Foundations of AI

5. Constraint Satisfaction Problems

CSPs as Search Problems, Solving CSPs, Problem Structure

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Contents

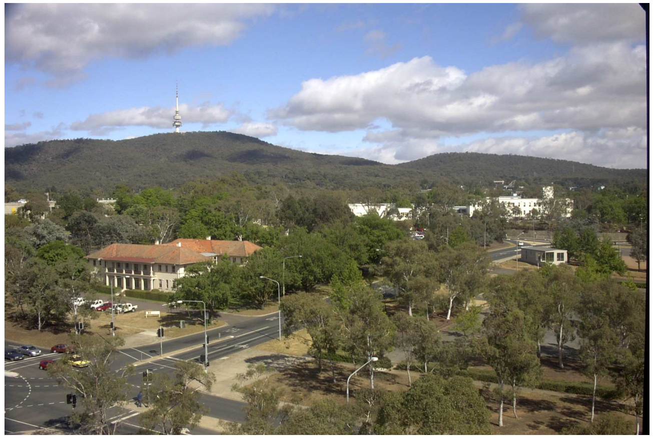
Constraint Satisfaction Problems

- In search problems, the state does not have a structure (everything is in the data structure) – in CSPs states are explicitly represented as variable assignments.
- A CSP consists of
  - a set of variables {*x1, x2, … xn*} to which
  - values {*d1, d2, ..,dk*} can be assigned
  - respecting a set of constraints over the variables
- A CSP is solved by a variable assignment that satisfies all given constraints
- *Formal representation language* with associated general inference algorithms

Example: Map-Coloring

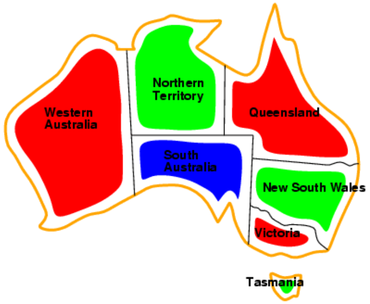
- Variables: *WA, NT, SA, Q, NSW, V, T*
- Values: {*red, green, blue*}
- Constraints: adjacent regions must have different colors, e.g. NSW $\neq$ V

Australian Capital Territory (ACT) and Canberra (inside NSW)



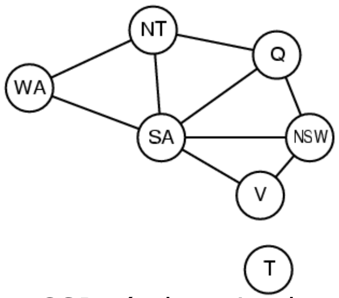
View of the Australian National University and Telstra Tower

One Solution



- Solution assignment:
  - { *WA = red, NT = green, Q = red, NSW = green, V = red, SA = blue, T = green* }
    - Perhaps in addition *ACT = blue*

Constraint Graph



- Works for binary CSPs (otherwise hypergraph)
- Nodes = variables, arcs = constraints
- Graph structure can be important (e.g., connected components)

Note: Our problem is 3-colorability for a planar graph

Variations

- Binary, ternary, or even higher arity
- Finite domains (*d* values) => d^n possible variable assignments
- Infinite domains (reals, integers)
  - *linear constraints* solvable (in P if real)
  - *nonlinear constraints* unsolvable

Applications

- Timetabling (classes, rooms, times)
- Configuration (hardware, cars, …)
- Spreadsheets
- Scheduling
- Floor planning
- Frequency assignments
- …

Backtracking Search over Assignments

- Assign values to variables step by step (order does not matter)
- Consider only one variable per search node!
- DFS with single-variable assignments is called backtracking search
- Can solve *n*-queens for $n \approx 25$

