Formal Properties of Constraint Calculi for Qualitative Spatial Reasoning

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Abstract

In the previous two decades, a number of qualitative constraint calculi have been developed, which are used to represent and reason about spatial configurations. A common property of almost all of these calculi is that reasoning in them can be understood as solving a binary constraint satisfaction problem over infinite domains. The main algorithmic method that is used is constraint propagation in the form of the path-consistency method. This approach can be applied to a wide range of different aspects of spatial reasoning. We describe how to make use of this representation and reasoning technique and point out the possible problems one might encounter.

1 Qualitative Spatial Representation and Reasoning

Representing spatial information and reasoning about this information is an important subproblem in many applications, such as geographical information systems (GIS), natural language understanding, robot navigation, and document interpretation. Often this information is only available qualitatively, for instance when a GIS query or integrity condition has to be specified (Sharma et al., 1994). Similarly, in document interpretation, the precise size and location of layout objects is not of interest, but the relative position of these objects matters (Walischewski, 1999).

A number of approaches to representing qualitative spatial information and reasoning about it are possible. A very early attempt at qualitative spatial representation and reasoning is Kuipers' (1978) TOUR model, which addresses the navigation problem using qualitative descriptions. Other approaches aim, for instance, to capture spatial notions using first-order logic (Randell and Cohn, 1989;

Eschenbach and Kulik, 1997), or even address representation and reasoning over spatio-temporal configurations (Muller, 1998).

All the mentioned approaches rely on quite expressive languages to talk about space. In contrast to them, there are approaches based on constraint satisfaction, which have a very limited expressiveness and usually reasonably good computational properties. The characteristic of these methods is that one has a system of (usually binary) relations, which is used to relate the objects of interest. For example, one can specify the relative position of layout objects using the relations *left* and *right* as well as *above* and *below*. Using this vocabulary, we can, for instance, state that an object A is *left & above* of an object B, which in turn is *right & above* of an object C. Having given these descriptions, it is obvious that the additional statement A *below* C is incompatible with what has been stated above.

Meanwhile there exist a large number of reasoning systems of this type. The first calculus in this family is Allen's (1983) *interval calculus*, which has originally been used for reasoning about qualitative temporal information. However, this one-dimensional calculus can also be interpreted spatially (Knauff et al., 1995). Furthermore, it can be generalized to two and more dimensions by projecting the objects of interest onto the axis of the coordinate system and describe the relationship between objects by the relationships between the projections (Guesgen, 1989; Balbiani et al., 1998). For example, the qualitative description of the relative position of layout objects sketched above can be done using the 2D version of Allen's interval calculus.

Other qualitative spatial reasoning systems are, for example, a calculus for reasoning about *topological relations* (Egenhofer, 1991; Randell et al., 1992b) often called **RCC8**, a calculus for reasoning about *cardinal directions* (Frank, 1991; Ligozat, 1998), and a calculus for reasoning about *orientations* (Freksa, 1992; Freksa and Zimmermann, 1992; Zimmermann and Freksa, 1993). Some more recent approaches are the *dipole calculus* (Moratz et al., 2000) and a calculus for describing *2D orientations using cyclic orderings* (Isli and Cohn, 2000). All of the mentioned approaches share the property that reasoning in these calculi can be done by constraint propagation over systems of binary (or sometimes ternary) relations with infinite domains.

2 Qualitative Reasoning Using Constraint Propagation

In order to describe the reasoning technique that is used in most of the qualitative spatial calculi, we will consider a particular simple calculus, the so-called **point calculus**. In this calculus, we can describe the relative positions of two points on the real axis. Obviously, there are three possibilities how two points can be related, namely, $a < b$, $a = b$, and $a > b$. If we want to describe indefinite information such as the fact that $a < b$ *or* $a > b$, it is also necessary to consider the *set-theoretic unions* (corresponding to the logical disjunctions) of the relations, e.g., $< \cup >$, which is more conventionally written as $\neq$. Considering all possible unions, we get $\leq, \geq, \neq$, and the **universal relation** $\top$ which holds for all pairs of numbers. Finally, we will also consider the **impossible relation** $\perp$ that holds between no pairs of numbers. Using this set of eight relations ($<, >, =, \leq, \geq, \neq, \top, \perp$), we can describe the relative position of four points a, b, c, and d:

