## Constraint Satisfaction Problems

Qualitative Representation and Reasoning

Malte Helmert and Stefan Wölfl
Albert-Ludwigs-Universität Freiburg
July 3/5/10/12/17, 2007

## Quantitative vs. Qualitative Representations

Spatio-temporal configurations can be described quantitatively by specifying the coordinates of the relevant objects:

Example: At time point 10.0 object $A$ is at position
(11.0, 1.0, 23.7), at time point 11.0 at position (15.2, 3.5, 23.7). From time point 0.0 to 11.0 , object $B$ is at position (15.2, 3.5, 23.7). Object $C$ is at time point 11.0 at position (300.9, 25.6, 200.0) and at time point 35.0 at (11.0, 1.0, 23.7).

Often, however, a qualitative description (using a finite vocabulary) is more adequate:

Example: Object $A$ hit object
$B$. Afterwards, object $C$ arrived.

Sometimes we want to reason with such descriptions.

Example: Object $C$ was not close to object $A$, when it hit object $B$.

## Constraint Satisfaction Problems

July 3/5/10/12/17, 2007 - Qualitative Representation and Reasoning
Qualitative Constraint Satisfaction Problems
Qualitative Constraint Languages
Constraint Propagation

## Tractability

Allen's Interval Algebra
Intervals and Relations Between Them
IA: Examples
IA: Example for Incompleteness
The Continuous Endpoint Class
The Continuous Endpoint Class
The Endpoint Subclass
The ORD-Horn Subclass
Solving Arbitrary Allen CSPs
RCC8
RCC8: Motivation
S. Wölf, PGWCmerit Basespit Relataigns Constraint Satisfaction Problems July 3/5/10/12/17, 2007 $2 / 73$

## Topology

Complexity

## Motivation

## Representation of Qualitative Knowledge

Intention: describe configurations in an infinite (continuous) domain using a finite vocabulary and reason about these descriptions

- Specification of a vocabulary: usually a finite set of relations (often binary) that are pairwise disjoint and jointly exhaustive
- Specification of a language: often sets of atomic formulae (constraint networks), perhaps restricted disjunction
- Specification of a formal semantics
- Analysis of computational properties and design of reasoning methods (often constraint propagation)
- Perhaps, specification of operational semantics for verifying whether a relation holds in a given quantitative configuration


## Applications in ...

- Natural language processing
- Specification of abstract spatio-temporal configurations
- Query languages for spatio-temporal information systems
- Layout descriptions of documents (and learning of such layouts)
- Action planning
- ...


## Some Reasoning Problems

$$
\{x(<\cup=) y, y(<\cup=) z, v(<\cup=) y, w>y, z(<\cup=) x\}
$$

- Satisfiability: Are there values for all time points such that all formulae are satisfied?
- Satisfiability with $v=w$ ?
- Finding a satisfying instantiation of all time points
- Deduction: Does $x\{=\} y$ follow logically?

$$
\text { Does } v \leq w \text { follow? }
$$

- Finding a minimal description: What are the most constrained relations that describe the same set of instantiations?


## Example: Qualitative Temporal Relations

Suppose, we want to talk about time instants (points) and binary relations over them.

- Vocabulary: $X=Y(X$ equals $Y), X<Y(X$ before $Y)$, and $X>Y$ ( $X$ after $Y$ )
- Language:
- Allow for disjunctions of basic relations to express indefinite information. Use unions of relations to express that. For instance, $<U=$ expresses $\leq$.
- $2^{3}$ different relations (including the impossible and the universal relation)
- Use sets of atomic formulae with these relations to describe configurations. For example:

$$
\{x=y, y(<\cup>) z\}
$$

- Semantics: Interpret the time point symbols and relation symbols over the rational (or real) numbers.

5. Wölfl, M. Helmert (Universität Freiburg) Constraint Satisfaction Problems

July 3/5/10/12/17, 2007

## Motivation

## From a Logical Point of View ...

In general, qualitatively described configurations are simple logical theories:

- Only sets of atomic formulae to describe the configuration
- Only existentially quantified variables (or constants)
- A fixed background theory that describes the semantics of the relations (e.g., dense linear orders)
- We are interested in satisfiability, model finding, and deduction


## Motivation Qualitative CSP

Let $\mathcal{B}$ be a finite set of (binary) relations on some (infinite) domain $D$ (elements of $\mathcal{B}$ are called base relations).
We require:

- The relations in $\mathcal{B}$ are JEPD, i. e., jointly exhaustive and pairwise disjoint.
- $\mathcal{B}$ is closed under converses.

Then:

- Let $\mathcal{A}$ be the set of relations that can be built by taking the unions of relations from $\mathcal{B}\left(\rightsquigarrow 2^{|\mathcal{B}|}\right.$ different relations).
- $\mathcal{A}$ is closed under converse, complement, intersection and union.
- Often, $\mathcal{A}$ is closed under composition of base relations, i. e., for all $B, B^{\prime} \in \mathcal{B}$,

$$
B \circ B^{\prime} \in \mathcal{A}
$$

Then, $\mathcal{A}$ is closed under composition of arbitrary relations.
But often this condition is not satisfied.
S. Wölfl, M. Helmert (Universität Freiburg) Constraint Satisfaction Problems July 3/5/10/12/17, $2007 \quad 9 / 73$
Motivation Qualitative CSP

## Reasoning Problems

Given a qualitative CSP:

CSP-Satisfiability (CSAT):

- Is the CSP satisfiable/solvable?

CSP-Entailment (CENT):

- Given in addition $x R y$ : Is $x R y$ satisfied in each solution of the CSP?

Computation of an equivalent minimal CSPs (CMIN):

- Compute for each pair $x, y$ of variables the strongest constrained (minimal) relation entailed by the CSP.


## Computing Operations on Relations

Let $\mathcal{A}$ be the system of relations over a set of base relations $\mathcal{B}$ that satisfies all the conditions above.
We may write relations as sets of base relations:

$$
B_{1} \cup \cdots \cup B_{n} \cong\left\{B_{1}, \ldots, B_{n}\right\}
$$

Then the operations on the relations can be computed as follows:
Composition:

$$
\left\{B_{1}, \ldots B_{n}\right\} \circ\left\{B_{1}^{\prime}, \ldots, B_{m}^{\prime}\right\}=\bigcup_{i=1}^{n} \bigcup_{j=1}^{m} B_{i} \circ B_{j}^{\prime}
$$

Converse:

$$
\left\{B_{1}, \ldots, B_{n}\right\}^{-1}=\left\{B_{1}^{-1}, \ldots, B_{n}^{-1}\right\}
$$

Complement:

$$
\overline{\left\{B_{1}, \ldots, B_{n}\right\}}=\left\{B \in \mathcal{B}: B \neq B_{i}, \text { for each } 1 \leq i \leq n\right\}
$$

Intersection and union are defined in the usual set-theoretical way.
S. Wölfl, M. Helmert (Universität Freiburg) $\quad$ Constraint Satisfaction Problems $\quad$ July 3/5/10/12/17, 2007 10 / 73

> Motivation Qualitative CSP

## Reductions between CSP Problems

Theorem
CSAT, CENT and CMIN are equivalent under polynomial Turing reductions.