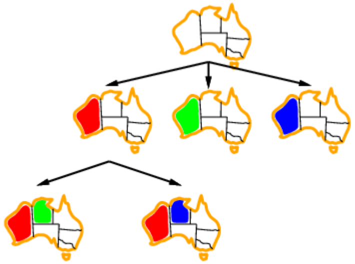
Algorithm

```
function BACKTRACKING-SEARCH(csp) returns solution/failure
   return RECURSIVE-BACKTRACKING([], csp)
function RECURSIVE-BACKTRACKING(assigned, csp) returns solution/failure
   if assigned is complete then return assigned
   var ← SELECT-UNASSIGNED-VARIABLE(VARIABLES[csp], assigned, csp)
   for each value in ORDER-DOMAIN-VALUES(var, assigned, csp) do
       if value is consistent with assigned according to CONSTRAINTS[csp] then
           result ← RECURSIVE-BACKTRACKING([var = value|assigned], csp)
           if result ≠ failure then return result
   end
   return failure
```

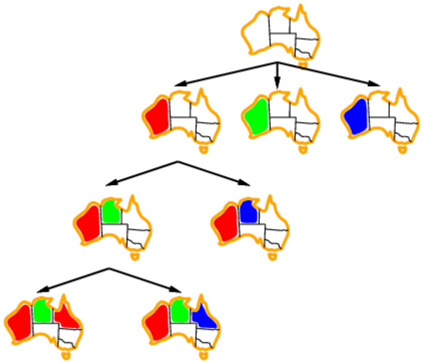

Example (1)

Example (2)

Example (3)



Example (4)



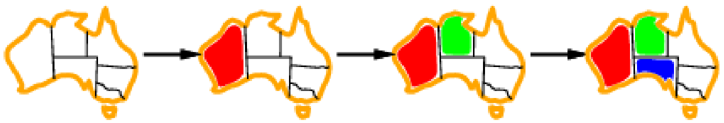
Improving Efficiency: CSP Heuristics & Pruning Techniques

- Variable ordering: Which one to assign first?
- Value ordering: Which value to try first?
- Try to detect failures early on
- Try to exploit problem structure

➢ Note: all this is not problem-specific!

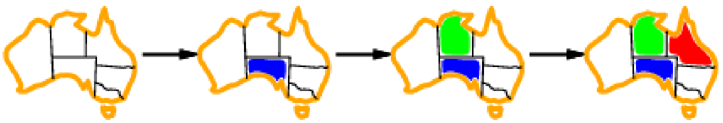
Variable Ordering: Most constrained first

- Most constrained variable:
  - choose the variable with the fewest remaining legal values
  - ➢ reduces branching factor!



Variable Ordering: Most Constraining Variable First

- Break ties among variables with the same number of remaining legal values:
  - choose variable with the most constraints on remaining unassigned variables
  - ➢ reduces branching factor in the next steps



Value Ordering: Least Constraining Value First

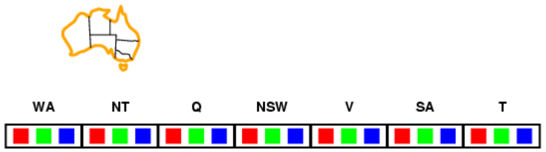
- Given a variable,
  - choose first a value that rules out the fewest values in the remaining unassigned variables
  - ➢ We want to find an assignment that satisfies the constraints (of course, does not help if unsat.)

Rule Out Failures Early On: Forward Checking

- Whenever a value is assigned to a variable, values that are now illegal for other variables are removed
- Implements what the ordering heuristics implicitly compute
- *WA* = *red*, then *NT* cannot become *red*
- If all values are removed for one variable, we can stop!

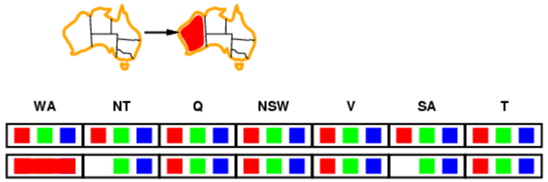
Forward Checking (1)

- Keep track of remaining values
- Stop if all have been removed



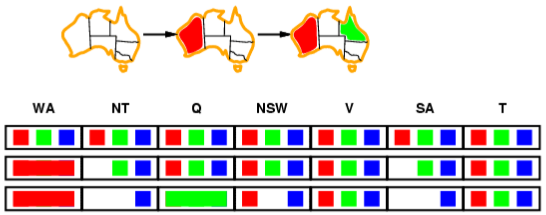
Forward Checking (2)

