Planning by satisfiability testing (May 2, 2005)

Planning in the propositional logic

- Early work on deductive planning viewed plans as proofs that lead to a desired goal (theorem).
- Planning as satisfiability testing was proposed in 1992.
  1. A propositional formula represents all length n action sequences from the initial state to a goal state.
  2. If the formula is satisfiable then a plan of length n exists.
- Satisfiability planning is the best approach to solve difficult planning problems.
  Heuristic search is often more efficient on very big but easy problems.
- Bounded model-checking in Computer Aided Verification was introduced in 1998 as an extension of satisfiability planning after the success of the latter had been noticed outside the AI community.

Planning in the propositional logic
Abstractly

1. Represent actions (= binary relations) as propositional formulae.
2. Construct a formula saying "execute one of the actions".
3. Construct a formula saying "execute a sequence of n actions, starting from the initial state, ending in a goal state."
4. Test the satisfiability of this formula by a satisfiability algorithm.
5. If the formula is satisfiable, construct a plan from a satisfying valuation.

Sets (of states) as formulae

Formulae on A as sets of states
We view formulae ϕ as representing sets of states $s : A \to \{0, 1\}$.

Example
Formula $a \vee b$ on the state variables a, b, c represents the set $\{010, 011, 100, 101, 110, 111\}$.

Relations/actions as formulae

Formulae on $A \cup A'$ as binary relations
Let $A = \{a_1, \ldots, a_n\}$ represent state variables in the current state, and $A' = \{a'_1, \ldots, a'_n\}$ state variables in the successor state.
Formulae ϕ on $A \cup A'$ represent binary relations on states: a valuation of $A \cup A' \to \{0, 1\}$ represents a pair of states $s : A \to \{0, 1\}$, $s' : A' \to \{0, 1\}$.

Example
Formula $(a \to a') \wedge ((a' \vee b) \to b')$ on a, b, a', b' represents the binary relation $\{(00, 00), (00, 01), (00, 11), (01, 01), (01, 11), (10, 11), (11, 11)\}$.

Matrices as formulae

Example (Formulae as relations as matrices)

Binary relation $\{(00, 00), (00, 01), (00, 11), (01, 01), (01, 11), (10, 11), (11, 11)\}$ can be represented as the adjacency matrix:

ab \ $a'b'$	00	01	10	11
00	1	1	0	1
01	0	1	0	1
10	0	0	0	1
11	0	0	0	1

Representation of big matrices is possible
For n state variables a formula (over $2n$ variables) represents an adjacency matrix of size $2^n \times 2^n$.
For $n = 20$, matrix size is $2^{20} \times 2^{20} \sim 10^6 \times 10^6$.

Actions/relations as propositional formulae
Example

$\phi = (a_1 \leftrightarrow \neg a'_1) \wedge (a_2 \leftrightarrow \neg a'_2)$ as a matrix

a_1a_2 \ $a'_1a'_2$	0 0	0 1	1 0	1 1
00	0	0	0	1
01	0	0	1	0
10	0	1	0	0
11	1	0	0	0

and as a conventional truth-table:

a_1	a_2	a'_1	a'_2	ϕ
0	0	0	0	0
0	0	0	1	0
0	0	1	0	0
0	0	1	1	1
0	1	0	0	0
0	1	0	1	0
0	1	1	0	1
0	1	1	1	0
1	0	0	0	0
1	0	0	1	1
1	0	1	0	0
1	0	1	1	0
1	1	0	0	1
1	1	0	1	0
1	1	1	0	0
1	1	1	1	0

Actions/relations as propositional formulae
Example

$(a_1 \leftrightarrow a'_2) \wedge (a_2 \leftrightarrow a'_3) \wedge (a_3 \leftrightarrow a'_1)$ represents the matrix:

	000	001	010	011	100	101	110	111
000	1	0	0	0	0	0	0	0
001	0	0	0	0	1	0	0	0
010	0	1	0	0	0	0	0	0
011	0	0	0	0	0	1	0	0
100	0	0	1	0	0	0	0	0
101	0	0	0	0	0	0	1	0
110	0	0	0	1	0	0	0	0
111	0	0	0	0	0	0	0	1

This action rotates the value of the state variables a_1, a_2, a_3 one step forward.

Deterministic vs. nondeterministic actions

Expressiveness of propositional logic

- For every operator there is a corresponding formula (see next slides!)
- Our current definition of operators does not allow expressing nondeterministic actions.
- In the propositional logic they can be expressed.

Example (A nondeterministic action)

The formula $\top$ describes the relation in which any state can be reached from any other state by this action.

