Constraint Satisfaction Problems

Tractable Constraint Languages

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Expressiveness vs. Complexity

- For some restricted constraint languages we know some polynomial time algorithms that solve each instance of that language
- Restricting constraint languages entails restricting expressiveness, i. e., the class of problems that can be expressed in the language

$\rightsquigarrow$ How can we weight expressiveness against performance and vice versa?

CSP Instances aka Constraint Networks

Definition

An instance of a constraint satisfaction problem (i. e., a constraint network) is a triple

$$P = \langle V, D, C \rangle \,,$$

where:

- V is a non-empty and finite set of variables,
- D is an arbitrary set (domain),
- C is a finite set of constraints $C_1, \ldots, C_q$, i. e., each constraint C_i is a pair (s_i, R_i), where s_i is a tuple of variables of length m_i and R_i is an m_i-ary relation on D
  (s_i: constraint scope; R_i: constraint relation).

Restricting the General CSP

The general CSP search problem is the following: Given an instance of a constraint satisfaction problem, P, determine if there exists solution to P, i. e., determine whether

$$\mathrm{Sol}(P) := \left\{(d_1, \ldots, d_n) \in D^n : a(v_i) = d_i \text{ for a solution } a \text{ of } P\right\}$$

(where n is the number of variables of V) is not empty.

Restricting the general CSP:

- structural restriction: consider just CSP instances with particular constraint scopes (e. g., where the network hypergraph has specific properties)
- relational restriction: consider just CSP instances, where the constraint relations have a specific form or specific properties

Constraint Language

Definition

A constraint language is an arbitrary set of relations, Γ, defined over some fixed domain (denoted by $D(\Gamma)$).

Definition

For a constraint language Γ, let $\mathbf{C}_\Gamma$ be the class of CSP instances $P = \langle V, D, C \rangle$ such that for each $(s, R) \in C$, $R \in \Gamma$.
$\mathbf{C}_\Gamma$ is referred to as the relational subclass associated with Γ.

Definition

A finite constraint language Γ is tractable if there exists a polynomial algorithm that solves all instances of $\mathbf{C}_\Gamma$.
An infinite constraint language Γ is tractable if each finite subset of the language is tractable.

Following, we present some examples:

Example: the CHIP language

CHIP is a constraint language for arithmetic and other constraints. Basic constraints in CHIP are so-called:

- domain constraints: unary constraints that restrict the domains of variables to a finite set of natural numbers
- arithmetic constraints: constraints of one of the forms

$$ax = by + c$$
$$ax \leq by + c$$
$$ax \geq by + c$$

  $(a, b, c \in \mathbb{N}, a \neq 0)$. If these equations are conceived of as relations, the resulting constraint language is tractable.

The language is still tractable if we allow for relations expressed by

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n \geq by + c$$
$$ax_1 \cdots x_n \geq by + c$$
$$(a_1x_1 \geq b_1) \vee \cdots \vee (a_nx_n \geq b_n) \vee (ay \geq b)$$

Example: Linear Equations

Let D be any field (e.g., the field of real numbers).
A linear relation on D is any relation defined by a linear equation

$$a_1 x_1 + \cdots + a_n x_n = r \qquad (a_1, \ldots, a_n, r \in D).$$

The language of all linear relations over D is tractable (apply Gaussian elimination).

Example: Relations on Ordered Finite Sets

Let D be an ordered and finite set.
Consider the binary disequality relation

$$\neq_D = \{(d_1, d_2) \in D^2 \ : \ d_1 \neq d_2\}$$

The class of CSP instances $\mathbf{C}_{\{\neq_D\}}$ corresponds to the graph colorability problem with $|D|$ colors.
$\mathbf{C}_{\{\neq_D\}}$ is tractable if $|D| \leq 2$, and intractable, otherwise.

The ternary betweenness relation over D is defined by:

$$B_D = \{(a, b, c) \in D^3 \ : \ a < b < c \vee c < b < a\}$$

$\mathbf{C}_{B_D}$ is tractable if $|D| \leq 4$, and intractable if $|D| \geq 5$.