- Keep track of remaining values
- Stop if all have been removed



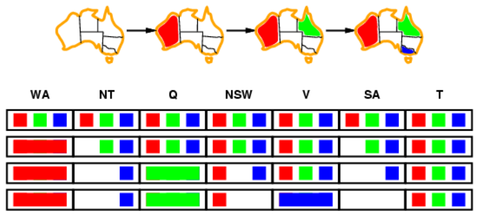
Forward Checking (3)

- Keep track of remaining values
- Stop if all have been removed



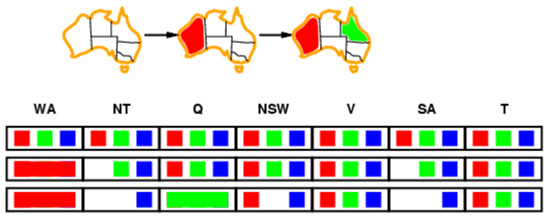
Forward Checking (4)

- Keep track of remaining values
- Stop if all have been removed



Forward Checking: Sometimes it Misses Something

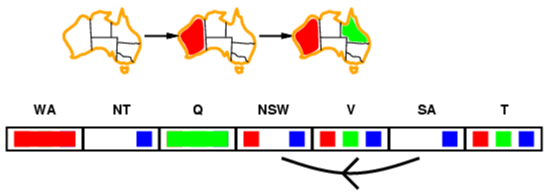
- Forward Checking propagates information from assigned to unassigned variables
- However, there is no propagation between unassigned variables



Arc Consistency

- A directed arc X → Y is "consistent" iff
  - for every value *x* of *X*, there exists a value *y* of *Y*, such that *(x,y)* satisfies the constraint between *X* and *Y*
- Remove values from the domain of *X* to enforce arc-consistency
- Arc consistency detects failures earlier
- Can be used as preprocessing technique or as a propagation step during backtracking

Arc Consistency Example



AC3 Algorithm

function AC-3(*csp*) **returns** the CSP, possibly with reduced domains
  inputs: *csp*, a binary CSP with variables $\{X_1, X_2, \ldots, X_n\}$
  local variables: *queue*, a queue of arcs, initially all the arcs in *csp*

  while *queue* is not empty **do**
    $(X_i, X_j) \leftarrow$ REMOVE-FIRST(*queue*)
    if REMOVE-INCONSISTENT-VALUES(X_i, X_j) **then**
      for each X_k **in** NEIGHBORS[X_i] **do**
        add (X_k, X_i) to *queue*

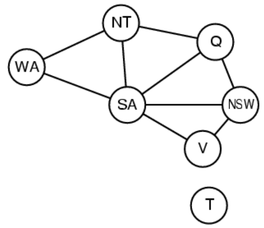
---

function REMOVE-INCONSISTENT-VALUES(X_i, X_j) **returns** true iff we remove a value
  removed ← *false*
  for each x **in** DOMAIN[X_i] **do**
    if no value y in DOMAIN[X_j] allows (x,y) to satisfy the constraint between X_i and X_j
      then delete x from DOMAIN[X_i]; *removed* ← *true*
  return *removed*

Properties of AC3

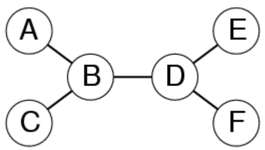
- AC3 runs in $O(d^3n^2)$ time, with n being the number of nodes and d being the maximal number of elements in a domain
- Of course, AC3 does not detect all inconsistencies (which is an NP-hard problem)

Problem Structure (1)



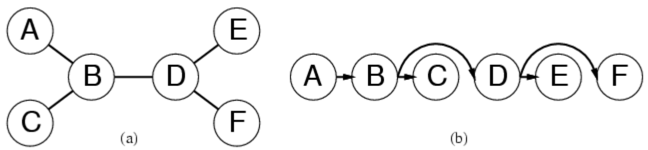
- CSP has two independent components
- Identifiable as connected components of constraint graph
- Can reduce the search space dramatically

Problem Structure (2): Tree-structured CSPs



- If the CSP graph is a tree, then it can be solved in $O(nd^2)$
  - General CSPs need in the worst case $O(d^n)$
- *Idea:* Pick root, order nodes, apply arc consistency from leaves to root, and assign values starting at root