Proof.
CSAT $\leq_{T}$ CENT and CENT $\leq_{T}$ CMIN are obvious.
CENT $\leq_{T}$ CSAT: We solve CENT (CSP $\models x R y$ ?) by testing satisfiability of the CSP extended by $x\{B\} y$ where $B$ ranges over all base relations. Let $B_{1}, \ldots, B_{k}$ be the relations for which we get a positive answer. Then $x\left\{B_{1}, \ldots, B_{k}\right\} y$ is entailed by the CSP.
CMIN $\leq{ }_{T}$ CENT: We use entailment for computing the minimal constraint for each pair of variables. Starting with the universal relation, we remove one base relation until we have a minimal relation that is still entailed.

Motivation Qualitative CSP
The Path Consistency Method

Given a qualitative CSP with $R_{v_{1}, v_{2}}=R_{v_{2}, v_{1}}^{-1}$. Then the path consistency method is to apply the operation

$$
R_{v_{1}, v_{2}} \leftarrow R_{v_{1}, v_{2}} \cap\left(R_{v_{1}, v_{3}} \circ R_{v_{3}, v_{2}}\right) .
$$

on all the constraints of the network until a fixpoint is reached.
The path consistency method guarantees...

- sometimes minimality
- sometimes satisfiability
- however sometimes the CSP is not satisfiable, even if the CSP contains only base relations

Wölfl, M. Helmert (Universität Freiburg) Constraint Satisfaction Problems

Some Properties of the Point Relations

Theorem
A path consistent CSP over the point relations is consistent.
In particular, the path consistency method decides consistency.

Theorem
A path consistent CSP over all point relations without $\{<,>\}$ is minimal.
Proofs later ...

Motivation Qualitative CSP

## Example: Point Relations

Composition table:

|  | $<$ | $=$ | $>$ |
| :---: | :---: | :---: | :---: |
| $<$ | $<$ | $<$ | $<,=,>$ |
| $=$ | $<$ | $=$ | $>$ |
| $>$ | $<,=,>$ | $>$ | $>$ |

Figure: Composition table for the point algebra. For example: $\{<\} \circ\{=\}=\{<\}$

- $\{<,=\} \circ\{<\}=\{<\}$
- $\{<,>\} \circ\{<\}=\{<,=,>\}$
- $\{<,=\}^{-1}=\{>,=\}$
- $\{<,=\} \cap\{>,=\}=\{=\}$
S. Wölfl, M. Helmert (Universität Freiburg) Constraint Satisfaction Problems


## A Pathological Relation System

Let $e, d, i$ be (self-converse) base relations between points on a circle:

- e: Rotation by 72 degrees (left or right)
- d: Rotation by 144 degrees (left or right)
- i: Identity

The following CSP is path consistent and contains only base relations, but it is not satisfiable:


## Qualitative Constraint Languages

## Qualitative Constraint Languages

From now on, let $D$ be a finite or infinite domain.
Definition
A partition scheme on $D$ is any non-empty, finite set $\Gamma$ of binary relations
on $D$ such that:

- 「 defines a partition of $D \times D$.
- $\Gamma$ contains the binary identity relation $\mathrm{id}_{D}$.
- 「 is closed under converses.


## Definition

A constraint language of binary relations on $D, \Gamma$, is said to be generated from a partition scheme $\Delta$, if $\Gamma$ consists of all finite unions of relations in $\Delta$.
Constraint languages in this sense will be referred to as qualitative constraint languages.
S. Wölfl, M. Helmert (Universität Freiburg) Constraint Satisfaction Problems

July 3/5/10/12/17, 2007

## Weak Composition

Let $\Gamma$ be a qualitative constraint language with partition scheme $\Delta$. For $R, S \in \Gamma$, define:

$$
R \circ_{w} S:=\bigcup\{T \in \Delta: T \cap(R \circ S) \neq \emptyset\}
$$

- $o_{w}$ is called weak composition of $R$ and $S$.


## Lemma

For all relations $R, S, T \in \Gamma$,

- $R \circ S \subseteq R \circ{ }_{w} S$;
- $T \cap(R \circ S)=\emptyset$ if and only if $T \cap\left(R \circ{ }_{w} S\right)=\emptyset$;
- $\left(R \circ_{w} S\right)^{-1}=S^{-1} \circ_{w} R^{-1}$;
- $R \circ_{w}(S \cup T)=\left(R \circ_{w} S\right) \cup\left(R \circ_{w} T\right)$.


## Qualitative Constraint Network

Let $\Gamma$ be a subset of a qualitative constraint language with partition scheme $\Delta$.

Definition
A qualitative constraint network over $\Gamma$ is a triple

$$
P=\langle V, D, C\rangle,
$$

where

- $V$ is a non-empty and finite set of variables,
- $D$ is an arbitrary non-empty set (domain),
- $C$ is a finite set of constraints $C_{1}, \ldots, C_{q}$, i. e., each constraint $C_{i}$ is a pair $\left(s_{i}, R_{i}\right)$, where $s_{i}$ is a pair of variables and $R_{i}$ is a binary relation contained in $\Gamma$.
S. Wölfl, M. Helmert (Universität Freiburg)

Constraint Satisfaction Problems

## Qualitative Constraint Languages <br> Weak Composition: Examples

## Example:

Consider a linear order on a domain with 2 elements $a<b$. The relations $R_{<}, R_{=}, R_{>}$define a partition schema on $D$. It holds:

$$
R_{<} \circ R_{<}=R_{>} \circ R_{>}=\emptyset, R_{<} \circ R_{>}=\{(a, a)\}, R_{>} \circ R_{<}=\{(b, b)\}
$$

but

$$
R_{<} \circ_{w} R_{<}=R_{>} \circ_{w} R_{>}=\emptyset, \quad R_{<} \circ_{w} R_{>}=R_{=}, \quad R_{>} \circ_{w} R_{<}=R_{=}
$$

Moreover,

$$
\left(R_{<} \circ_{w} R_{>}\right) \circ_{w} R_{>}=R_{=} \circ_{w} R_{>}=R_{>} \neq \emptyset=R_{<} \circ_{w} \emptyset=R_{<} \circ_{w}\left(R_{>} \circ_{w} R_{>}\right) .
$$

## Example:

Consider a linear order on a domain with 3 elements $a<b<c$. Then

$$
R_{<} \circ R_{<}=\{(a, c)\} \quad \text { but } \quad R_{<} \circ_{w} R_{<}=R_{<}
$$