Example: Connected Row-Convex Relations

Let $D = \{d_1, \ldots, d_n\}$ be an ordered and finite set.
For a binary relation R over D, the matrix representation of R is an $n \times n$ 0,1-matrix M, where $M_{ij} = 1$ iff $(d_i, d_j) \in R$.
The pruned matrix representation of R results from the matrix representation of R, when we remove all rows and columns in which only 0's occur.
R is connected row-convex, if in the pruned matrix representation of R, the pattern of 1's is connected along each column, along each row, and forms a connected 2-dimensional region.

For example,

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The constraint language on any class of connected row-convex relations is tractable.

Example: Boolean Constraints

Let $D = \{d_0, d_1\}$.
The class of CSP instances $\mathbf{C}_{N_D}$, where

$$N_D = D^3 \setminus \{(d_0, d_0, d_0), (d_1, d_1, d_1)\}$$

is the not-all-equal relation over D, is intractable.
$\mathbf{C}_{N_D}$ corresponds to the not-all-equal satisfiability problem (NAE-3SAT), which is known to be NP-hard.

The class of CSP instances $\mathbf{C}_{T_D}$, where

$$T_D = \{(d_0, d_0, d_1), (d_0, d_1, d_0), (d_1, d_0, d_0)\},$$

is intractable.
$\mathbf{C}_{N_D}$ corresponds to the one-in-three satisfiability problem (1-in-3 SAT).

Example: 0/1/all-Relations

Let D be an arbitrary finite set. A relation R over D is called 0/1/all-relation if one of the following conditions holds:

- R is unary;
- $R = D_1 \times D_2$ for subsets D_1, D_2 of D;
- $R = \{(d, \pi(d)) : d \in D_1\}$, for some subset $D_1 \subseteq D$ and some permutation π of D_1;
- $R = \{(a, b) \in D_1 \times D_2 : a = d_1 \vee b = d_2\}$, for some subsets D_1, D_2 of D and some elements $d_1 \in D_1, d_2 \in D_2$.

The language defined by all 0/1/all-relations is tractable.
(It is even maximal tractable in the sense that if we add any binary relation over D that is not a 0/1/all-relation, then the resulting constraint language becomes intractable).

max-Closed Relations

Let $(D, <)$ be a linear order. Define $\max : D \times D \rightarrow D$ in the usual way, i. e., $\max(a, b) = a$ if $a > b$, and $\max(a, b) = b$, otherwise.
We extend max to a function that can be applied to tuples, i. e., we define $\max : D^k \times D^k \rightarrow D^k$ by

$$\max((a_1, \dots, a_k), (b_1, \dots, b_k)) := (\max(a_1, b_1), \dots, \max(a_k, b_k)).$$

Definition

An n-ary relation R over D is max-closed if for all $(a_1, \dots, a_n)$, $(b_1, \dots, b_n) \in R$,

$$\max((a_1, \dots, a_n), (b_1, \dots, b_n)) \in R.$$

max-Closed Relations and Tractability

Lemma

Let Γ be a constraint language with max-closed relations only. Then $\mathbf{C}_\Gamma$ is tractable.

Proof.

Enforce generalized arc-consistency. If any domain of the resulting network is empty, the network is inconsistent. Otherwise, set each variable to its maximal value, ... □

Example: max-Closed Relations

Consider the CHIP language. All relations of CHIP are max-closed. Hence any set of equations can be solved by establishing arc-consistency.
For example, consider a CSP instance with domain $\{1, \dots, 5\}$, variables $\{v, w, x, y, z\}$, and equations

$$w \neq 3, \ \ z \neq 5, \ \ 3v \leq z, \ \ y \geq z + 2,$$
$$3x + y + z \geq 5w + 1, \ \ wz \geq 2y.$$

Enforcing arc-consistency results in:

$$D(v) = \{1\}, \ \ D(w) = \{4\}, \ \ D(x) = \{3, 4, 5\},$$
$$D(y) = \{5\}, \ \ D(z) = \{3\}.$$

Hence

$$v \mapsto 1, w \mapsto 4, x \mapsto 5, y \mapsto 5, z \mapsto 3$$

is a solution of the constraint network.

