable with C from the KB:

$$KB \models \varphi \text{ implies } \mathrm{KB} \vdash_C \varphi$$

Resolution: Idea

We want a way to derive new formulae that does not depend on testing every interpretation.

Idea: We attempt to show that a set of formulae is unsatisfiable.

Condition: All formulae must be in CNF.

But: In most cases, the formulae are close to CNF (and there exists a fast satisfiability-preserving transformation – Theoretical Computer Science course).

Nevertheless: In the worst case, this derivation process requires an exponential amount of time (this is, however, probably unavoidable).

Resolution: Representation

Assumption: All formulae in the KB are in CNF.

Equivalently, we can assume that the KB is a *set of clauses*.

Due to commutativity, associativity, and idempotence of $\vee$, clauses can also be understood as sets of literals. The empty set of literals is denoted by $\square$.

Set of clauses: Δ

Set of literals: C, D

Literal: l

Negation of a literal: $\bar{l}$

An interpretation I satisfies *C* iff there exists $l \in C$ such that $I \models l$. I satisfies Δ if for all $C \in \Delta : I \models C$, i.e., $I \not\models \square, I \not\models \{\square\}$, $I \models \{\}$, for all I.

The Resolution Rule

$$\frac{C_1 \cup \{l\}, C_2 \cup \{\bar{l}\}}{C_1 \cup C_2}$$

$C_1 \cup C_2$ are called resolvents of the parent clauses $C_1 \cup \{l\}$ and $C_1 \cup \{\bar{l}\}$. l and $\bar{l}$ are the resolution literals.

Example: $\{a,b,\neg c\}$ resolves with $\{a,d,c\}$ to $\{a,b,d\}$.

Note: The resolvent is not equivalent to the parent clauses, but it follows from them!

Notation: $R(\Delta) = \Delta \cup \{C \mid C \text{ is a resolvent of two clauses from } \Delta\}$

Derivations

We say *D* can be derived from Δ using resolution, i.e.,

$$\Delta \vdash D,$$

if there exist $C_1, C_2, C_3, \ldots, C_n = D$ such that

$$C_i \in R(\Delta \cup \{C_1, \ldots, C_{i-1}\}), \text{ for } 1 \leq i \leq n.$$

Lemma (soundness) If $\Delta \vdash D$, *then* $\Delta \models D$.

Proof idea: Since all $D \in R(\Delta)$ *follow logically from* Δ, the lemma results through induction over the length of the derivation.

Completeness?

Is resolution also complete? I.e. is

$$\Delta \models \varphi \text{ implies } \Delta \vdash \varphi$$

valid? Only for clauses. Consider:

$$\{\{a,b\}, \{\neg b,c\}\} \models \{a,b,c\} \not\vdash \{a,b,c\}$$

But it can be shown that resolution is refutation-complete:

$$\Delta \text{ is unsatisfiable implies } \Delta \vdash \square,$$

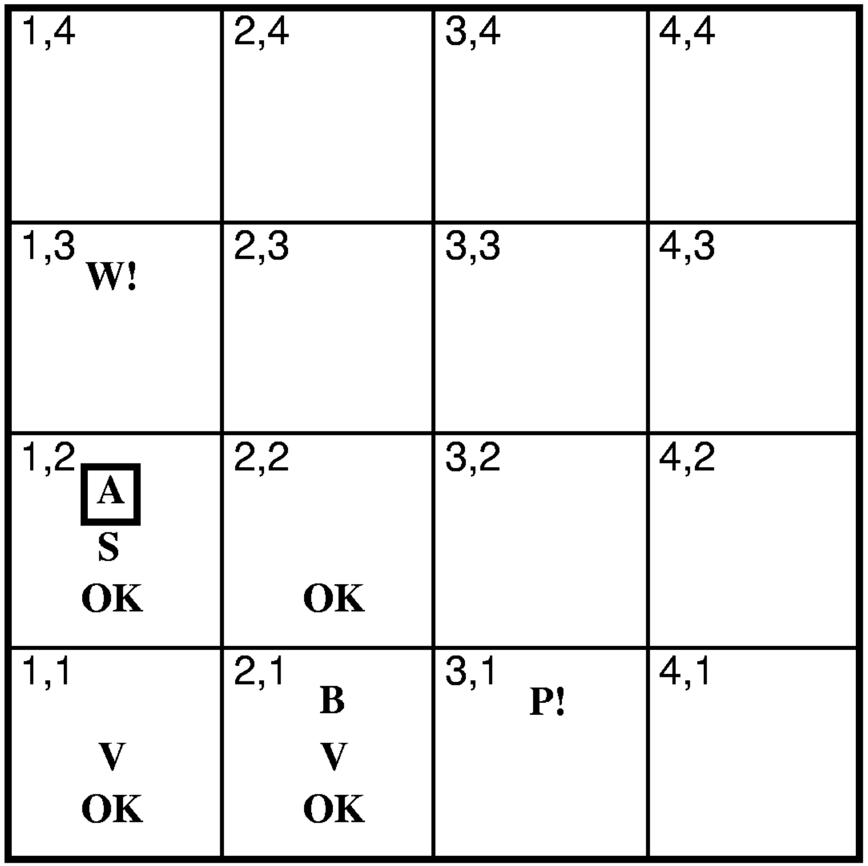
Theorem: Δ is unsatisfiable iff $\Delta \vdash \square$,

With the help of the contradiction theorem, we can show that $KB \models \varphi$.