$$\Theta = \{a \leq b, b \leq c, b < d, d < c\}. \quad (1)$$

Now one can ask what additional relationships follow from Θ and whether it is possible to find real numbers that satisfy all the relationships simultaneously. For the given description Θ, one sees, for example, that we cannot add the formula $a \geq c$ and have still all relationships satisfied by some real numbers. The reason is that from $b < d$ and $d < c$, we can deduce that $b < c$ which leads together with $a \leq b$ to $a < c$. This is clearly incompatible with $a \geq c$, leading us to conclude that we cannot add this formula.

2.1 Constraint Systems and Constraint Propagation

An algorithmic method for answering the above questions is *constraint propagation* (Mackworth, 1987), or more specifically the *path-consistency* method (Montanari, 1974). A **constraint system** is given by

- a finite set of **variables** $V = \{v_1, \ldots, v_n\}$, where each variable has a **domain** $dom(v_i)$;
- a set of **constraints** $C = \{C_1 \times \ldots C_k\}$, where each constraint is defined over a subset of the variables, denoted by $var(C_j)$. The cardinality of $var(C_j)$ is the *arity* of the constraint C_j. Constraints can be understood as subsets of the cross product over the domains of the variables in $var(C_j)$.

We say that an **assignment** of values to variables $A = \{v_1 \leftarrow w_1, \ldots, v_l \leftarrow w_l\}$ (with $v_i \in V, w_i \in dom(v_i)$) **satisfies** a constraint C_k, if it assigns values to all variables in $var(C_k)$ such that the tuple of values for the variables in $var(C_k)$ is a tuple of the constraint C_k. A **solution** of a constraint system is an assignment to all variables in V such that all constraints are satisfied. And this is most of the time the most interesting question: Is a given constraint system satisfiable?[1] Considering our small example (1) again, it is easy to see that the following assignment is a solution:

$$\{a \leftarrow 1, b \leftarrow 2, c \leftarrow 2.5, d \leftarrow 2.3\}. \quad (2)$$

In general, we observe that constraint systems based on the point calculus have the following properties:

1. the constraints are all *binary*;
2. the domains are *infinite*.

The first property is shared by most constraint satisfaction problems. In particular, it means that most of the techniques from the area of constraint solving are applicable (Mackworth, 1987). The second property, however, distinguishes qualitative reasoning problems from most constraint satisfaction problems, which are usually defined over finite domains. Nevertheless, it is possible to apply the *path-consistency method* (Montanari, 1974) (also called *3-consistency method*). This method eliminates relations between two variables by considering the relation to a third variable. For instance, in Θ we made the statement $b \leq c$. Taking now d into consideration, we see that there are the two constraints $b < d$ and $d < c$, from which we can conclude that $b = c$ can be ruled out, i.e., the only possible remaining relation between b and c is $<$. The path-consistency method now repeats the process for every triple of variables until it is impossible to further restrict any relation between two variables. Figure 1 visualizes which relations can be inferred in the case of our example.

After we terminate, the constraint system is said to be *3-consistent*. This means that for any partial assignment that satisfies the constraint between two variables, the partial assignment can be extended to a third variable such that all constraints between the three variables are satisfied. In our particular case – for the point calculus – we can even conclude that if no impossible relation has been

[1] In fact, the satisfiability question can be considered as the central one because all other interesting reasoning problems in the context of constraint systems can be reduced in polynomial time to satisfiability (Golumbic and Shamir, 1993).