## Qualitative Constraint Languages

## Non-Associative Relation Algebras

## Definition

A non-associative relation algebra is a set $A$ with

- binary operations $\sqcap$, $\sqcup$, and ;,
- unary operations - and ${ }^{-}$, and
- distinct elements 0,1 , and $\delta$ such that
(a) $(A, \sqcap, \sqcup,-, 0,1)$ is a Boolean algebra.
(b) For all elements $a, b$ and $c$ of $A$ :

$$
\begin{aligned}
a ;(b \sqcup c) & =(a ; b) \sqcup(a ; c) \\
\delta ; a & =a ; \delta=a \\
\left(a^{-}\right)^{-} & =a \text { and }(-a)^{-}=-\left(a^{-}\right) \\
(a \sqcup b)^{-} & =a^{-} \sqcup b^{-} \\
(a ; b)^{-} & =b^{-} ; a^{-} \\
(a ; b) \sqcap c^{-}=0 & \text { if and only if }(b ; c) \sqcap a^{-}=0
\end{aligned}
$$

S. Wölfl, M. Helmert (Universität Freiburg) Constraint Satisfaction Problems

## Algebraically Closed Networks

A qualitative network $P=\langle V, D, C\rangle$ is normalized, if

- for each pair of variables $x, y, C$ contains at least one constraint $((x, y), R)$;
- for each constraint $((x, x), R)$ in $C, R=i d_{D}$;
- for constraints $((x, y), R)$ and $((y, x), S)$ in $C, R=S^{-1}$.

In what follows we will always assume that constraint networks are normalized.

## Definition

A qualitative constraint network $P$ is algebraically closed (or: a-closed), if for all constraints $((x, y), R),((x, z), S)$, and $((z, y), T)$ of $P$, it holds:

$$
R \subseteq S \circ_{w} T
$$

Note: If $P$ is algebraically closed, then $R=R \cap\left(S \circ_{w} T\right)$.

## Qualitative Languages and Algebras

Let $\Gamma$ be a qualitative constraint language with partition scheme $\Delta$. As spelled out before, each relation $R$ in $\Gamma$ can be represented by a finite disjunction of "base relations" $B_{1}, \ldots, B_{k} \in \Delta$. In what follows we identify $R$ with the set of its base relations

$$
\left\{B_{1}, \ldots, B_{k}\right\}
$$

## Lemma

For each partition scheme $\Delta$, the tuple

$$
\left\langle 2^{\Delta}, \cap, \cup, \circ_{w}, \mathbf{C}_{\Delta},{ }^{-1}, \emptyset, \Delta, \mathrm{id}_{\Delta}\right\rangle
$$

defines a non-associative relation algebra.
S. Wölfl, M. Helmert (Universität Freiburg) Constraint Satisfaction Problems

[^0]
## Constraint Propagation

Following, we present two constraint propagation algorithms.
The path consistency algorithm can only be used if the underlying partition scheme is closed under composition, i. e., if for each pair of relations $R, S \in \Delta, R \circ S$ is a (finite) union of a subset of $\Delta$.
The algebraic closure algorithm is a variant of the path consistency algorithm. Instead of ordinary composition of relations, we use weak composition.
Since weak composition is an upper approximation of composition only, the algebraic closure algorithm may not result in a path-consistent network.

Let $P=\langle V, D, C\rangle$ be a (normalized) qualitative constraint network. Let Table $[i, j]$ be a $n \times n$-matrix ( $n$ : number of variables), in which we record the constraints between the variables.

## Qualitative Constraint Languages Constraint Propagation

## Path Consistency Algorithm

## EnforcePathConsistency ( $P$ ):

Input: a qualitative network $P=\langle V, D, C\rangle$
Output: "inconsistent", or an equivalent, path-consistent network $P^{\prime}$

$$
\operatorname{Paths}(i, j)=\{(i, j, k): 1 \leq k \leq n, k \neq i, j\} \cup
$$

$$
\{(k, i, \bar{j}): \overline{1} \leq k \leq n, k \neq i, j\}
$$

Queue : $=\bigcup_{i, j} \operatorname{Paths}(i, j)$
while $Q \neq \emptyset$

$$
\text { select and delete }(i, k, j) \text { from } Q
$$

$T:=$ Table $[i, j] \cap($ Table $[i, k] \circ$ Table $[k, j])$
if $T=\emptyset$
return "inconsistent"
elseif $T \neq$ Table $[i, j]$
Table $[i, j]:=T$
Table $[j, i]:=T^{-1}$
Queue $:=$ Queue $\cup \operatorname{Paths}(i, j)$
return $P^{\prime}$ with the refined constraints as recorded in Table
S. Wölfl, M. Helmert (Universität Freiburg) Constraint Satisfaction Problems

## Computing on the Symbolic Level

Let $\Gamma$ be a qualitative constraint language with partition scheme $\Delta$.
We suppose that we have determined (by some formal proof or some computation) the (weak) composition table for $\Delta$, i. e.,

$$
{ }^{\circ}(w): \Delta \times \Delta \rightarrow 2^{\Delta} .
$$

Let now $B$ be a finite set of symbols (bijective with $\Delta$ ).
Then $2^{B}$ is a Boolean algebra, from which we obtain a (non-associative) relation algebra, if we extend $\circ_{(w)}$ to a function

$$
\circ_{(w)}: 2^{B} \times 2^{B} \rightarrow 2^{B} .
$$

Now we can perform all the operations needed in the path consistency/a-closure algorithm on the symbolic level.

## Algebraic Closure Algorithm

## EnforceAlgClosure ( $P$ ):

Input: a qualitative network $P=\langle V, D, C\rangle$
Output: "inconsistent", or an equivalent algebraically closed network $P^{\prime}$

$$
\begin{aligned}
& \operatorname{Paths}(i, j)=\{(i, j, k): 1 \leq k \leq n, k \neq i, j\} \cup \\
&\{(k, i, j): 1 \leq k \leq n, k \neq i, j\}
\end{aligned}
$$

Queue : $=\bigcup_{i, j} \operatorname{Paths}(i, j)$
while $Q \neq \emptyset$
select and delete $(i, k, j)$ from $Q$
$T:=$ Table $[i, j] \cap\left(\right.$ Table $[i, k] \circ_{w}$ Table $\left.[k, j]\right)$
if $T=\emptyset$
return "inconsistent"
elseif $T \neq$ Table $[i, j]$
Table $[i, j]:=T$
Table $[j, i]:=T^{-1}$
Queue $:=$ Queue $\cup \operatorname{Paths}(i, j)$
return $P^{\prime}$ with the refined constraints as recorded in Table
S. Wölfl, M. Helmert (Universität Freiburg) Constraint Satisfaction Problems

## Path Consistency and Tractability

Let $\Gamma$ be a subset of a qualitative constraint language with a partition scheme $\Delta$ that is closed under composition.
Let $\widehat{\Gamma}$ be smallest superset of $\Gamma$ that is closed under intersection, converses, and composition.