Resolution: Overview

- Resolution is a refutation-complete proof process. There are others (Davis-Putnam Procedure, Tableaux Procedure, …).
- In order to implement the process, a strategy must be developed to determine which resolution steps will be executed and when.
- In the worst case, a resolution proof can take exponential time. This, however, very probably holds for all other proof procedures.
- For CNF formulae in propositional logic, the Davis-Putnam Procedure (backtracking over all truth values) is probably (in practice) the fastest complete process that can also be taken as a type of resolution process.

Where is the Wumpus?
The Situation



- **A** = *Agent*
- **B** = *Breeze*
- **G** = *Glitter, Gold*
- **OK** = *Safe square*
- **P** = *Pit*
- **S** = *Stench*
- **V** = *Visited*
- **W** = *Wumpus*

Where is the Wumpus? Knowledge of the Situation

B = Breeze, S = Stench, $B_{i,j}$ = there is a breeze in (i,j)

$\neg S_{1,1}$	$\neg B_{1,1}$
$\neg S_{2,1}$	$B_{2,1}$
$S_{1,2}$	$\neg B_{1,2}$

Knowledge about the wumpus and smell:

R_1: $\neg S_{1,1} \Rightarrow \neg W_{1,1} \wedge \neg W_{1,2} \wedge \neg W_{2,1}$

R_2: $\neg S_{2,1} \Rightarrow \neg W_{1,1} \wedge \neg W_{2,1} \wedge \neg W_{2,2} \wedge \neg W_{3,1}$

R_3: $\neg S_{1,2} \Rightarrow \neg W_{1,1} \wedge \neg W_{1,2} \wedge \neg W_{2,2} \wedge \neg W_{1,3}$

R_4: $S_{1,2} \Rightarrow W_{1,3} \vee W_{1,2} \vee W_{2,2} \vee W_{1,1} \ldots$

To show: KB $\models W_{1,3}$

Clausal Representation of the Wumpus World

Situational knowledge:

$\neg S_{1,1}$, $\neg S_{2,1}$, $\neg S_{1,2}$, ...

Knowledge of rules:

Knowledge about the wumpus and smell:

R_1: $S_{1,1} \vee \neg W_{1,1}$, $S_{1,1} \vee \neg W_{1,2}$, $S_{1,1} \vee \neg W_{2,1}$

R_2: ..., $S_{2,1} \vee \neg W_{2,2}$, ...

R_3: ...

R_4: $\neg S_{1,2} \vee W_{1,3} \vee W_{1,2} \vee W_{2,2} \vee W_{1,1}$

...

Negated goal formula: $\neg W_{1,3}$

Resolution Proof for the Wumpus World

Resolution:

$\neg W_{1,3}$, $\neg S_{1,2} \vee W_{1,3} \vee W_{1,2} \vee W_{2,2} \vee W_{1,1}$

$\longrightarrow \neg S_{1,2} \vee W_{1,2} \vee W_{2,2} \vee W_{1,1}$

$S_{1,2}$, $\neg S_{1,2} \vee W_{1,2} \vee W_{2,2} \vee W_{1,1}$

$\longrightarrow W_{1,2} \vee W_{2,2} \vee W_{1,1}$

$\neg S_{1,1}$, $S_{1,1} \vee \neg W_{1,1}$

$\longrightarrow \neg W_{1,1}$

$\neg W_{1,1}$, $W_{1,2} \vee W_{2,2} \vee W_{1,1}$

$\longrightarrow W_{1,2} \vee W_{2,2}$

...

$\neg W_{2,2}$, $W_{2,2}$

$\longrightarrow \square$

From Knowledge to Action

We can now infer new facts, but how do we translate knowledge into action?

Negative selection: Excludes any provably dangerous actions.

$$\mathsf{A}_{1,1} \wedge \mathsf{East}_A \wedge \mathsf{W}_{2,1} \Rightarrow \neg\ \mathsf{Forward}$$

Positive selection: Only suggests actions that are provably safe.

$$\mathsf{A}_{1,1} \wedge \mathsf{East}_A \wedge \neg \mathsf{W}_{2,1} \Rightarrow \mathsf{Forward}$$

Differences?

From the suggestions, we must still select an action.

Problems with Propositional Logic

Although propositional logic suffices to represent the wumpus world, it is rather involved.

1. Rules must be set up for each square.
   R_1: $\neg S_{1,1} \Rightarrow \neg W_{1,1} \wedge \neg W_{1,2} \wedge \neg W_{2,1}$
   R_2: $\neg S_{2,1} \Rightarrow \neg W_{1,1} \wedge \neg W_{2,1} \wedge \neg W_{2,2} \wedge \neg W_{3,1}$
   R_3: $\neg S_{1,2} \Rightarrow \neg W_{1,1} \wedge \neg W_{1,2} \wedge \neg W_{2,2} \wedge \neg W_{1,3}$
   ...

We need a time index for each proposition to represent the validity of the proposition over time → further expansion of the rules.

→ More powerful logics exist, in which we can use object variables.

→ First-Order Predicate Logic

Summary

- Rational agents require knowledge of their world in order to make rational decisions.
- With the help of a declarative (knowledge-representation) language, this knowledge is represented and stored in a knowledge base.
- We use propositional logic for this (for the time being).
- Formulae of propositional logic can be valid, satisfiable or unsatisfiable.
- The concept of logical implication is important.
- Logical implication can be mechanized by using an inference calculus → resolution.
- Propositional logic quickly becomes impractical when the world becomes too large (or infinite).