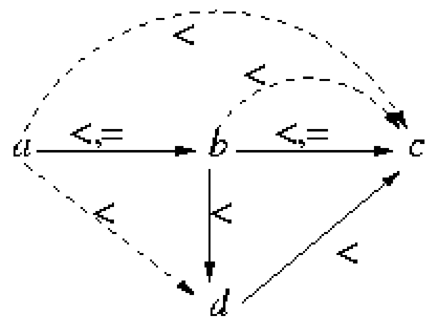

Figure 1: Visualization of the constraint system $\Theta = \{a \leq b, b \leq c, b < d, d < c\}$ and the relations derived by the path-consistency method, which are symbolized by dashed arrows.

generated in the process, then the system of constraints has a solution (Vilain and Kautz, 1986). This, however, is not necessarily the case for all qualitative calculi. For example, for Allen's interval calculus, a 3-consistent constraint system not containing the impossible relation may not have a solution.[2]

2.2 Operations on Relations, Relation Algebras, and Constraint Algebras

In order to capture what we have described above on a more abstract and formal level, we will consider *systems of relations* in the following. We will use infix notation such as aRb to express that the variables a and b are constrained by the relation R. We further use the notation $(x, y) \in R$ if we want to express that the pair of values (x, y), where x and y are elements of the domain of some variables, is in the relation R.

When we want to describe the above process of enforcing *path-consistency*, we need two *operations* on relations. First, we need the **composition** of two relations R and S, symbolically $R \circ S$:

$$R \circ S \stackrel{\text{def}}{=} \{(x, y) \mid \exists z \colon (x, z) \in R \wedge (z, y) \in S\}. \tag{3}$$

If we now denote the constraint between the variables v_i and v_j by C_{ij}, then the operation of *restricting* the relation between v_i and v_j by taking a third variable v_k

[2]Allen (1983) already gave an example of a 3-consistent network that is unsatisfiable.

into account can be described as follows:

$$C'_{ij} \leftarrow C_{ij} \cap (C_{ik} \circ C_{kj}), \tag{4}$$

where C'_{ij} denotes the constraint between v_i and v_j after applying the operation.

It is now clear, what to do with pairs of constraints of the form aRb and bSc. However, what do we do when we only have constraints of the form aRb and cSb? In this case, we first have to determine the *converse* of the relation S. The **converse** of a binary relation R, symbolically $R^{\smile}$, is defined as follows:

$$R^{\smile} \stackrel{\text{def}}{=} \{(x, y) \mid (y, x) \in R\}. \tag{5}$$

In the following, we will assume that for any binary constraint between v_i and v_j in a constraint system, we have

$$C_{ij} = C_{ji}^{\smile}. \tag{6}$$

Obviously, the operation described by (4) never eliminates a pair of values from a constraint that would be a possible solution. What is less obvious, however, is how to represent the new constraint C'_{ij} in cases where one allows for infinite domains. In case of the point calculus above it appears to be the case that a new constraint is always one of the eight point relations. However, under which conditions is this the case? From the above, it is clear that we need a set of relations,[3] which is *closed* under *converse*, *finite intersection*, and *composition*. These requirements result from (6) and (4). In addition, we want the universal relation and the identity relation to be part of the relation system. The former is necessary to state that there is no constraint, the latter in order to make conjunctive statement. A set of binary relations $\mathbf{R}$ containing the universal and the identity relation together with the above mentioned operations is called **constraint algebra** if it is closed under the operations.

It can be easily verified that the point calculus satisfies these condition, provided the relations are interpreted over the rationale or real numbers. However, if we interpret them over the integers, then the relations are not closed under composition.[4]

[3]For practical purposes, the set should be finite because we want the path-consistency method to terminate.

[4]Consider, e.g., the composition of $< \circ <$, which should be identical to $<$ – at least there is no other relation which could be identical to $< \circ <$. However, if we consider $1 < 2$, then it is clear that there is integer between 1 and 2, i.e., we have $< \circ < \subseteq <$, but $< \not\subseteq < \circ <$.

From a mathematical point of view, we are talking about a substructure of a *proper relation algebra* (PRA) (Maddux, 1991), where a PRA has to be additionally closed under complement and finite union. The interesting point about proper relation algebras is that they have an axiomatic counter-part, where the properties of the relation system is described purely axiomatically. In particular, a number of results in this area of algebraic logic can be transferred to qualitative reasoning about space and time (Hirsch, 1997; Hirsch, 2000; Ladkin and Maddux, 1994).

However, relation algebras are a bit more powerful than is needed for constraint propagation. For this reason, one may miss important properties when viewing the relation systems as PRAs. In particular, when one aims for a subsystem of a relation system that permits better computational properties, it may be important that the relation system is *not closed* under disjunction and complement. As a matter of fact, as described below, there exist often subsets of the relation systems for which 3-consistency is sufficient to decide satisfiability. These subsets are, however, not closed under union and complement. For these reasons, it seems preferable to consider *constraint algebras* instead of relation algebras.