## Lemma

There exists a polynomial time reduction from $\mathbf{C}_{\widehat{\Gamma}}$ to $\mathbf{C}_{\Gamma}$.
In particular, it holds:

- $\Gamma$ is tractable if and only if $\widehat{\Gamma}$ is tractable.
- Enforcing path consistency decides consistency over $\widehat{\Gamma}$ if and only if it does so over $\Gamma$.


## Proof idea.

Each relation in $\widehat{\Gamma}$ stems from a finite number of compositions, intersections, and conversions applied to relations in $\bar{\Gamma}$. Hence each constraint network over $\widehat{\Gamma}$ can be transformed step-by-step into an equivalent network over $\Gamma$. In the case where a relation results from composing other relations, we need to introduce some fresh variables.

## Qualitative Constraint Languages Tractability

## Algebraic Closure and Tractability

Let now $\Gamma$ be a subset of a qualitative constraint language with a partition scheme $\Delta$ (not necessarily closed under composition).
From now on, let $\widehat{\Gamma}$ always be smallest superset of $\Gamma$ that is closed under intersection, converses, and weak composition.
Lemma (Ligozat \& Renz 2005)
If enforcing a-closure decides consistency for atomic networks (i. e., for qualitative networks over $\Delta$ ), then $\mathbf{C}_{\widehat{\Gamma}}$ is polynomial-time reducible to $\mathbf{C}_{\Gamma}$. In particular, if a-closure decides consistency for atomic networks, then

- $\Gamma$ is tractable if and only if $\hat{\Gamma}$ is so;
- enforcing a-closure decides consistency over $\Gamma$ if and only if a-closure decides consistency over $\widehat{\Gamma}$.

Wölfl, M. Helmert (Universität Freiburg)
Constraint Satisfaction Problems

## IA: The Base Relations

To determine all possible relation between Allen intervals, we determine how one can order the four points of two intervals:

| Relation | Symbol | Name |
| :---: | :---: | :--- |
| $\left\{(X, Y): X^{-}<X^{+}<Y^{-}<Y^{+}\right\}$ | $\prec$ | before |
| $\left\{(X, Y): X^{-}<X^{+}=Y^{-}<Y^{+}\right\}$ | m | meets |
| $\left\{(X, Y): X^{-}<Y^{-}<X^{+}<Y^{+}\right\}$ | $\circ$ | overlaps |
| $\left\{(X, Y): X^{-}=Y^{-}<X^{+}<Y^{+}\right\}$ | s | starts |
| $\left\{(X, Y): Y^{-}<X^{-}<X^{+}=Y^{+}\right\}$ | f | finishes |
| $\left\{(X, Y): Y^{-}<X^{-}<X^{+}<Y^{+}\right\}$ | d | during |
| $\left\{(X, Y): Y^{-}=X^{-}<X^{+}=Y^{+}\right\}$ | $\equiv$ | equal |

and the converse relations (obtained by exchanging $X$ and $Y$ )

## Allen's Interval Calculus

- Allen's interval calculus (IA): time intervals and binary relations over them
- Let $\langle\mathbb{R},<\rangle$ be the linear order on the real numbers (conceived of as the flow of time).
Then, the domain $D$ of Allen's calculus is the set of all intervals

$$
X=\left(X^{-}, X^{+}\right) \in \mathbb{R}^{2}, \text { where } X^{-}<X^{+}
$$

(naïve approach)

- Relations between concrete intervals, e. g.:
$(1.0,2.0)$ strictly before $(3.0,5.5)$
$(1.0,3.0)$ meets $(3.0,5.5)$
$(1.0,4.0)$ overlaps $(3.0,5.5)$
S. Wölff, M. Helmert (Universität Freiburg) Constraint Satisfaction Problems

Allen's Interval Algebra Intervals and Relations Between Them
IA: The 13 Base Relations Graphically

S. Wölfl, M. Helmert (Universität Freiburg) Constraint Satisfaction Problems July 3/5/10/12/17, 2007

## IA: Partition Scheme and Composition

## Lemma

The 13 base relations of Allen's interval calculus define a partition scheme on the set of all Allen intervals.

In what follows:

- IA: the qualitative constraint language generated from all base relations of Allen's interval calculus (contains $2^{13}=8192$ relations)
- IA-B: the subclass of IA containing base relations only


## Lemma

The set of base relations of Allen's interval calculus is closed under composition.

Wölfl, M. Helmert (Universität Freiburg) Constraint Satisfaction Problems


Allen's Interval Algebra IA: Example for Incompleteness
IA: Example for Incompleteness

S. Wölfl, M. Helmert (Universität Freiburg)

Constraint Satisfaction Problems

## Allen's Interval Algebra IA: Example for Incompleteness

## IA: NP-Hardness

Theorem (Kautz \& Vilain)
Deciding satisfiability over IA is NP-hard.
Proof.
Reduction from 3-colorability (the original proof uses 35 at).
Let $G=(V, E), V=\left\{v_{1}, \ldots, v_{n}\right\}$ be an instance of 3-colorability.
Then we use the intervals $\left\{v_{1}, \ldots, v_{n}, 1,2,3\right\}$ with the following constraints:

| 1 | $\{\mathrm{~m}\}$ | 2 |  |
| :---: | :---: | :---: | :--- |
| 2 | $\{\mathrm{~m}\}$ | 3 |  |
| $v_{i}$ | $\left\{\mathrm{~m},=\mathrm{m}^{-1}\right\}$ | 2 | $\forall v_{i} \in V$ |
| $v_{i}$ | $\left\{\mathrm{~m}, \mathrm{~m}^{-1}, \prec, \succ\right\}$ | $v_{j}$ | $\forall\left(v_{i}, v_{j}\right) \in E$ |

This constraint system is satisfiable iff $G$ can be colored with 3 colors.

Wölfl, M. Helmert (Universität Freiburg)
Constraint Satisfaction Problems

## IA: The Continuous Endpoint Class

Continuous Endpoint Class IA-C: the subset of IA consisting of those relations with a clause form containing only unit clauses, where $\neg(a=b)$ is forbidden.

Example: All basic relations and, e.g., $\{d, o, s\}$, because

$$
\begin{aligned}
\pi(X\{\mathrm{~d}, \mathrm{o}, \mathrm{~s}\} Y)= & \left\{X^{-}<X^{+}, Y^{-}<Y^{+}\right. \\
& X^{-}<Y^{+}, X^{+}>Y^{-} \\
& \left.X^{+}<Y^{+}\right\}
\end{aligned}
$$



The set IA-C contains 83 relations. It is closed under intersection, composition, and converses (it is a sub-algebra wrt. these three operations on relations). This can be shown by using a computer program.