A sufficient (but not necessary) condition for determinism

Formula has the form $(\phi_1 \leftrightarrow a'_1) \wedge \cdots \wedge (\phi_n \leftrightarrow a'_n)$ where $A = \{a_1, \ldots, a_n\}$ and ϕ_i have no occurrences of propositions in A'.

Deterministic vs. nondeterministic actions

Example

Example

An action that is applicable if a is false, and that randomly sets values to state variables b and c:

	$a'b'c'$	$a'b'c'$	$a'b'c'$	$a'b'c'$	$a'b'c'$	$a'b'c'$	$a'b'c'$	$a'b'c'$
abc	000	001	010	011	100	101	110	111
000	1	1	1	1	0	0	0	0
001	1	1	1	1	0	0	0	0
010	1	1	1	1	0	0	0	0
011	1	1	1	1	0	0	0	0
100	0	0	0	0	0	0	0	0
101	0	0	0	0	0	0	0	0
110	0	0	0	0	0	0	0	0
111	0	0	0	0	0	0	0	0

Corresponding formula: $\neg a \wedge \neg a'$

Translating operators into formulae

- Any operator can be translated into a propositional formula.
- Translation takes polynomial time.
- Resulting formula has polynomial size.
- Use in planning algorithms. Two main applications are
  1. Planning as Satisfiability
  2. Progression & regression for state sets as used in symbolic state-space traversal, as typically implemented with the help of binary decision diagrams.

Translating operators into formulae

Definition

Let $o = \langle c, e \rangle$ be an operator and A a set of state variables.
Define $\tau_A(o)$ as the conjunction of

$$c \qquad (1)$$

$$\bigwedge_{a \in A} (\mathit{EPC}_a(e) \vee (a \wedge \neg \mathit{EPC}_{\neg a}(e))) \leftrightarrow a' \qquad (2)$$

$$\bigwedge_{a \in A} \neg(\mathit{EPC}_a(e) \wedge \mathit{EPC}_{\neg a}(e)) \qquad (3)$$

(2) says that the new value of a, represented by a', is 1 if the old value was 1 and it did not become 0, or it became 1.
(3) says that none of the state variables is assigned both 0 and 1. This together with c determine whether the operator is applicable.

Translating operators into formulae

Example

Example

Let the state variables be $A = \{a, b, c\}$.
Consider operator $\langle a \vee b, (b \rhd a) \wedge (c \rhd \neg a) \wedge (a \rhd b) \rangle$.
The corresponding propositional formula is

$$\begin{aligned}(a \vee b) &\wedge ((b \vee (a \wedge \neg c)) \leftrightarrow a') \\ &\wedge ((a \vee (b \wedge \neg \bot)) \leftrightarrow b') \\ &\wedge ((\bot \vee (c \wedge \neg \bot)) \leftrightarrow c') \\ &\wedge \neg(b \wedge c) \wedge \neg(a \wedge \bot) \wedge \neg(\bot \wedge \bot) \\ \equiv& \\ (a \vee b) &\wedge ((b \vee (a \wedge \neg c)) \leftrightarrow a') \\ &\wedge ((a \vee b) \leftrightarrow b') \\ &\wedge (c \leftrightarrow c') \\ &\wedge \neg(b \wedge c)\end{aligned}$$

Translating operators into formulae

Example

Example

Let $A = \{a, b, c, d, e\}$ be the state variables.
Consider operator $\langle a \wedge b, c \wedge (d \rhd e) \rangle$.
The formula $\tau_A(o)$ after simplifications is

$$(a \wedge b) \wedge (a \leftrightarrow a') \wedge (b \leftrightarrow b') \wedge c' \wedge (d \leftrightarrow d') \wedge ((d \vee e) \leftrightarrow e')$$

Correctness

Lemma

Let s and s' be states and o an operator. Let $v : A \cup A' \rightarrow \{0, 1\}$ be a valuation such that

1. *for all $a \in A$, $v(a) = s(a)$, and*
2. *for all $a \in A$, $v(a') = s'(a)$.*

Then $v \models \tau_A(o)$ if and only if $s' = app_o(s)$.

Planning as satisfiability

1. Encode operator sequences of length 0, 1, 2, ... as formulae Φ_0^{seq}, Φ_1^{seq}, Φ_2^{seq}, ... (see next slide...)
2. Test satisfiability of Φ_0^{seq}, Φ_1^{seq}, Φ_2^{seq}, ...
3. If a satisfying valuation v is found, a plan can constructed from v.