Boolean Constraint Languages

The key result in the literature on tractable constraint languages is Schaefer's Dichotomy Theorem (1978).

Definition

A Boolean constraint language is a constraint language over the two-element domain $D = \{0, 1\}$.

Schaefer's theorem states that any Boolean constraint language is either tractable or NP-complete. Moreover, it provides a classification of all tractable constraint languages.

Definition

An arbitrary constraint language Γ is NP-complete if $\mathbf{C}_\Delta$ is NP-complete for some finite subset $\Delta \subseteq \Gamma$.

Schaefer's Theorem

Theorem (Schaefer 1978)

Let Γ be a Boolean constraint language. Then Γ is tractable if at least one of the following conditions is satisfied:

1. *Each relation in Γ contains the tuple $(0, \ldots, 0)$.*
2. *Each relation in Γ contains the tuple $(1, \ldots, 1)$.*
3. *Each relation in Γ is definable by a formula in CNF s. t. each conjunct has at most one negative literal.*
4. *Each relation in Γ is definable by a formula in CNF s. t. each conjunct has at most one positive literal.*
5. *Each relation in Γ is definable by a formula in CNF s. t. each conjunct has at most two literals.*
6. *Each relation in Γ is the set of solutions of a system of linear equations over the finite field with 2 elements.*

In all other cases, Γ is NP-complete.

Algorithm Selector

Let Γ be a Boolean constraint language.

- Class 1: any CSP instance P can be solved by simply assigning 0 to each variable of P.
- Class 2: cf. Class 1 ($v \mapsto 1$).
- Class 6: any CSP instance P can be solved by applying the Gaussian elimination procedure.
- Class 5: any CSP instance P can be solved by resolution: in this case $\mathbf{C}_\Gamma$ corresponds to the 2-SAT satisfiability problem and this can be solved efficiently by resolution.
- Class 4: any CSP instance P can be solved by unit resolution: here $\mathbf{C}_\Gamma$ corresponds to the Horn-SAT satisfiability problem, which can be solved efficiently by unit resolution.
- Class 3: cf. Class 4 ("anti-Horn").

Gadgets

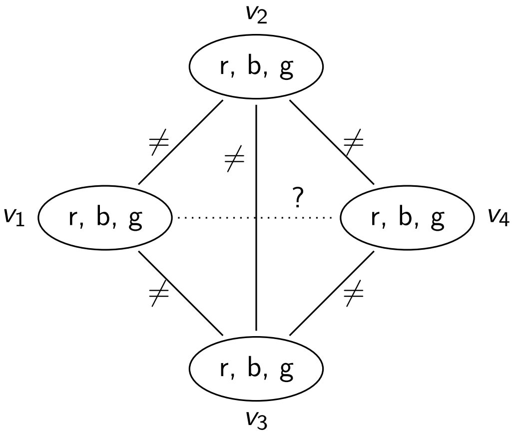
Definition

Let Γ be constraint language and R be a relation on $\Gamma(D)$.
R is expressible in Γ if there exists a CSP instance $P \in \mathbf{C}_\Gamma$ and a sequence of variables $v_1, \ldots, v_n$ such that

$$R = \pi_{v_1,\ldots,v_n}(\mathrm{Sol}(P)).$$

P is referred to as a gadget for expressing R in $\mathbf{C}_\Gamma$, the sequence $v_1, \ldots, v_n$ as construction site for R.

Example



Which relation is expressed by the edge (v_1, v_4)?

Relational Clones

Expressiveness can also be reformulated in the following way:
Let Γ, Γ' be constraint languages (def. on the same domain D).

Definition

Γ' is a relational clone of Γ if Γ' contains each relation expressible by a FO-formula with

- relations from $\Gamma \cup \{=_D\}$,
- conjunctions, and
- existential quantification.

(Formulae of this form are called primitive positive formulae.)

Definition

Let Γ be a constraint language. $\langle\Gamma\rangle$ denotes the smallest relational clone containing Γ, the clone generated by Γ.