## Allen's Interval Algebra The Continuous Endpoint Class

## IA: Clause Representation

Following, we will look at polynomial special cases, i.e., subclasses of the qualitative constraint language IA.

For this we start from a natural translation of interval relations/constraints (of the form $X R Y$ ) into clause formulas over atoms of the form $a$ op $b$, where:

- $a, b \in\left\{X^{-}, X^{+}, Y^{-}, Y^{+}\right\} ;$and
- $o p \in\{<,>,=, \leq, \geq\}$.

Example: All base relations can be expressed as unit clauses.

## Lemma

Let $P$ be a constraint network over $I A$, and let $\pi(P)$ be the translation of $P$ into clause form.
$P$ is satisfiable iff $\pi(P)$ is satisfiable over the rational numbers.
S. Wölfl, M. Helmert (Universität Freiburg)

Constraint Satisfaction Problems

IA: Consistency for IA-C

## Following we prove:

## Lemma

Each 3-consistent interval CSP over IA-C is globally consistent
From this we can conclude:
Theorem (van Beek)
Applied to networks over IA-C, enforcing path consistency decides satisfiability and solves the minimal label problem.

Corollary
A path-consistent interval constraint network containing base relations only is satisfiable.

## Helly's Theorem

## Definition

A set $M \subseteq \mathbb{R}^{n}$ is convex iff for all pairs of points $a, b \in M$, all points on the line connecting $a$ and $b$ belong to $M$.

Theorem (Helly)
Let $F$ be a family of at least $n+1$ convex sets in $\mathbb{R}^{n}$. If all sub-families of $F$ with $n+1$ sets have a non-empty intersection, then $\bigcap F \neq \emptyset$.

IA: Strong $n$-Consistency (2): Instantiating the $k$ th Variable

## Proof (Part 2).

The instantiation of the $k-1$ variables $X_{i}$ to $\left(s_{i}, e_{i}\right)$ restricts the instantiation of $X_{k}$.
Note: Since $R_{i j} \in \mathrm{IA}-\mathcal{C}$ by assumption, these restrictions can be expressed by inequalities of the form

$$
s_{i}<X_{k}^{+} \wedge e_{j} \geq X_{k}^{-} \wedge \ldots
$$

Such inequalities define convex subsets in $\mathbb{R}^{2}$.
$\rightsquigarrow$ Consider sets of 3 inequalities ( $=3$ convex sets).

## IA: Strong $n$-Consistency (1)

Proof of the lemma
We prove the claim by induction over $k$ with $k \leq n$.
Base case: $k=1,2,3 \quad \sqrt{ }$
Induction assumption: Assume strong $k-1$-consistency (and non-emptiness of all relations)
Induction step: From the assumption, it follows that there is an instantiation of $k-1$ variables $X_{i}$ to pairs $\left(s_{i}, e_{i}\right) \in \mathbb{R}^{2}$ satisfying the constraints $R_{i j}$ between the $k-1$ variables.
We have to show that we can extend the instantiation to any $k$ th variable.
S. Wölfl, M. Helmert (Universität Freiburg)

Constraint Satisfaction Problems

## IA: Strong $n$-Consistency (3): Using Helly's Theorem

Proof (Part 3).
Case 1: All 3 inequalities mention only $X_{k}^{-}$(or mention only $X_{k}^{+}$). Then it suffices to consider only 2 of these inequalities (the strongest). Because of 3 -consistency, there exists at least 1 common point satisfying these 3 inequalities
Case 2: The inequalities mention $X_{k}^{-}$and $X_{k}^{+}$, but it does not contain the inequality $X_{k}^{-}<X_{k}^{+}$. Then there are at most 2 inequalities with the same variable and we have the same situation as in Case 1.
Case 3: The set contains the inequality $X_{k}^{-}<X_{k}^{+}$. In this case, only three intervals (incl. $X_{k}$ ) can be involved and by the same argument as above there exists a common point.
$\rightsquigarrow$ With Helly's Theorem, it follows that there exists a consistent instantiation for all subsets of variables.
$\rightsquigarrow$ Strong $k$-consistency for all $k \leq n$.

## Allen's Interval Algebra The Endpoint Subclass

## IA: The Endpoint Subclass

Endpoint Subclass: IA-P is the subclass that permits a clause form containing only unit clauses ( $a \neq b$ is now allowed).
Example: all basic relations and $\{\mathrm{d}, \mathrm{o}\}$ since

$$
\begin{aligned}
\pi(X\{\mathrm{~d}, \mathrm{o}\} Y)= & \left\{X^{-}<X^{+}, Y^{-}<Y^{+}\right. \\
& X^{-}<Y^{+}, X^{+}>Y^{-}, X^{-} \neq Y^{-} \\
& \left.X^{+}<Y^{+}\right\}
\end{aligned}
$$



Y

Theorem (Vilain \& Kautz 86, Ladkin \& Maddux 88)
The path consistency method decides satisfiability over IA-P
S. Wölfl, M. Helmert (Universität Freiburg) Constraint Satisfaction Problems July 3/5/10/12/17, 2007 45 / 73

IA: The ORD-Horn Subclass (2)

Lemma
IA-H is closed under intersection, composition, and converses.

Theorem
The path consistency method decides satisfiability over IA-H.

IA-H contains 886 relations.
Question: Is IA- $\mathcal{H}$ a maximal subalgebra?

## Allen's Interval Algebra The ORD-Horn Subclass

## IA: The ORD-Horn Subclass

ORD-Horn Subclass: IA- $\mathcal{H}$ is the subclass of IA that permits a clause form containing only Horn clauses, where only the following literals are allowed:

$$
a \leq b, a=b, a \neq b
$$

$\neg a \leq b$ is not allowed!
Example: all $R \in \mathrm{IA}-\mathcal{P}$ and $\left\{\mathrm{o}, \mathrm{s}, \mathrm{f}^{-1}\right\}$ :

$$
\pi\left(X\left\{\mathrm{o}, \mathrm{~s}, \mathrm{f}^{-1}\right\} Y\right)=\left\{\begin{array}{l}
X^{-} \leq X^{+}, X^{-} \neq X^{+} \\
\\
Y^{-} \leq Y^{+}, Y^{-} \neq Y^{+} \\
\\
X^{-} \leq Y^{-} \\
\\
X^{-} \leq Y^{+}, X^{-} \neq Y^{+} \\
\\
Y^{-} \leq X^{+}, X^{+} \neq Y^{-} \\
\\
X^{+} \leq Y^{+} \\
\\
\end{array} X^{-} \neq Y^{-} \vee X^{+} \neq Y^{+}\right\} .
$$

S. Wölfl, M. Helmert (Universität Freiburg)

Constraint Satisfaction Problems

## Allen's Interval Algebra The ORD-Horn Subclass

## IA: The ORD-Horn Subclass (3)

A computer-aided case analysis leads to the following result:
Lemma
There are only two minimal sub-algebras that strictly contain IA- $\mathcal{H}: \mathcal{X}_{1}, \mathcal{X}_{2}$

$$
\begin{aligned}
& N_{1}=\left\{\mathrm{d}, \mathrm{~d}^{-1}, \mathrm{o}^{-1}, \mathrm{~s}^{-1}, \mathrm{f}\right\} \in \mathcal{X}_{1} \\
& N_{2}=\left\{\mathrm{d}^{-1}, \mathrm{o}, \mathrm{o}^{-1}, \mathrm{~s}^{-1}, \mathrm{f}^{-1}\right\} \in \mathcal{X}_{2}
\end{aligned}
$$

The clause forms of these relations contain "proper" disjunctions!
Theorem
The satisfiability problem over $I A-\mathcal{H} \cup\left\{N_{i}\right\}$ is NP-complete.
Lemma
$I A-\mathcal{H}$ is the only maximal tractable subclass that contains all base relations of IA.