Planning as satisfiability

Definition (Transition relation in CPC)

For $\langle A, I, O, G\rangle$ define

$$\mathcal{R}_1(A, A') = \bigvee_{o \in O} \tau_A(o).$$

Definition (Bounded-length plans in CPC)

Existence of plans length t is represented by a formula over propositions $A^0 \cup \cdots \cup A^t$ where $A^i = \{a^i | a \in A\}$ for all $i \in \{0, \ldots, t\}$ as

$$\Phi_t^{seq} = \iota^0 \wedge \mathcal{R}_1(A^0, A^1) \wedge \mathcal{R}_1(A^1, A^2) \wedge \cdots \wedge \mathcal{R}_1(A^{t-1}, A^t) \wedge G^t$$

where $\iota^0 = \bigwedge\{a^0 | a \in A, I(a) = 1\} \cup \{\neg a^0 | a \in A, I(a) = 0\}$ and G^t is G with propositions a replaced by a^t.

Planning as satisfiability

Example

Example

Consider

$$I \models b \wedge c$$
$$G = (b \wedge \neg c) \vee (\neg b \wedge c)$$
$$o_1 = \langle \top, (c \rhd \neg c) \wedge (\neg c \rhd c)\rangle$$
$$o_2 = \langle \top, (b \rhd \neg b) \wedge (\neg b \rhd b)\rangle.$$

Formula for plans of length 3 is

$$\begin{aligned}
&(b^0 \wedge c^0)\\
&\wedge(((b^0 \leftrightarrow b^1) \wedge (c^0 \leftrightarrow \neg c^1)) \vee ((b^0 \leftrightarrow \neg b^1) \wedge (c^0 \leftrightarrow c^1)))\\
&\wedge(((b^1 \leftrightarrow b^2) \wedge (c^1 \leftrightarrow \neg c^2)) \vee ((b^1 \leftrightarrow \neg b^2) \wedge (c^1 \leftrightarrow c^2)))\\
&\wedge(((b^2 \leftrightarrow b^3) \wedge (c^2 \leftrightarrow \neg c^3)) \vee ((b^2 \leftrightarrow \neg b^3) \wedge (c^2 \leftrightarrow c^3)))\\
&\wedge((b^3 \wedge \neg c^3) \vee (\neg b^3 \wedge c^3)).
\end{aligned}$$

Planning as satisfiability

Existence of (optimal) plans

Theorem

Let Φ_t^{seq} be the formula for $\langle A, I, O, G\rangle$ and plan length t. The formula Φ_t^{seq} is satisfiable if and only if there is a sequence of states $s_0, \ldots, s_t$ and operators $o_1, \ldots, o_t$ such that $s_0 = I$, $s_t \models G$ and $s_i = app_{o_i}(s_{i-1})$ for all $i \in \{1, \ldots, t\}$.

Consequence

If $\Phi_0^{seq}, \Phi_1^{seq}, \ldots, \Phi_{i-1}^{seq}$ are unsatisfiable and Φ_i^{seq} is satisfiable, then the length of shortest plans is i.

Satisfiability planning with Φ_i^{seq} yields optimal plans, like heuristic search with admissible heuristics and optimal algorithms like A$*$ or IDA$*$.

Planning as satisfiability

Plan extraction

All satisfiability algorithms give a valuation v that satisfies Φ_i^{seq} upon finding out that Φ_i^{seq} is satisfiable.

This makes it possible to construct a plan.

Constructing a plan from a satisfying valuation

Let v be a valuation so that $v \models \Phi_t^{seq}$. Then define $s_i(a) = v(a^i)$ for all $a \in A$ and $i \in \{0, \ldots, t\}$.

The ith operator in the plan is $o \in O$ if $app_o(s_{i-1}) = s_i$. Notice: There may be more than one such operator, any of them may be chosen.

Planning as satisfiability

Example, continued

Example

One valuation that satisfies Φ_3^{seq}:

	time i			
	0	1	2	3
b^i	1	1	0	0
c^i	1	0	0	1

Notice:

1. Also a plan of length 1 exists.
2. Plans of length 2 do not exist.

Conjunctive normal form

Many satisfiability algorithms require formulas in the conjunctive normal form: transformation by repeated applications of the following equivalences.

$$\begin{aligned}
\neg(\phi \vee \psi) &\equiv \neg\phi \wedge \neg\psi\\
\neg(\phi \wedge \psi) &\equiv \neg\phi \vee \neg\psi\\
\neg\neg\phi &\equiv \phi\\
\phi \vee (\psi_1 \wedge \psi_2) &\equiv (\phi \vee \psi_1) \wedge (\phi \vee \psi_2)
\end{aligned}$$

The formula is conjunction of clauses (disjunctions of literals).

