Nondeterministic effects: definition

Effects are recursively defined as follows.

1. a and $\neg a$ for state variables $a \in P$ are effects.
2. $e_1 \wedge \cdots \wedge e_n$ is an effect if $e_1, \ldots, e_n$ are effects (the special case with $n = 0$ is the empty conjunction $\top$.)
3. $c \triangleright e$ is an effect if c is a formula over P and e is an effect.
4. $p_1 e_1 | \cdots | p_n e_n$ is an effect if $e_1, \ldots, e_n$ for $n \geq 0$ are effects, $p_i > 0$ for all $i \in \{1, \ldots, n\}$ and $\sum_{i=1}^{n} p_i = 1$.

Nondeterministic effects: semantics

Let $\langle c, e \rangle$ be an operator over P and s a state. The operator is applicable in s if $s \models c$. Define $[e]_s$ recursively as follows.

1. $[a]_s = \{\{a\}\}$ and $[\neg a]_s = \{\{\neg a\}\}$ for $a \in P$.
2. $[e_1 \wedge \cdots \wedge e_n]_s = \{\bigcup_{i=1}^{n} f_i | f_1 \in [e_1]_s, \ldots, f_n \in [e_n]_s\}$.
3. $[c' \triangleright e']_s = [e']_s$ if $s \models c'$ and $[c' \triangleright e']_s = \{\emptyset\}$ otherwise.
4. $[p_1 e_1 | \cdots | p_n e_n]_s = [e_1]_s \cup \cdots \cup [e_n]_s$

The successor states are determined by members of $[e]_s$.

Nondeterministic effects: probabilistic semantics

1. $[a]_s = \{\langle 1, \{a\} \rangle\}$ and $[\neg a]_s = \{\langle 1, \{\neg a\} \rangle\}$ for $a \in P$.
2. $[e_1 \wedge \cdots \wedge e_n]_s = \{\langle \Pi_{i=1}^{n} p_i, \bigcup_{i=1}^{n} f_i \rangle | \langle p_1, f_1 \rangle \in [e_1]_s, \ldots, \langle p_n, f_n \rangle \in [e_n]_s\}$.
3. $[c' \triangleright e']_s = [e']_s$ if $s \models c'$ and $[c' \triangleright e']_s = \{\langle 1, \emptyset \rangle\}$ otherwise.
4. $[p_1 e_1 | \cdots | p_n e_n]_s = \{\langle p_1 \cdot p, e \rangle | \langle p, e \rangle \in [e_1]_s\} \cup \cdots \cup \{\langle p_n \cdot p, e \rangle | \langle p, e \rangle \in [e_n]_s\}$

Nondeterministic effects: probabilistic semantics

Consider a state s with $s \models A \wedge B \wedge C$. Successor states under e and their probabilities are determined by members of $[e]_s$.

Let $[e]_s = \{\langle 0.3, \{A, \neg B\} \rangle, \langle 0.5, \{\neg B\} \rangle, \langle 0.2, \{\neg B, \neg C\} \rangle\}$.

With e state s has two successor states s_1 and s_2 such that

$$\begin{aligned} s_1 &\models A \wedge \neg B \wedge C \\ s_2 &\models A \wedge \neg B \wedge \neg C \end{aligned}$$

Probability of reaching s_1 is 0.8 and that of s_2 is 0.2.

Nondeterministic effects: examples

$a \rhd (0.5(\neg a \wedge b \wedge c)|0.5(a \wedge \neg b \wedge \neg c))$

$0.5(a \rhd (a \wedge b \wedge c))|0.5(a \rhd (b \wedge \neg c))$

$0.5(a \rhd (a \wedge b \wedge c))|0.5\top$

$(0.2a|0.8b) \wedge (0.3c|0.7d) \wedge (0.4e|0.6f)$

$(0.2a|0.8b) \wedge (0.3\neg a|0.7d)$ **not well-defined**

Nondeterministic effects: more equivalences

$$\begin{aligned}
c \rhd (p_1e_1|\cdots|p_ne_n) &\equiv p_1(c \rhd e_1)|\cdots|p_n(c \rhd e_n) \\
e \wedge (p_1e_1|\cdots|p_ne_n) &\equiv p_1(e \wedge e_1)|\cdots|p_n(e \wedge e_n) \\
p_1(p'_1e'_1|\cdots|p'_ne'_n)|p_2e_2|\cdots|p_ne_n &\equiv (p_1p'_1)e'_1|\cdots|(p_1p'_n)e'_n|p_2e_2|\cdots|p_ne_n
\end{aligned}$$

Nondeterministic effects: normal form I

DEFINITION: Let $e_1, \ldots, e_n$ be any deterministic effects in normal form as defined earlier. Then e obtained from $e_1, \ldots, e_n$ with $\wedge$ and $|$ is in normal form I.

EXAMPLES:

$(0.5((a \rhd b) \wedge (b \rhd \neg c) \wedge (d \rhd c))|0.5(a \rhd b)) \wedge (d \rhd \neg c)$
$(0.5(a \rhd b)|0.4(b \rhd q)|0.1(j \rhd e)) \wedge (0.3(\neg a \rhd \neg b)|0.7(c \rhd \neg c))$

THEOREM: Any effect can be transformed into normal form I in polynomial time.

Nondeterministic effects: normal form II

DEFINITION: Let $e_1, \ldots, e_n$ for $n \geq 0$ be any deterministic effects in normal form as defined earlier. Then $e = p_1e_1|\cdots|p_ne_n$ is in normal form II.

EXAMPLES:

$(a \rhd b) \wedge (c \rhd \neg d)$
$0.3((a \rhd b) \wedge (c \rhd \neg d))|0.2(g \rhd \neg h)|0.5((b \rhd a) \wedge ((f \vee q) \rhd h))$

THEOREM: Any effect can be transformed into normal form II.

Nondeterministic effects: size of normal form II

Size increase in transforming into normal form II may be exponential.

Let $Q = \{q_1, \dots, q_n\}$ be a set of state variables. Effect that can produce every valuation of Q in one step is

$$(0.5q_1|0.5\neg q_1) \wedge (0.5q_2|0.5\neg q_2) \wedge \cdots \wedge (0.5q_n|0.5\neg q_n).$$

The effect in normal form II is

$$2^{-n}(q_1 \wedge q_2 \wedge \cdots \wedge q_n)|2^{-n}(\neg q_1 \wedge q_2 \wedge \cdots \wedge q_n)|\cdots|2^{-n}(\neg q_1 \wedge \neg q_2 \wedge \cdots \wedge \neg q_n).$$