## IA: Solving General Allen CSPs

- Backtracking algorithm using path consistency as a forward-checking method
- Method works on tractable fragments of Allen's calculus: split relations into relations of a tractable fragment, and backtrack over these.
- Refinements and evaluation of different heuristics
$\rightsquigarrow$ Which tractable fragment should one use?


## IA: Branching Factors

- If the labels are split into base relations, then on average a label is split into


## 6.5 relations

- If the labels are split into pointizable relations $(\mathcal{P})$, then on average a label is split into


### 2.955 relations

- If the labels are split into ORD-Horn relations $(\mathcal{H})$, then on average a label is split into


### 2.533 relations

$\rightsquigarrow$ A difference of 0.422 which becomes significant, when applied to extremely hard instances
S. Wölfl, M. Helmert (Universität Freiburg)

Constraint Satisfaction Problems

## RCC8: Motivation

We may want to state qualitative relationships between regions in space, for example:

- "Region $X$ touches region $Y$ "
- "Germany and Switzerland have a common border"
- "Freiburg is located in Baden-Württemberg"


## RCC8: Possible Applications

- This can be useful when only partial information is available:
- We may know that region $X$ is not connected with region $Y$ without knowing the shape and location of $X$ and $Y$.
- We may want to query a database:
- Show me all countries bordering the Mediterranean!
- We may want to state integrity constraints:
- An island has to be located in the interior of a sea.

RCC8 RCC8: Base Relations

## RCC8: Qualitative Relations Between Regions

Eight relations between regions:


- Regions are some "reasonable" non-empty subsets of space.
- DC (disconnected) means that the two regions do not share any point at all.
- EC (externally connected) means that they only share borders.
- PO (partially overlapping) means that the two regions share interior points.
- TPP (tangential proper part) means that one region is a subset of the other sharing some points on the borders.
- NTPP (non-tangential proper part) same, but without sharing any bordering points.
S. Wölfl, M. Helmert (Universität Freiburg)

Constraint Satisfaction Problems

## Point-Set Topology

Point-set topology is a mathematical theory that deals with properties of space independent of size and shape.

In topology, we can define notions such as

- interior and exterior points of regions,
- isolated points of regions,
- boundaries of regions,
- connected components of regions,
- connected regions,
- ...


## RCC8: Intuition

RCC8 RCC8: Base Relations

## Topology

## Definition

A topological space is a pair $\mathcal{T}=(S, \mathcal{O})$, where

- $S$ is a non-empty set (the universe), and
- $\mathcal{O}$ is a set of subsets of $S$ (the open sets)
such that the following conditions hold:
- $\emptyset \in \mathcal{O}$ and $S \in \mathcal{O}$.
- If $O_{1} \in \mathcal{O}$ and $O_{2} \in \mathcal{O}$, then $O_{1} \cap O_{2} \in \mathcal{O}$.
- If $\left(O_{i}\right)_{i \in I}$ is a (possibly infinite) family of elements from $\mathcal{O}$, then

$$
\bigcup_{i \in I} O_{i} \in \mathcal{O}
$$

Example: In Euclidean space, a set $O$ is open if for each point $x \in O$ there is a ball surrounding $x$ that is contained in $O$.

RCC8 Topology

## Terminology \& Notation

Definition
Let $X \subseteq S$ and $x \in S$.

- A set $N \subseteq S$ is a neighborhood of a point $x$ if there is an open set $O \in \mathcal{O}$ such that $x \in O \subseteq N$.
- $x \in S$ is an interior point of $X$ if there is a neighborhood $N$ of $x$ such that $N \subseteq X$.
- $x \in S$ is a touching point of $X$ if every neighborhood of $x$ has a non-empty intersection with $X$.


## Notation:

- $\operatorname{int}(X)$ is the set of interior points of $X$ (the interior of $X$ ).
- $\operatorname{cls}(X)$ is the set of touching points of $X$ (the closure of $X$ ).
- A set is closed if $X=\operatorname{cls}(X)$.
S. Wölfl, M. Helmert (Universität Freiburg) Constraint Satisfaction Problems

RCC8: What Is a Region?

$A$ and $D$ are reasonable regions, $B, C$, and $E$ are not
In other words, $X$ is a region iff it is non-empty

$$
x \neq \emptyset
$$

and regular closed, i.e., the closure of an open set:

$$
X=\operatorname{cls}(\operatorname{int}(X))
$$

It is not necessary that a region is internally connected.

RCC8 Topology

## Interior and Closure Operators

The function $\operatorname{int}(\cdot)$ is an interior operator:

1. $\operatorname{int}(S)=S$
2. $\operatorname{int}(X) \cap \operatorname{int}(Y)=\operatorname{int}(X \cap Y)$
3. $\operatorname{int}(X) \subseteq X$
4. $\operatorname{int}(\operatorname{int}(X))=\operatorname{int}(X)$

Note:

- $X$ is open iff $X=\operatorname{int}(X)$
- $\operatorname{cls}(X)=S \backslash \operatorname{int}(S \backslash X)$
$\rightsquigarrow$ It can be seen that these relations define a partition scheme.


## Defining the RCC8-Relations

Let $S$ be a topological space. Then define the following relations on Reg:

$$
\begin{aligned}
\mathrm{DC}(X, Y) & :=X \cap Y=\emptyset \\
\mathrm{EC}(X, Y) & :=X \cap Y \neq \emptyset \wedge \operatorname{int} X \cap \operatorname{int} Y=\emptyset \\
\mathrm{PO}(X, Y) & :=\operatorname{int} X \cap \operatorname{int} Y \neq \emptyset \wedge X \nsubseteq Y \wedge Y \nsubseteq X \\
\mathrm{EQ}(X, Y) & :=X=Y \\
\operatorname{TPP}(X, Y) & :=X \subseteq Y \wedge X \nsubseteq \operatorname{int} Y \\
\operatorname{NTPP}(X, Y) & :=X \subseteq \operatorname{int} Y
\end{aligned}
$$

RCC8 Topology

## RCC8: From Regions to Boolean Algebras

Let Reg denote the set of all regular closed set of some fixed topological space.
For $X, Y \in \operatorname{Reg} \cup\{\emptyset\}$ define:

$$
\begin{aligned}
-X & :=\operatorname{cls}(S \backslash X) \\
X \sqcup Y & :=X \cup Y \\
X \sqcap Y & :=\operatorname{cls}(\operatorname{int}(X \cap Y))
\end{aligned}
$$

By these definition, we obtain a Boolean algebra.