Example

$(A \vee \neg B \vee C) \wedge (\neg C \vee \neg B) \wedge A$

The unit resolution rule

Unit resolution

From $l_1 \vee l_2 \vee \cdots \vee l_n$ (here $n \geq 1$) and $\overline{l_1}$ infer $l_2 \vee \cdots \vee l_n$.

Example

From $a \vee b \vee c$ and $\neg a$ infer $b \vee c$.

Unit resolution: a special case

From A and $\neg A$ we get the empty clause $\bot$ ("disjunction consisting of zero disjuncts").

Unit subsumption

The clause $l_1 \vee l_2 \vee \cdots \vee l_n$ can be eliminated if we have the unit clause l_1.

The Davis-Putnam procedure

- The first efficient decision procedure for any logic (Davis, Putnam, Logemann & Loveland, 1960/62).
- Based on binary search through the valuations of a formula.
- Unit resolution and unit subsumption help pruning the search tree.
- The currently most efficient satisfiability algorithms are variants of the Davis-Putnam procedure
  (Although there is currently a shift toward viewing these procedures as performing more general resolution: clause-learning.)

Satisfiability test by the Davis-Putnam procedure

1. Let C be a set of clauses.
2. For all clauses $l_1 \vee l_2 \vee \cdots \vee l_n \in C$ and $\overline{l_1} \in C$, remove $l_1 \vee l_2 \vee \cdots \vee l_n$ from C and add $l_2 \vee \cdots \vee l_n$ to C.
3. For all clauses $l_1 \vee l_2 \vee \cdots \vee l_n \in C$ and $l_1 \in C$, remove $l_1 \vee l_2 \vee \cdots \vee l_n$ from C. (UNIT SUBSUMPTION)
4. If $\bot \in C$, return FALSE.
5. If C contains only unit clauses, return TRUE.
6. Pick some $a \in A$ such that $\{a, \neg a\} \cap C = \emptyset$
7. Recursive call: if $C \cup \{a\}$ is satisfiable, return TRUE.
8. Recursive call: if $C \cup \{\neg a\}$ is satisfiable, return TRUE.
9. Return FALSE.

Planning as satisfiability

Example: plan search with Davis-Putnam

Consider the problem from a previous slide, with two operators each inverting the value of one state variable, for plan length 3.

$$
\begin{aligned}
&(b^0 \wedge c^0)\\
&\wedge(((b^0 \leftrightarrow b^1) \wedge (c^0 \leftrightarrow \neg c^1)) \vee ((b^0 \leftrightarrow \neg b^1) \wedge (c^0 \leftrightarrow c^1)))\\
&\wedge(((b^1 \leftrightarrow b^2) \wedge (c^1 \leftrightarrow \neg c^2)) \vee ((b^1 \leftrightarrow \neg b^2) \wedge (c^1 \leftrightarrow c^2)))\\
&\wedge(((b^2 \leftrightarrow b^3) \wedge (c^2 \leftrightarrow \neg c^3)) \vee ((b^2 \leftrightarrow \neg b^3) \wedge (c^2 \leftrightarrow c^3)))\\
&\wedge((b^3 \wedge \neg c^3) \vee (\neg b^3 \wedge c^3)).
\end{aligned}
$$

Planning as satisfiability

Example: plan search with Davis-Putnam

To obtain a short CNF formula, we introduce auxiliary variables o_1^i and o_2^i for $i \in \{1,2,3\}$ denoting operator applications.

$$
\begin{array}{ll}
b^0 & o_1^1 \rightarrow ((b^0 \leftrightarrow b^1) \wedge (c^0 \leftrightarrow \neg c^1))\\
c^0 & o_2^1 \rightarrow ((b^0 \leftrightarrow \neg b^1) \wedge (c^0 \leftrightarrow c^1))\\
o_1^1 \vee o_2^1 & o_1^2 \rightarrow ((b^1 \leftrightarrow b^2) \wedge (c^1 \leftrightarrow \neg c^2))\\
o_1^2 \vee o_2^2 & o_2^2 \rightarrow ((b^1 \leftrightarrow \neg b^2) \wedge (c^1 \leftrightarrow c^2))\\
o_1^3 \vee o_2^3 & o_1^3 \rightarrow ((b^2 \leftrightarrow b^3) \wedge (c^2 \leftrightarrow \neg c^3))\\
(b^3 \wedge \neg c^3) \vee (\neg b^3 \wedge c^3) & o_2^3 \rightarrow ((b^2 \leftrightarrow \neg b^3) \wedge (c^2 \leftrightarrow c^3))
\end{array}
$$

