Constraint Satisfaction Problems

Constraint Networks

Malte Helmert and Stefan Wölfl

Albert-Ludwigs-Universität Freiburg

April 24, 2007

Constraint Satisfaction Problems

April 24, 2007 — Constraint Networks

Constraint Networks

Recall:

Definition

A constraint network is a triple

$$\mathcal{C} = \langle V, \mathrm{dom}, C \rangle$$

where:

- V is a non-empty and finite set of variables.
- dom is a function that assigns a non-empty (value) set (domain) to each variable $v \in V$.
- C is a set of relations over variables of V (constraints), i. e., each constraint is a relation $R_{v_1,\dots,v_n}$ over some variables $v_1, \dots, v_n$ in V.

Without loss of generality, we can assume that, for each subset S of variables, C contains only one constraint with scope S. The set of scopes $\{S_1, \dots S_t\}$ is called network scheme.

Example: 4-Queens Problem

Consider variables $v_1, \dots, v_4$ (associated to the columns of a 4×4-chess board). Each of these variables v_i has as its domain $\{1, \dots, 4\}$ (conceived of as the row positions of a queen in column i).

	v_1	v_2	v_3	v_4
1				
2				
3				
4				

Define then binary constraints (thus encoding possible queen movements):

$R_{v_1,v_2} := \{(1,3),(1,4),(2,4),(3,1),(4,1),(4,2)\}$

$R_{v_1,v_3} := \{(1,2),(1,4),(2,1),(2,3),(3,2),(3,4),(4,1),(4,3)\}$

...

Instantiation, Solution

Let $\mathcal{C} = \langle V, \mathrm{dom}, C\rangle$ be a constraint network.

Definition

(a) An instantiation of a subset V' of V is an assignment $a : V' \to \bigcup_{v \in V'} \mathrm{dom}(v_i)$ with $a(v_i) \in \mathrm{dom}(v_i)$.

(b) An instantiation a is a partial solution if a satisfies each constraint with scope $S \subseteq V'$ (we also say: a is consistent relative to $\mathcal{C}$).

(c) For an instantiation a of a subset $V' = \{v_1, \ldots, v_k\}$ and a constraint R with scope $S \subseteq V'$, let

$$\overline{a}[S] := (a(v_1), \ldots, a(v_k)).$$

(d) A solution is an instantiation of all variables in V that is consistent relative to $\mathcal{C}$.

Instantiation, Solution

Note:

(a) An instantiation of variables in $V' \subseteq V$, a, is consistent relative to $\mathcal{C}$ iff

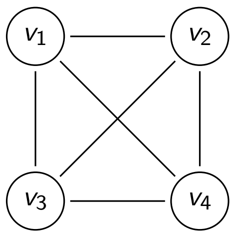
$$\overline{a}[S] \in R, \quad \text{for each constraint } R \text{ with } S \subseteq V'.$$

(b) Not each partial solution is part of a (full) solution, i. e., there may be partial solutions of a constraint network that cannot be extended to a solution. For the 4-queens problem, for example,

	v_1	v_2	v_3	v_4
1	q			
2			q	
3				
4		q		

Example: Primal Constraint Graph

The primal constraint graph for the 4-queens problem is a complete graph:



Its dual constraint graph? Its hypergraph?

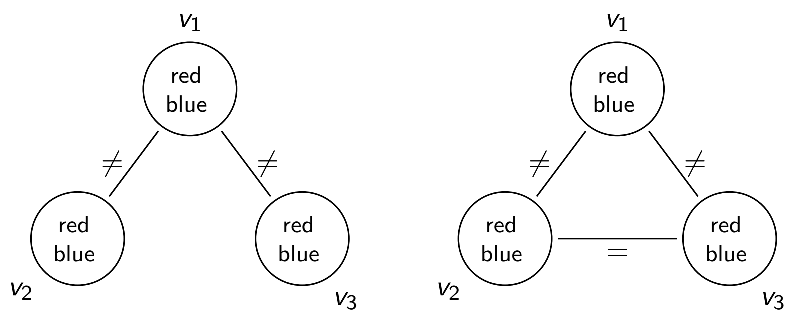
Equivalence

Let $\mathcal{C}$ and $\mathcal{C}'$ be constraint networks on the same set of variables and on the same domains for each variable.

Definition

$\mathcal{C}$ and $\mathcal{C}'$ are equivalent if each solution of $\mathcal{C}$ is a solution of $\mathcal{C}'$, and vice versa.

Example:



Tighter Networks and Equivalence

Let $\mathcal{C}$ and $\mathcal{C}'$ be constraint networks on the same set of variables and on the same domains for each variable. We assume that for each set of variables S, both networks contain at most one constraint with scope S.

Definition

$\mathcal{C}'$ is at least as tight as $\mathcal{C}$ if for each constraint R' of $\mathcal{C}'$ with scope S, it holds:

(a) $\mathcal{C}$ has no constraint with scope S, or

(b) $R' \subseteq R$, where R is the constraint of $\mathcal{C}$ with scope S.

Note: $\mathcal{C}'$ may be at least as tight as $\mathcal{C}$ although $|C'| > |C|$.

Intersection of Networks

Let $\mathcal{C}$ and $\mathcal{C}'$ be constraint networks as above.

Definition

The intersection of $\mathcal{C}$ and $\mathcal{C}'$, $\mathcal{C} \cap \mathcal{C}'$, is the network defined by intersecting for each scope S of constraints $R_S \in C$ and $R'_S \in C'$ the respective relations, i. e.,

$$R''_S := R_S \cap R'_S.$$

If for a scope S only one of the networks contains a constraint, then we set:

$$R''_S := R_S \quad (\text{or } := R'_S, \text{ resp.})$$

Lemma

If $\mathcal{C}$ and $\mathcal{C}'$ are equivalent networks, then $\mathcal{C} \cap \mathcal{C}'$ is equivalent to both networks and at least as tight as both networks.

Minimal Network

Definition

Let $\mathcal{C}_0$ be a constraint network and let $\mathcal{C}_1 \dots, \mathcal{C}_k$ be the set of *all* constraint network (defined on the same set of variables and the same domains) that are equivalent to $\mathcal{C}_0$.

$$\bigcap_{1 \leq i \leq k} \mathcal{C}_i$$

is the minimal network of $\mathcal{C}_0$.

The minimal network is equivalent to and as least as tight as all the constraint networks $\mathcal{C}_i$. There is no network equivalent to $\mathcal{C}_0$ that is tighter than the minimal network.

Projecting Relations

Let R_S be a relation with scope $S = \{v_1, \dots, v_k\}$ (we can conceive of R_S as a constraint network ...).

Definition

The projection network of R_S, $\mathrm{Proj}(R_S)$, is the constraint network defined by

$$\begin{aligned} V &:= S \\ \mathrm{dom}(v_i) &:= \pi_{v_i}(R_S) \\ R_{v_i,v_j} &:= \pi_{v_i,v_j}(R_S) \end{aligned}$$

Note: The projection network is an upper approximation by binary networks in the following sense:

Lemma

Any solution of R_S (as a network) defines a solution of $\mathrm{Proj}(R_S)$.

Binary Representation

Definition

A relation R_S with scope S has a binary representation if the relation (conceived of as a network) is equivalent to $\mathrm{Proj}(R_S)$.

From the fact that a relation has a binary representation, it does not follow that all its projections have binary representations as well (Exercise!).

Definition

A relation R_S with scope S is binary decomposable if the relation itself and all its projections to subsets of S (with at least 3 elements) have a binary representation.

Literature

Rina Dechter.
Constraint Processing,
Chapter 2, Morgan Kaufmann, 2003