RCC8 Topology

## Boolean Connection Algebras

## Definition

A connection algebra is a Boolean algebra $B$ together with a binary relation $C$ on $B$ such that the following conditions are satisfied:

- $x \neq 0 \Leftrightarrow x C x$
- $x C y \Rightarrow y C x$
- $x \neq 0,1 \Rightarrow x C-x$
- $x C y \cup z \Leftrightarrow x C y$ or $x C z$
- $x \neq 0,1 \Rightarrow$ not $x C y$, for some $y \neq 0,1$
. Wölfl, M. Helmert (Universität Freiburg) Constraint Satisfaction Problems


## RCC8: From Topologies to Connection Algebras

If the underlying topological space is regular and connected, i. e.,

- Hausdorff and for each $x \in S$ and closed subset $A \subset S$ with $x \notin A$, there exist disjoint open neighborhoods of $x$ and $A$;
- the only sets that are open and closed are $\emptyset$ and $S$;
then

$$
x \mathrm{C} y \Longleftrightarrow x \cap y \neq \emptyset
$$

defines a connection algebra on $\operatorname{Reg} \cup\{\emptyset\}$.
RCC8 Topology
Defining the RCC8-Relations (2)

Let $B$ be a connection algebra. Then we can define the RCC8 relations on $B \backslash\{0\}$ as follows:

$$
\begin{aligned}
X \mathrm{DC} Y & :=\operatorname{not} X \mathrm{C} Y \\
X \mathrm{P} Y & :=(X, Y) \notin \mathrm{C} \circ \mathrm{DC} \\
X \text { PP } Y & :=X \mathrm{P} Y \wedge X \neq Y \\
X \text { O } Y & :=(X, Y) \in \mathrm{P}^{-1} \circ \mathrm{P} \\
X \text { PO } Y & :=X \mathrm{O} Y \wedge \operatorname{not} X \mathrm{P} Y \wedge \operatorname{not} Y \mathrm{P} X \\
X \text { EC } Y & :=X \mathrm{C} Y \wedge \text { not } X \mathrm{O} Y \\
X \text { TPP } Y & :=X \mathrm{PP} Y \wedge(X, Y) \in \mathrm{EC} \circ \mathrm{EC} \\
X \text { NTPP } Y & :=X \mathrm{PP} Y \wedge \text { not } X \mathrm{TPP} Y
\end{aligned}
$$

RCC8 Complexity

## RCC8: Complexity

## Using a reduction from 3-SAT, it can be shown:

Theorem
Testing satisfiability over arbitrary RCC8 relations is NP-hard.

Using a translation into S4-modal logics, one can show:
Theorem
Testing satisfiability over arbitrary RCC8 relations is NP-complete.

## RCC8: Composition Table

| - | DC | EC | PO | TPP | NTPP | TPP ${ }^{-1}$ | NTPP ${ }^{-1}$ | EQ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| DC | * | $\begin{aligned} & \hline \hline \mathrm{DC,EC} \\ & \text { PO,TPP } \\ & \text { NTPP } \end{aligned}$ | $\begin{gathered} \hline \hline \text { DC,EC } \\ \text { PO,TPP } \\ \text { NTPP } \end{gathered}$ | $\begin{gathered} \hline \hline \mathrm{DC,EC} \\ \text { PO,TPP } \\ \text { NTPP } \end{gathered}$ | $\begin{aligned} & \hline \hline \mathrm{DC,EC} \\ & \text { PO,TPP } \\ & \text { NTPP } \end{aligned}$ | DC | DC | DC |
| EC | $\begin{gathered} \text { DC,EC } \\ \text { PO, TPP }^{-1} \\ \text { NTPP }^{-1} \end{gathered}$ | $\begin{gathered} \mathrm{DC}, \mathrm{EC} \\ \mathrm{PO}, \mathrm{TPP} \\ \mathrm{TPP}^{-1}, \mathrm{EQ} \end{gathered}$ | $\begin{array}{\|c} \hline \mathrm{DC}, \mathrm{EC} \\ \text { PO,TPP } \\ \text { NTPP } \\ \hline \end{array}$ | $\begin{aligned} & \text { EC,PO } \\ & \text { TPP } \\ & \text { NTPP } \end{aligned}$ | $\begin{gathered} \text { PO } \\ \text { TPP } \\ \text { NTPP } \end{gathered}$ | DC,EC | DC | EC |
| PO | $\begin{gathered} \text { DC,EC } \\ \text { PO,TPP }^{-1} \\ \text { NTPP }^{-1} \\ \hline \end{gathered}$ | $\begin{array}{\|l\|l\|} \hline \text { DC,EC } \\ \mathrm{PO}_{1}, \mathrm{TPP}^{-1} \\ \text { NTPP }^{-1} \\ \hline \end{array}$ | * | $\begin{gathered} \text { PO } \\ \text { TPP } \\ \text { NTPP } \end{gathered}$ | $\begin{gathered} \text { PO } \\ \text { TPP } \\ \text { NTPP } \end{gathered}$ | $\begin{gathered} \text { DC,EC } \\ \hline \text { PO, TPP } \\ \text { NTPP }^{-1} \\ \hline \end{gathered}$ | $\begin{array}{\|c\|} \hline \text { DC,EC } \\ \text { PO,TPP }^{-1} \\ \text { NTPP }^{-1} \\ \hline \end{array}$ | PO |
| TPP | DC | DC,EC | $\begin{array}{\|c} \hline \text { DC,EC } \\ \text { PO,TPP } \\ \text { NTPP } \end{array}$ | TPP <br> NTPP | NTPP | $\begin{array}{\|c\|} \hline \mathrm{DC}, \mathrm{EC} \\ \mathrm{PO}, \mathrm{TPP} \\ \mathrm{TPP}^{-1}, \mathrm{EQ} \end{array}$ | $\begin{gathered} \hline \text { DC,EC } \\ \text { PO, TPP }^{-1} \\ \text { NTPP }^{-1} \end{gathered}$ | TPP |
| NTPP | DC | DC | $\begin{array}{\|c} \hline \text { DC,EC } \\ \text { PO,TPP } \\ \text { NTPP } \end{array}$ | NTPP | NTPP | $\begin{aligned} & \text { DC,EC } \\ & \text { PO,TPP } \\ & \text { NTPP } \end{aligned}$ | * | NTPP |
| TPP ${ }^{-1}$ | $\begin{gathered} \hline \mathrm{DC}, \mathrm{EC} \\ \mathrm{PO}, \mathrm{TPP}^{-1} \\ \text { NTPP }^{-1} \\ \hline \end{gathered}$ | $\begin{gathered} \mathrm{EC}, \mathrm{PO} \\ \text { TPP }^{-1} \\ \text { NTPP }^{-1} \\ \hline \end{gathered}$ | $\begin{array}{\|c} \hline \text { PO } \\ \text { TPP }^{-1} \\ \text { NTPP }^{-1} \\ \hline \end{array}$ | $\begin{gathered} \hline \text { PO,EQ } \\ \text { TPP } \\ \text { TPP }^{-1} \\ \hline \end{gathered}$ | $\begin{gathered} \hline \text { PO } \\ \text { TPP } \\ \text { NTPP } \\ \hline \end{gathered}$ | $\begin{gathered} \operatorname{TPP}^{-1} \\ \mathrm{NTPP}^{-1} \end{gathered}$ | NTPP ${ }^{-1}$ | TPP ${ }^{-1}$ |
| NTPP ${ }^{-1}$ | $\begin{gathered} \hline \text { DC,EC } \\ \text { PO,TPP }^{-1} \\ \text { NTPP }^{-1} \\ \hline \end{gathered}$ | $\begin{gathered} \hline \text { PO } \\ \text { TPP }^{-1} \\ \text { NTPP }^{-1} \end{gathered}$ | $\begin{array}{\|c\|} \hline \text { PO } \\ \text { TPP }^{-1} \\ \text { NTPP }^{-1} \\ \hline \end{array}$ | $\begin{array}{\|c\|} \hline \text { PO }^{-1} \\ \text { TPP }^{-1} \\ \text { NTPP }^{-1} \\ \hline \end{array}$ | $\begin{array}{\|l\|} \hline \text { PO,TPP }{ }^{-1} \\ \text { TPP,NTPP } \\ \text { NTPP }^{-1}, \mathrm{EQ} \\ \hline \end{array}$ | NTPP ${ }^{-1}$ | NTPP ${ }^{-1}$ | NTPP ${ }^{-1}$ |
| EQ | DC | EC | PO | TPP | NTPP | TPP ${ }^{-1}$ | NTPP ${ }^{-1}$ | EQ |