Planning as satisfiability

Example: plan search with Davis-Putnam

We rewrite the formulae for operator applications by using the equivalence $\phi \rightarrow (l \leftrightarrow l') \equiv ((\phi \wedge l \rightarrow l') \wedge (\phi \wedge \bar{l} \rightarrow \overline{l'}))$.

$$
\begin{array}{llll}
b^0 & o_1^1 \wedge b^0 \rightarrow b^1 & o_1^2 \wedge b^1 \rightarrow b^2 & o_1^3 \wedge b^2 \rightarrow b^3\\
c^0 & o_1^1 \wedge \neg b^0 \rightarrow \neg b^1 & o_1^2 \wedge \neg b^1 \rightarrow \neg b^2 & o_1^3 \wedge \neg b^2 \rightarrow \neg b^3\\
o_1^1 \vee o_2^1 & o_1^1 \wedge c^0 \rightarrow \neg c^1 & o_1^2 \wedge c^1 \rightarrow \neg c^2 & o_1^3 \wedge c^2 \rightarrow \neg c^3\\
o_1^2 \vee o_2^2 & o_1^1 \wedge \neg c^0 \rightarrow c^1 & o_1^2 \wedge \neg c^1 \rightarrow c^2 & o_1^3 \wedge \neg c^2 \rightarrow c^3\\
o_1^3 \vee o_2^3 & o_2^1 \wedge b^0 \rightarrow \neg b^1 & o_2^2 \wedge b^1 \rightarrow \neg b^2 & o_2^3 \wedge b^2 \rightarrow \neg b^3\\
b^3 \vee c^3 & o_2^1 \wedge \neg b^0 \rightarrow b^1 & o_2^2 \wedge \neg b^1 \rightarrow b^2 & o_2^3 \wedge \neg b^2 \rightarrow b^3\\
\neg c^3 \vee \neg b^3 & o_2^1 \wedge c^0 \rightarrow c^1 & o_2^2 \wedge c^1 \rightarrow c^2 & o_2^3 \wedge c^2 \rightarrow c^3\\
 & o_2^1 \wedge \neg c^0 \rightarrow c^1 & o_2^2 \wedge \neg c^1 \rightarrow c^2 & o_2^3 \wedge \neg c^2 \rightarrow c^3
\end{array}
$$

Planning as satisfiability with parallel plans

- Efficiency of satisfiability planning is strongly dependent on the plan length because satisfiability algorithms have runtime $O(2^n)$ where n is the formula size, and formula sizes are linearly proportional to plan length.
- Formula sizes can be reduced by allowing several operators in parallel.
- On many problems this leads to big speed-ups.
- However there are no guarantees of optimality.

Parallel operator application

Formal definition

We consider the possibility of executing several operators simultaneously.

Definition

Let T be a set of operators and s a state.
Define $app_T(s)$ as the state that is obtained from s by making the literals in $\bigcup_{\langle c,e\rangle \in T}[e]_s$ true.
For $app_T(s)$ to be defined, we require that $s \models c$ for all $o = \langle c,e\rangle \in T$ and $\bigcup_{\langle c,e\rangle \in T}[e]_s$ is consistent.

Parallel operator application

Representation in CPC

Consider the formula $\tau_A(o)$ representing operator $o = \langle c, e\rangle$

$$
\begin{aligned}
&c \wedge\\
&\textstyle\bigwedge_{a \in A}((EPC_a(e) \vee (a \wedge \neg EPC_{\neg a}(e))) \leftrightarrow a') \wedge\\
&\textstyle\bigwedge_{a \in A} \neg(EPC_a(e) \wedge EPC_{\neg a}(e)).
\end{aligned}
$$

This can be logically equivalently be written as follows.

$$
\begin{aligned}
&c \wedge\\
&\textstyle\bigwedge_{a \in A}(EPC_a(e) \rightarrow a') \wedge\\
&\textstyle\bigwedge_{a \in A}(EPC_{\neg a}(e) \rightarrow \neg a') \wedge\\
&\textstyle\bigwedge_{a \in A}((a \wedge \neg EPC_{\neg a}(e)) \rightarrow a') \wedge\\
&\textstyle\bigwedge_{a \in A}((\neg a \wedge \neg EPC_a(e)) \rightarrow \neg a')
\end{aligned}
$$

This separates the changes from non-changes. This is the basis of the translation for parallel actions for which we do not say that executing a given operator directly means that unrelated state variables retain their old value.