Example

Consider a Boolean constraint language with the following relations:

$$R_1 = \{(0,1),(1,0),(1,1)\} \qquad R_2 = \{(0,0),(0,1),(1,0)\}.$$

The relational clone generated by the set of these two relations contains all 16 binary Boolean relations. For example:

$$\begin{array}{ll}
R_3 := \{(0,1),(1,0)\} & R_1(v_1,v_2) \wedge R_2(v_1,v_2) \\
R_4 := \{(0,0),(1,0),(1,1)\} & \exists y(R_1(v_1,y) \wedge R_2(y,v_2)) \\
R_5 := \{(0,0),(1,1)\} & v_1 = v_2 \\
R_6 := \{(0,0)\} & R_2(v_1,v_2) \wedge R_5(v_1,v_2) \\
R_7 := \{(1,1)\} & R_1(v_1,v_2) \wedge R_5(v_1,v_2) \\
R_8 := \{(0,1)\} & \exists y(R_6(v_1,y) \wedge R_1(y,v_2)) \\
\quad \ldots &
\end{array}$$

Reducibility I

Theorem

Let Γ be a set of relations on a fixed domain D, and let Δ be a finite subset of $\langle\Gamma\rangle$. Then there exists a polynomial time reduction from $\mathbf{C}_\Delta$ to $\mathbf{C}_\Gamma$.

Proof.

Let $\Delta = \{S_1, \dots, S_k\}$ be a finite set of relations, where each S_j is expressible by a pp-formula with relations from Γ and the relation $=_D$. For each S_j fix such a formula $\phi_j(x_1, \dots, x_{r_j})$, where r_j is the arity of S_j. Without loss of generality, we may assume that each $\phi_j(x_1, \dots, x_{r_j})$ has the form

$$\exists u_1 \dots u_m (R_1(w^1_1, \dots, w^1_{k_1}) \wedge \cdots \wedge R_n(w^n_1, \dots, w^n_{k_n})) \tag{1}$$

where $w^1_1, \dots, w^1_{k_1}, \dots, w^n_1, \dots, w^n_{k_n} \in \{x_1, \dots, x_{r_j}, u_1, \dots, u_m\}$ for some auxiliary variables $u_1, \dots, u_m$, and $R_1, \dots, R_n \in \Gamma \cup \{=_D\}$. ...

...

Reducibility II

Let $P = \langle V, D, C \rangle$ be an arbitrary instance in $\mathbf{C}_\Delta$. Initially, set $V' := V, D' := D, C' := C$. For each constraint (s, R) (where $s = (v_1, \dots, v_r)$) of P, proceed as follows:

1. add the auxiliary variables $u_1, \dots, u_m$ to V' (always add new variables, rename variables if necessary (also in (1)))
2. remove (r, R) from C' and instead add to C' the constraints (cf. (1)):

$$((w^1_1, \dots, w^1_{k_1}), R_1), \dots, (w^n_1, \dots, w^n_{k_n}, R_n)$$

The CSP instance P' obtained by this procedure is contained in $\mathbf{C}_{\Gamma \cup \{=_D\}}$ and is obviously equivalent to P. Furthermore, from P' we can obtain a CSP instance P'' in $\mathbf{C}_\Gamma$ by deleting constraints of the form $((v_i, v_j), =_D)$ and replacing any occurrence of v_j by v_i. Obviously, both transformation can be done in polynomial time. □

Reducibility III

Corollary

A constraint language Γ is tractable if and only if its relational clone $\langle\Gamma\rangle$ is tractable. Γ is NP-complete if and only if $\langle\Gamma\rangle$ is NP-complete.

Corollary

Let Γ be a constraint language and let R be a relation.
R is expressible in Γ if and only if $R \in \langle\Gamma\rangle$.

The Indicator Problem

Let $k \geq 1$ be a fixed natural number.
Let $s = (x_1, \ldots, x_m)$ be a list of k-tuples in D^k.
Let R be an n-ary relation on D.

We say, that s matches R if $n = m$ and if for each $1 \leq i \leq k$, the n-tuple $(x_1[i], \ldots, x_n[i])$ is in R.