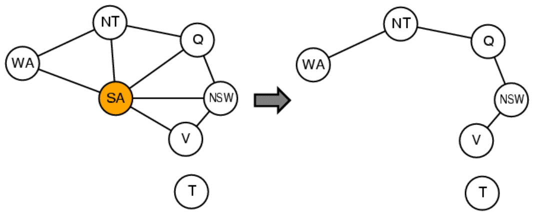
Problem Structure (2): Tree-structured CSPs



- Apply arc-consistency to (X_i, X_k), when X_i is the parent of X_k, for all $k=n$ *downto* 2.
- Now one can start at X_1 assigning values from the remaining domains without creating any conflict in one sweep through the tree!
- Algorithm linear in n

Problem Structure (3): Almost Tree-structured

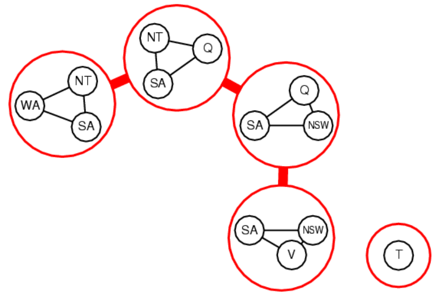
- Conditioning: Instantiate a variable and prune values in neighboring variables



- Cutset conditioning: Instantiate (in all ways) a set of variables in order to reduce the graph to a tree (note: finding minimal cutset is NP-hard)

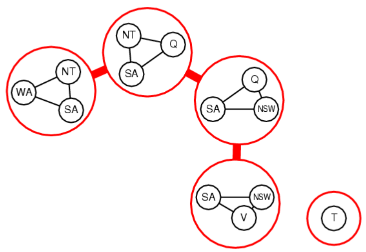
Another Method: Tree Decomposition (1)

- Decompose problem into a set of connected sub-problems, where two sub-problems are connected when they share a constraint
- Solve sub-problems independently and combine solutions



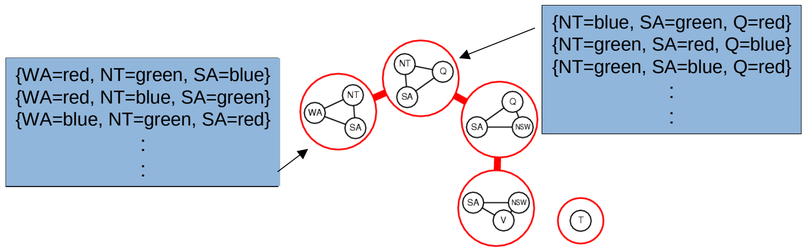
Another Method: Tree Decomposition (2)

- A tree decomposition must satisfy the following conditions:
  - Every variable of the original problem appears in at least one sub-problem
  - Every constraint appears in at least one sub-problem
  - If a variable appears in two sub-problems, it must appear in all sub-problems on the path between the two sub-problems
  - The connections form a tree



Another Method: Tree Decomposition (3)

- Consider sub-problems as new mega-nodes, which have values defined by the solutions to the sub-problems
- Use technique for tree-structured CSP to find an overall solution (constraint is to have identical values for the same variable).



Tree Width

- Tree width of a tree decomposition = size of largest sub-problem minus 1
- Tree width of a graph is minimal tree width over all possible tree decompositions
- If a graph has tree width *w* and we know a tree decomposition with that width, we can solve the problem in $O(nd^{w+1})$
- Finding a tree decomposition with minimal tree width is NP-hard

Summary & Outlook

- CSPs are a special kind of search problem:
  - states are value assignments
  - goal test is defined by constraints
- Backtracking = DFS with one variable assigned per node. Other intelligent backtracking techniques possible
- Variable/value ordering heuristics can help dramatically
- Constraint propagation prunes the search space
- Path-consistency is a constraint propagation technique for triples of variables
- Tree structure of CSP graph simplifies problem significantly
- Cutset conditioning and tree decomposition are two ways to transform part of the problem into a tree
- CSPs can also be solved using local search