The explanatory frame axioms

The formulae say that the only explanation for a changing its value is the application of one operator.

$$
\begin{aligned}
&\textstyle\bigwedge_{a \in A}((a \wedge \neg a') \rightarrow EPC_{\neg a}(e))\\
&\textstyle\bigwedge_{a \in A}((\neg a \wedge a') \rightarrow EPC_a(e))
\end{aligned}
$$

When several operators could be applied in parallel, we have to consider all operators as possible explanations.

$$
\begin{aligned}
&\textstyle\bigwedge_{a \in A}((a \wedge \neg a') \rightarrow ((o_1 \wedge EPC_{\neg a}(e_1)) \vee \cdots \vee (o_n \wedge EPC_{\neg a}(e_n))\\
&\textstyle\bigwedge_{a \in A}((\neg a \wedge a') \rightarrow ((o_1 \wedge EPC_a(e_1)) \vee \cdots \vee (o_n \wedge EPC_a(e_n)))
\end{aligned}
$$

where $T = \{o_1, \ldots, o_n\}$ and $e_1, \ldots, e_n$ are the respective effects.

Parallel actions
Formula in CPC

Definition
Let T be a set of operators. Let $\tau_A(T)$ denote the conjunction of formulae

$$(o \rightarrow c) \wedge$$
$$\bigwedge_{a \in A}(o \wedge EPC_a(e) \rightarrow a') \wedge$$
$$\bigwedge_{a \in A}(o \wedge EPC_{\neg a}(e) \rightarrow \neg a')$$

for all $\langle c, e\rangle \in T$ and

$$\bigwedge_{a \in A}((a \wedge \neg a') \rightarrow ((o_1 \wedge EPC_{\neg a}(e_1)) \vee \cdots \vee (o_n \wedge EPC_{\neg a}(e_n))$$
$$\bigwedge_{a \in A}((\neg a \wedge a') \rightarrow ((o_1 \wedge EPC_{a}(e_1)) \vee \cdots \vee (o_n \wedge EPC_{a}(e_n)))$$

where $T = \{o_1, \ldots, o_n\}$ and $e_1, \ldots, e_n$ are the respective effects.

Correctness

The formula $\tau_A(T)$ exactly matches the definition of $app_T(s)$.

Lemma
Let s and s' be states and T a set of operators. Let $v : A \cup A' \cup T \rightarrow \{0, 1\}$ be a valuation such that

1. *for all $o \in T$, $v(o) = 1$,*
2. *for all $a \in A$, $v(a) = s(a)$, and*
3. *for all $a \in A$, $v(a') = s'(a)$.*

Then $v \models \tau_A(T)$ if and only if $s' = app_T(s)$.

Parallel actions
Meaning in terms of interleavings

Example
The operators $\langle a, \neg b\rangle$ and $\langle b, \neg a\rangle$ may be executed simultaneously resulting in a state satisfying $\neg a \wedge \neg b$.
But this state is not reachable by the two operators sequentially, because executing any one operator makes the precondition of the other false.

Step plans
Formal definition

Definition (Step plans)
For a set of operators O and an initial state I, a step plan for O and I is a sequence $T = \langle T_0, \ldots, T_{l-1}\rangle$ of sets of operators for some $l \geq 0$ such that there is a sequence of states $s_0, \ldots, s_l$ (the execution of T) such that

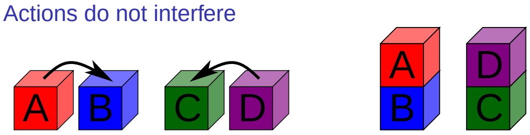
1. $s_0 = I$,
2. for all $i \in \{0, \ldots, l-1\}$ and every total ordering $o_1, \ldots, o_n$ of T_i, $app_{o_1;\ldots;o_n}(s_i)$ is defined and equals s_{i+1}.

Step plans
Tractable subclass

- Finding arbitrary step plans is difficult: even testing whether a set T of operators is executable in all orders is co-NP-hard.
- Representing the executability test exactly as a propositional formula seems complicated: doing this test exactly would seem to cancel the benefits of parallel plans.
- Instead, all work on parallel plans so far has used a sufficient but not necessary condition that can be tested in polynomial-time.
- This is a simple syntactic test: is the result of executing o_1 and o_2 in any state both in order $o_1; o_2$ and in $o_2; o_1$ the same.

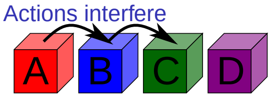
Interference
Example

Actions do not interfere



Actions can be taken simultaneously.

Actions interfere



If A is moved first, B won't be clear and cannot be moved.