Let now Γ be a fixed constraint language. Set $I_k(\Gamma) = \langle V, D, C \rangle$, where

$$V := D^k$$
$$C := \{(s, R) \; : \; s \text{ matches } R\}$$

Note: $I_k(\Gamma) \in \mathbf{C}_\Gamma$ and contains constraints from Γ on every possible scope which matches some relation in Γ.

Definition

$I_k(\Gamma)$ is said to be the indicator problem of order k for Γ.

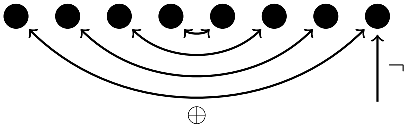
Example: ¬, ⊕

Consider the Boolean constraint language containing the unary relation ¬ and the exclusive-or relation ⊕, i. e.,

$$R_{\oplus} = \{(0,1),(1,0)\} \quad \text{and} \quad R_{\neg} = \{(0)\}.$$

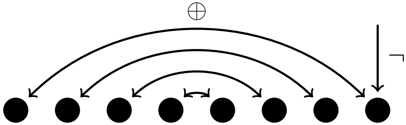
The 3-rd order indicator problem of this language is:

1	1	1	1	0	0	0	0
1	1	0	0	1	1	0	0
1	0	1	0	1	0	1	0



Example (cont'd): $\neg, \oplus$

Solutions of this indicator problem:



1	1	1	1	0	0	0	0
1	1	0	0	1	1	0	0
1	0	1	0	1	0	1	0

Solutions

1	0	1	1	0	0	1	0
1	0	1	0	1	0	1	0
1	0	0	1	0	1	1	0
1	0	0	0	1	1	1	0
1	1	1	1	0	0	0	0
1	1	1	0	1	0	0	0
1	1	0	1	0	1	0	0
1	1	0	0	1	1	0	0

Expressiveness and the Indicator Problem

Theorem (Jeavons (1998))

Let Γ *be a constraint language over some finite domain* D *and let* $R = \{t_1, \ldots, t_k\}$ *be any* n*-ary relation on* D*. Equivalent are:*

(a) R *is expressible in* Γ *(i. e.,* $R \in \langle \Gamma \rangle$*).*

(b) $I_k(\Gamma)$ *is a gadget for expressing* R *with construction site* $(v_1, \ldots, v_n)$*, where for each* $1 \leq i \leq n$*,*

$$v_i := (t_1[i], \ldots, t_k[i]).$$

Proof.

The direction from (b) to (a) is trivial, since $I_k(\Gamma)$ is contained in $\mathbf{C}_\Gamma$. The other direction will be proved later. □

Example: $\neg, \oplus$

Problem: Is the implication expressible in the Boolean language $\{\neg, \oplus\}$?

Consider the 3rd indicator problem (since $R_\Rightarrow$ has three elements $(1,1),(0,1),(0,0)$). Consider the variables $v = (1,0,0)$ and $w = (1,0,1)$:

1	1	**1**	**1**	0	0	0	0
1	1	**0**	**0**	1	1	0	0
1	0	**1**	**0**	1	0	1	0

Solutions

1	0	**1**	**1**	0	0	1	0
1	0	**1**	**0**	1	0	1	0
1	0	**0**	**1**	0	1	1	0
1	0	**0**	**0**	1	1	1	0
1	1	**1**	**1**	0	0	0	0
1	1	**1**	**0**	1	0	0	0
1	1	**0**	**1**	0	1	0	0
1	1	**0**	**0**	1	1	0	0

From this we obtain that $\pi_{(v,w)}(I_3(\Gamma)) = D \times D \neq R_\Rightarrow$.
Thus, the implication is not expressible.

Polymorphisms

Let f be a k-ary operation, i. e., a function $f : D^k \to D$.
For any collection of n-tuples, $t_1, \ldots, t_k \in D^n$, let $f(t_1, \ldots, t_k)$ be defined as the n-tuple:

$$\big(f(t_1[1], \ldots, t_k[1]), \ldots, f(t_1[n], \ldots, t_k[n])\big).$$

Definition

Let $f : D^k \to D$ be a k-ary operation, and R be an n-ary relation.
f is a polymorphism of R (or: R is invariant under f) if for all $t_1, \ldots, t_k \in R$, $f(t_1, \ldots, t_k) \in R$.