RCC8 Constraint Reasoning

## RCC8: Constraint Propagation

- As in Allen's interval algebra, we may want to use constraint propagation instead of translating everything to modal logic.
- We need a composition table...
- ... which could be computed using the modal logic encoding (and in fact, this has been done).
- Based on this table, we can then apply the algebraic closure algorithm
- .... and ask ourselves for which fragment of RCC8 it is complete.


## RCC8: Is the Composition Table Extensional?

It can easily be verified that already in the 2-dimensional case, the set of base relations is not closed under composition:

- Consider EC $\circ$ TPP and $X$ NTPP $S$, where $S$ denotes the universal region.
- Consider EC o EC and a donut-like region $X$ with "hole" $Y$.

Lemma (Düntsch et al. 2001)
In each connection algebra, the relation algebra generated by the RCC8 base relations contains at least 25 atomic relations.

Lemma (Li et al. 2006)
In each model associated to some Euclidean space $\mathbb{R}^{n}$, the relation algebra generated by the RCC8 base relations contains an infinite strictly decreasing sequence of relations.

RCC8 Tractable Fragments

## RCC8：Tractable Fragments？

## Theorem（Li 2006）

Enforcing algebraic closure on atomic RCC8 constraint network decides satisfiability．
－As in the case of Allen＇s interval calculus，we may ask for maximal tractable subsets ．．．
－Again，one can identify relations that can be encoded by Horn formulae．．．
－ 148 Horn relations $\mathcal{H}_{8}$ ，which forms again a maximal subset．
－There are 2 additional maximal subsets that allow for poly． satisfiability testing！

S．Wölfl，M．Helmert（Universität Freiburg） Constraint Satisfaction Problems

## Literature II

围 Alan K．Mackworth．
Constraint satisfaction．
In S．C．Shapiro，editor，Encyclopedia of Artificial Intelligence，pages 205－211
Wiley，Chichester，England， 1987.
囯 Alan K．Mackworth．
Consistency in networks of relations．
Artificial Intelligence，8：99－118， 1977
Ugo Montanari．
Networks of constraints：fundamental properties and applications to picture processing．
Information Science，7：95－132， 1974.
園 B．Nebel and H．－J．Bürckert．
Reasoning about temporal relations：A maximal tractable subclass of Allen＇s interval algebra，
Journal of the ACM，42（1）：43－66， 1995.

## Literature I

固 J．F．Allen．
Maintaining knowledge about temporal intervals．
Communications of the ACM，26（11）：832－843，November 1983.
Also in Readings in Knowledge Representation．
國 P．van Beek and R．Cohen．
Exact and approximate reasoning about temporal relations．
Computational Intelligence，6：132－144， 1990.
䁬 B．Bennett．
Spatial Reasoning with propositional logic
Principles of Knowledge Representation and Reasoning：Proceedings of the 4th International Conference（KR－94），1994，51－62．
Peter B．Ladkin and Roger Maddux．
On binary constraint networks．
Journal of the ACM，41：435－469，1994，
－Sanjiang Li and Mingsheng Ying．
Extensionality of the RCC8 Composition Table．
Fundamentae INformaticae XX，1－23， 2006.
S．Wölfl，M．Helmert（Universität Freiburg）
Constraint Satisfaction Problems

## Literature III

屋 B．Nebel．
Solving hard qualitative temporal reasoning problems：Evaluating the efficiency of using the ORD－horn class．
CONSTRAINTS，1（3）： 175
R R．Hirsch．
Tractable approximations for temporal constraint handling
Artificial Intelligence，116：287－295， 2000.
（Contains the pathological set of relations．）
A．Krokhin，P．Jeavons and P．Jonsson． A complete classification of complexity in Allen＇s algebra in the presence of a non－trivial basic relation．
Proc．17th Int．Joint Conf．on AI（IJCAI－01），83－88，Seattle，WA， 2001.
國 J．Renz \＆B．Nebel．
On the complexity of qualitative spatial reasoning：A maximal tractable fragment of the Region Connection Calculus．
Proceedings of the 15th International Joint Conference on Artificial Intelligence （IJCAI＇97），August 1997，522－527．

Literature IV
( J. Renz \& B. Nebel,
Efficient Methods for Qualitative Spatial Reasoning,
Proceedings of the 13th European Conference on Artificial Intelligence (ECAI'98),
August 1998, 562-566


[^0]:    Qualitative Constraint Languages Constraint Propagation