Interference
Auxiliary definition: affects

Definition (Affect)
Let A be a set of state variables and $o = \langle c, e\rangle$ and $o' = \langle c', e'\rangle$ operators over A. Then o affects o' if there is $a \in A$ such that

1. a is an atomic effect in e and a occurs in a formula in e' or it occurs negatively in c', or
2. $\neg a$ is an atomic effect in e and a occurs in a formula in e' or it occurs positively in c'.

Example
$\langle c, d\rangle$ affects $\langle \neg d, e\rangle$ and $\langle e, d \triangleright f\rangle$.
$\langle c, d\rangle$ does not affect $\langle d, e\rangle$ nor $\langle e, \neg c\rangle$.

Interference

Definition (Interference)
Operators o and o' interfere if o affects o' or o' affects o.

Example
$\langle c, d\rangle$ and $\langle \neg d, e\rangle$ interfere.
$\langle c, d\rangle$ and $\langle e, f\rangle$ do not interfere.

Interference

Lemma

Let s be a state and T a set of operators so that $app_T(s)$ is defined and no two operators interfere.
Then $app_T(s) = app_{o_1;\ldots;o_n}(s)$ for any total ordering $o_1, \ldots, o_n$ of T.

The translation for parallel plans in CPC

Definition

Define $\mathcal{R}_2(A, A', O)$ as the conjunction of $\tau_A(O)$ and

$$\neg(o \wedge o')$$

for all $o \in O$ and $o' \in O$ such that o and o' interfere and $o \neq o'$.

Planning as satisfiability

Existence of plans

Definition (Bounded-length plans in CPC)

Existence of parallel plans length t is represented by a formula over propositions $A^0 \cup \cdots \cup A^t \cup O^1 \cup \cdots \cup O^t$ where $A^i = \{a^i | a \in A\}$ for all $i \in \{0, \ldots, t\}$ and $O^i = \{o^i | o \in O\}$ for all $i \in \{1, \ldots t\}$ as

$$\Phi_t^{par} = \iota^0 \wedge \mathcal{R}_2(A^0, A^1, O^1) \wedge \cdots \wedge \mathcal{R}_2(A^{t-1}, A^t, O^t) \wedge G^t$$

where $\iota^0 = \bigwedge\{a^0 | a \in A, I(a) = 1\} \cup \{\neg a^0 | a \in A, I(a) = 0\}$ and G^t is G with propositions a replaced by a^t.

Planning as satisfiability

Existence of plans

Theorem

Let Φ_t^{par} be the formula for $\langle A, I, O, G\rangle$ and plan length t. The formula Φ_t^{par} is satisfiable if and only if there is a sequence of states $s_0, \ldots, s_t$ and sets $O_1, \ldots, O_t$ of non-interfering operators such that $s_0 = I$, $s_t \models G$ and $s_i = app_{O_i}(s_{i-1})$ for all $i \in \{1, \ldots, t\}$.

Why is optimality lost?

For parallel plans there is no guarantee for smallest number of operators

That a plan has the smallest number of time points does not guarantee that it has the smallest number of actions.

- Satisfiability algorithms return any satisfying valuation of Φ_i^{par}, and this does not have to be the one with the smallest number of operators.
- There could be better solutions with more time points.

Why is optimality lost?

Example

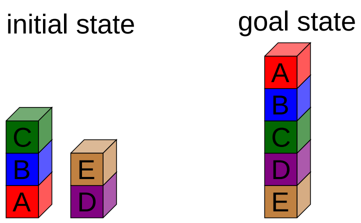
Let I be a state such that $s \models \neg c \wedge \neg d \wedge \neg e \wedge \neg f$.
Let $G = c \wedge d \wedge e$.
Let

$$\begin{aligned} o_1 &= \langle \top, c\rangle \\ o_2 &= \langle \top, d\rangle \\ o_3 &= \langle \top, e\rangle \\ o_4 &= \langle \top, f\rangle \\ o_5 &= \langle f, c \wedge d \wedge e\rangle \end{aligned}$$

Now $\{o_1, o_2, o_3\}$ is a plan with one step, and $\{o_4\}; \{o_5\}$ is a plan with two steps. The first one has less time steps and corresponds to a satisfying valuation of both Φ_1^{par} and Φ_2^{par}.

Planning as satisfiability

Example



The Davis-Putnam procedure solves the problem quickly:

- Formulae for lengths 1 to 4 shown unsatisfiable without any search.
- Formula for plan length 5 is satisfiable: 3 nodes in the search tree.
- Plans have 5 to 7 operators, optimal plan has 5.