Polymorphisms and Invariant Relations

Let Γ be a set of relations on a fixed domain D, and let F be a set of operations on D. Then define:

$\mathrm{Pol}(\Gamma)$:	the set of operations on D that preserve each relation in Γ
$\mathrm{Inv}(F)$:	the set of relations on D that are invariant under each operation of F

Lemma

Pol and Inv define anti-monotone functions, and are related by the following Galois connection:

$$\Gamma \subseteq \mathrm{Inv}(F) \iff F \subseteq \mathrm{Pol}(\Gamma).$$

In particular, it holds:

$$\Gamma \subseteq \mathrm{Inv}(\mathrm{Pol}(\Gamma)) \quad \textit{and} \quad F \subseteq \mathrm{Pol}(\mathrm{Inv}(F)).$$

The Indicator Problem and Polymorphisms

Lemma

Let Γ be a constraint language. The solutions of the k-th indicator problem $I_k(\Gamma)$ are precisely the k-ary polymorphisms of Γ.

Proof.

Apply the definitions ... □

Expressiveness and Polymorphisms

Lemma

Let Γ be a constraint language over some domain D. If $f : D^k \to D$ is a polymorphism of each $R \in \Gamma$, then f is a polymorphism of each $R \in \langle\Gamma\rangle$.

Proof.

Induction on primitive positive formula (cf. blackboard). □

Expressiveness and the Indicator Problem (Part 2)

The following lemma completes the proof of Jeavons' theorem:

Lemma

Let $R = \{t_1, \dots, t_k\}$ be an n-ary relation (over some finite domain D).
For $1 \leq i \leq n$, set $v_i := (t_1[i], \dots, t_k[i])$.
If R is expressible in Γ, then $R = \pi_{v_1,\dots,v_n}(\mathrm{Sol}(I_k(\Gamma)))$.

Proof.

Blackboard.

□

Expressiveness and Invariants

Theorem

For any constraint language Γ *over some finite domain* D*,*

$$\langle \Gamma \rangle = \mathrm{Inv}(\mathrm{Pol}(\Gamma))$$

Proof.

$\subseteq$ is clear. For the converse let R be an n-ary relation that is invariant for each polymorphism of Γ. We have to show that $R \in \langle \Gamma \rangle$. Let $R = \{t_1, \ldots, t_k\}$ and consider the k-th indicator problem of Γ. First define $v_i := (t_1[i], \ldots, t_k[i])$ $(1 \leq i \leq n)$, then consider $R_t = \pi_{v_1,\ldots,v_n}(\mathrm{Sol}(I_k(\Gamma)))$. By one of the lemmas above, R is expressible if $R = R_t$.
$R_t \subseteq R$ follows from the facts that every solution of $I_k(\Gamma)$ is a k-ary polymorphism and that each polymorphism of Γ preserves R. For $R \subseteq R_t$, consider t_j in R. Now the j-th projection function $p_j : D^k \to D$ is a polymorphism. Hence $t_j = p_j(t_1, \ldots, t_k) \in R$. □

Expressiveness, Polymorphisms, and Complexity

Corollary

A relation R on a finite domain is expressible by a constraint language if and only if $\mathrm{Pol}(\Gamma) \subseteq \mathrm{Pol}(\{R\})$.

Corollary

Let Γ and Δ be a constraint languages on a finite domain. If Δ is finite and $\mathrm{Pol}(\Gamma) \subseteq \mathrm{Pol}(\Delta)$, *then $\mathbf{C}_\Delta$ is polynomial-time reducible to $\mathbf{C}_\Gamma$.*

Operations

Following, we study k-ary operations $f : D^k \to D$.