Planning as satisfiability

Example

```
v0.9 13/08/1997 19:32:47
30 propositions 100 operators
Length 1
Length 2
Length 3
Length 4
Length 5
branch on -clear(b)[1] depth 0
branch on clear(a)[3] depth 1
Found a plan.
  0 totable(e,d)
  1 totable(c,b) fromtable(d,e)
  2 totable(b,a) fromtable(c,d)
  3 fromtable(b,c)
  4 fromtable(a,b)
Branches 2 last 2 failed 0; time 0.0
```

Planning as satisfiability

Example: valuations after unit propagation, after branching

```
     ON                  ON
  CLEARaaaabbbbccccddddeeeeTABLE
  abcdebcdeacdeabdeabceabcdabcde

0 FFTFTFFFFTFFFFTFFFFFFFFFTTFFTF
1 F TTTFFFFTFFFF FFFFFFFFFFTF TT
2    TFFFFF  FFF FFFFFTFFFFT  FT
3    FF FFF  FFFFTFFFFTFFFF  FFT
4   FFF FFFFTFFFFTFFFFTFFFF FFFT
5  FFFFTFFFFTFFFFTFFFFTFFFFFFFFFT


0 FFTFTFFFFTFFFFTFFFFFFFFFTTFFTF
1 FFTTTFFFFTFFFFTFFFFFFFFFFTFFTT
2 F TTFFFFFTFFFF FFFFFTFFFFTF FT
3  TTFFFFFF FFFFFTFFFFTFFFFT FFT
4 TTFFFFFFFFTFFFFTFFFFTFFFFTFFFT
5 TFFFFTFFFFTFFFFTFFFFTFFFFFFFFFT


0 FFTFTFFFFTFFFFTFFFFFFFFFTTFFTF
1 FFTTTFFFFTFFFFTFFFFFFFFFFTFFTT
2 FTTTFFFFFTFFFFFFFFFFTFFFFTFTFT
3 TTTFFFFFFFFFFFFTFFFFTFFFFTTFFT
4 TTFFFFFFFFTFFFFTFFFFTFFFFTFFFT
5 TFFFFTFFFFTFFFFTFFFFTFFFFFFFFFT
```


Planning as satisfiability

Example: valuations after unit propagation, after branching

```
             012345   012345   012345
    clear(a) FF       FFF TT   FFFTTT
    clear(b) F    F   FF TTF   FFTTTF
    clear(c) TT  FF   TTTTFF   TTTTFF
    clear(d) FTTFFF   FTTFFF   FTTFFF
    clear(e) TTFFFF   TTFFFF   TTFFFF
     on(a,b) FFF  T   FFFFFT   FFFFFT
     on(a,c) FFFFFF   FFFFFF   FFFFFF
     on(a,d) FFFFFF   FFFFFF   FFFFFF
     on(a,e) FFFFFF   FFFFFF   FFFFFF
     on(b,a) TT  FF   TTT FF   TTTFFF
     on(b,c) FF  TT   FFFFTT   FFFFTT
     on(b,d) FFFFFF   FFFFFF   FFFFFF
     on(b,e) FFFFFF   FFFFFF   FFFFFF
     on(c,a) FFFFFF   FFFFFF   FFFFFF
     on(c,b) T  FFF   TT FFF   TTFFFF
     on(c,d) FFFTTT   FFFTTT   FFFTTT
     on(c,e) FFFFFF   FFFFFF   FFFFFF
     on(d,a) FFFFFF   FFFFFF   FFFFFF
     on(d,b) FFFFFF   FFFFFF   FFFFFF
     on(d,c) FFFFFF   FFFFFF   FFFFFF
     on(d,e) FFTTTT   FFTTTT   FFTTTT
     on(e,a) FFFFFF   FFFFFF   FFFFFF
     on(e,b) FFFFFF   FFFFFF   FFFFFF
     on(e,c) FFFFFF   FFFFFF   FFFFFF
     on(e,d) TFFFFF   TFFFFF   TFFFFF
  ontable(a) TTT  F   TTTTTF   TTTTTF
  ontable(b) FF  FF   FFF FF   FFFTFF
  ontable(c) F  FFF   FF FFF   FFTFFF
  ontable(d) TTFFFF   TTFFFF   TTFFFF
  ontable(e) FTTTTT   FTTTTT   FTTTTT
```


Planning as satisfiability

Example: valuation for operators after plan has been found

```
               01234
fromtable(a,b) ....T
fromtable(b,c) ...T.
fromtable(c,d) ..T..
fromtable(d,e) .T...
  totable(b,a) ..T..
  totable(c,b) .T...
  totable(e,d) T....
```