Definition

- f is idempotent, if for each $x \in D$, $f(x, \ldots, x) = x$.
- Given $k = 3$, f is a majority operation, if for all $x, y \in D$,

$$f(x,x,y) = f(x,y,x) = f(y,x,x) = x.$$

- Given $k = 3$, f is a Mal'tsev operation, if for all $x, y \in D$,

$$f(y,y,x) = f(x,y,y) = x.$$

- f is conservative, if for all $x_1, \ldots, x_k \in D$,

$$f(x_1, \ldots, x_k) \in \{x_1, \ldots, x_k\}.$$

Operations (cont'd)

Definition

- Given $k = 2$, f is a semi-lattice operation, if it is
  - associative (i. e., $f(x, f(y, z)) = f(f(x, y), z)$),
  - commutative (i. e., $f(x, y) = f(y, x)$), and
  - idempotent.
- Given $k = 3$ and an Abelian group structure on D, f is affine, if for all $x, y, z \in D$,

$$f(x, y, z) = x - y + z.$$

- Given $k \geq 3$, f is a near-unanimity operation, if for all $x, y \in D$,

$$f(y, x, \dots, x) = f(x, y, x \dots, x) = \dots = f(x, \dots, x, y) = x.$$

Operations (cont'd)

Definition

- f is essentially unary, if there exists an $1 \leq i \leq k$ and a unary non-constant operation g on D such that for all $x_1, \ldots, x_k \in D$,

$$f(x_1, \ldots, x_k) = g(x_i).$$

  If g is the identity operation, then f is called a projection.

- Given $k \geq 3$, f is a semi-projection if f is not an projection and there exists an $1 \leq i \leq k$, such that for all $x_1, \ldots, x_k \in D$ with $|\{x_1, \ldots, x_k\}| < k$,

$$f(x_1, \ldots, x_k) = x_i.$$

A Necessary Condition for Tractability

Theorem

Given $P \neq NP$, any tractable constraint language Γ over a finite domain has a solution to an indicator problem $I_k(\Gamma)$ that defines

- *a constant operation,*
- *a majority operation,*
- *an idempotent binary operation,*
- *an affine operation, or*
- *a semi-projection.*

Boolean CSPs

The complexity of any language over a domain of size 2 can be determined by considering the solutions of its 3rd order indicator problem. The problem is intractable unless this indicator problem has one of the following six solutions:

Variables									
1	1	1	1	0	0	0	0		
1	1	0	0	1	1	0	0		
1	0	1	0	1	0	1	0		
Solutions								**Schaefer class**	**Name**
0	0	0	0	0	0	0	0	1	Constant 0
1	1	1	1	1	1	1	1	2	Constant 1
1	1	1	1	1	1	1	0	3	Anti-Horn
1	0	0	0	0	0	0	0	4	Horn-SAT
1	1	1	0	1	0	0	0	5	2-SAT
1	0	0	1	0	1	1	0	6	Linear

Example: $\neg, \oplus$

●	●	●	●	●	●	●	●
1	1	1	1	0	0	0	0
1	1	0	0	1	1	0	0
1	0	1	0	1	0	1	0
Solutions							
1	0	1	1	0	0	1	0
1	0	1	0	1	0	1	0
1	0	0	1	0	1	1	0
1	0	0	0	1	1	1	0
1	1	1	1	0	0	0	0
1	**1**	**1**	**0**	**1**	**0**	**0**	**0**
1	1	0	1	0	1	0	0
1	1	0	0	1	1	0	0

Sufficient Conditions: Semi-Lattice Operations

In what follows let Γ be always be a constraint language over a finite domain D. We present some sufficient criteria for (in-) tractability.

Theorem

If $\mathrm{Pol}(\Gamma)$ *contains a semi-lattice operation, then*

- Γ *is tractable, and*
- *each instance of* $\mathbf{C}_\Gamma$ *can be solved by enforcing generalized arc-consistency.*

Examples

Example 1:
If Γ is the Boolean constraint language containing all relations expressible by conjunctions of Horn clauses, then

$$\wedge : \{0,1\}^2 \rightarrow \{0,1\}$$

is a semi-lattice operation that is a polymorphism of Γ.

Example 2:
If D is ordered, then max is a semi-lattice operation, which is a polymorphism of each set of max-closed relations.

Sufficient Conditions: Conservative Operations

Theorem

If Pol(Γ) *contains a conservative and commutative operation, then* Γ *is tractable.*

Note: If Γ contains all unary relations on D, then all operations in Pol(Γ) are conservative.

Sufficient Conditions: Near-Unanimity Operations

Theorem

If Pol(Γ) *contains a k-ary near-unanimity operation, then*

- Γ *is tractable.*
- *Each instance of* $\mathbf{C}_\Gamma$ *can be solved by enforcing strong k-consistency.*

Proof.

Blackboard. □

Examples

Example 3:

Let Γ be the Boolean constraint language that consists of all relations definable by a PL-formula in CNF s. t. each conjunct has at most two literals.

Then

$$d(x, y, z) := (x \wedge y) \vee (y \wedge z) \vee (x \wedge z)$$

is a near-unanimity operation on $\{0, 1\}$ and a polym. of Γ.

Example 4:

The 0/1/all relations are invariant under the ternary operation

$$d(x, y, z) := \begin{cases} x & \text{if } y \neq z \\ y & \text{else} \end{cases}$$

which is a near-unanimity operation.

Sufficient Conditions: Mal'tsev Operations

Theorem

If Pol(Γ) *contains a k-ary Mal'tsev operation, then* $\mathbf{C}_\Gamma$ *is tractable.*

Note: Affine relations are Mal'tsev operations.

Reduced Constraint Languages

Lemma

Let Γ be a constraint language over D, and let f be a unary operation on $\mathrm{Pol}(\Gamma)$*. Let $f(\Gamma)$ be the set of all $f(R) := \{f(t) : t \in R\}$ with $R \in \Gamma$. Then, $\mathbf{C}_\Gamma$ is polynomial-time equivalent to $\mathbf{C}_{f(\Gamma)}$.*

Definition

A constraint language Γ is reduced if all its unary polymorphisms are surjective.

Note: Each constraint language can be transformed into a reduced language. For this find all unary polymorphisms by generating and solving the 1st order indicator problem. Choose one of these polymorphisms f with a minimal number of values in its range.

A Sufficient Condition for Intractability

Theorem

Let Γ be a constraint language over a finite domain. If $\mathrm{Pol}(\Gamma)$ contains only essentially unary operations, then $\mathbf{C}_\Gamma$ is NP-complete.

Proof idea:

We can assume that Γ is reduced. One can show that

- $\neq_D$ is in $\mathrm{Inv}(\mathrm{Pol}(\Gamma))$;
- if $|D| = 2$, $\mathrm{Inv}(\mathrm{Pol}(\Gamma))$ contains the not-all-equal relation:

$$D^3 \setminus \{(x, x, x) \,:\, x \in D\}$$

which ensures that $\mathbf{C}_\Gamma$ intractable. □

Towards a Classification

It can be shown that for any reduced constraint language Γ on a finite domain D, one of the following conditions holds:

- $\mathrm{Pol}(\Gamma)$ contains a constant operation;
- $\mathrm{Pol}(\Gamma)$ contains a ternary near-unanimity operation;
- $\mathrm{Pol}(\Gamma)$ contains a Mal'tsev operation;
- $\mathrm{Pol}(\Gamma)$ contains an idempotent binary operation;
- $\mathrm{Pol}(\Gamma)$ contains a semi-projection;
- $\mathrm{Pol}(\Gamma)$ contains essentially unary operations only.

Maximal and Maximal Tractable Languages

Definition

- A constraint language Γ is maximal tractable, if it is tractable and for each relation $R \notin \Gamma$, $\Gamma \cup \{R\}$ is intractable.
- A constraint language Γ is maximal, if there is a relation $R \notin \langle\Gamma\rangle$ and each proper extension of $\langle\Gamma\rangle$ contains all relations on D.

Note: If Γ is a maximal language that is tractable, then $\langle\Gamma\rangle$ is maximal tractable.

Maximality vs. Tractability

Theorem

Let Γ be a constraint language on some finite domain D, and let f be a k-ary operation such that $\langle\Gamma\rangle = \mathrm{Inv}(\{f\})$.
Then $\langle\Gamma\rangle$ is maximal tractable, if

- *f is a constant operation;*
- *f is a ternary near-unanimity operation;*
- *f is a semi-lattice operation;*
- *f is an affine operation.*

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