2.3 Build Your Own Constraint Algebra

One may now be interested in what one needs to set up one's own constraint algebra. The best way to start is to divide all the possible binary relations into a set of *jointly exhaustive, pairwise disjoint* (JEPD) relations, containing $=$ as one relation. For example, for the point calculus, we started with the relations $<, =, >$. As a next step, one should make sure that the set of JEPD relations is closed under converse – so that we can guarantee condition (6). JEPD relations that are closed under converse are usually called **base relations**.

As a next step, we can create the entire relation system by building all unions of the base relations – and adding the empty relation $\bot$. Finally, we have to make sure that the set is closed under converse, intersection and composition. Converse is easy because the following equation holds:

$$(R_1 \cup \ldots \cup R_k) = R_1^{\smile} \cup \ldots \cup R_k^{\smile}. \tag{7}$$

So, we get closure under converse for free. Similarly, closure under intersection comes for free because each relation is the disjoint union of some base relations. Closure under composition is not that easy, however. The only simplification we can achieve is that it suffices to check the compositions of base relations because the following equation holds:

$$(R_1 \cup \ldots \cup R_k) \circ (S_1 \cup \ldots \cup S_l) = \bigcup_{i=1}^{k} \bigcup_{j=1}^{l} (R_i \circ S_j). \quad (8)$$

Incidentally, this equation tells us also how to compute the composition as required in the path-consistency operation (4). We reduce arbitrary compositions to compositions over base relations and combine these by building the union. So, the only thing that we have to know when we want to reason in our constraint calculus are the results of all compositions of base relations. For this purpose, one usually computes the so-called **composition table**, which contains just these results. For the point calculus, this table is given in Table 1.

$R \circ S$	$<$	$=$	$>$
$<$	$<$	$<$	$<,=,>$
$=$	$<$	$=$	$>$
$>$	$<,=,>$	$>$	$>$

Table 1: Composition Table for the Point Calculus

Such a composition table is usually generated manually by enumerating for each entry the possible relations. In order to make sure that one indeed gets all possible relations listed in an entry (and not more or less), formal proofs are necessary. This can be done manually or by using computer support (Randell et al., 1992a; Bennett, 1994). A prerequisite to both, however, is that one has a precise and formal semantics for the relations.

One might wonder whether it is really necessary to get formal at this point. Does it really happen that somebody gets a composition table wrong? Or is it possible that a relation systems generated from JEPD relations that are closed under converse is not closed under composition? And isn't it then possible to refine the relation system so that it becomes closed under composition? As mentioned above, it is indeed possible that a relation system is not closed under composition. For instance, the point calculus is not closed under composition if it is interpreted over the integers. Moreover, there is no finite refinement that leads to closure under composition.

While the point calculus over the integers is certainly not an interesting case, there are interesting relation systems that are not closed under composition. The ternary Double Cross calculus (Freksa, 1992), for instance, turns out to have this property. In addition, there is no finite refinement of this relation system that leads to closure under composition (Scivos and Nebel, 2001).

So what can we do if the relation system is not closed under composition? In this case, we can use an operation often called *weak composition* that determines the most specialized relation containing the result of the ordinary composition. Using such weak compositions, we can still apply the path-consistency operation (4) and eliminate impossible relations. However, the result will not necessarily be a 3-consistent constraint system – and for this reason we will not be able to make any further guarantees.

2.4 Computational Complexity and Completeness

Up to this point, the design, analysis, and usage of a constraint algebra is mostly straightforward. In particular, we know that by applying the path-consistency method we get sound inferences. The interesting and non-trivial part is to find out whether or under which condition 3-consistency is sufficient to decide satisfiability. As we already stated above, for the point calculus this is the case. However, for most of the other constraint algebras, this is not the case. Instead, the satisfiability problem is NP-hard, and since path-consistency runs in polynomial time, it cannot be complete for these algebras. In fact, one usually can identify cases when a 3-consistent constraint system does not contain the empty relation but is nevertheless unsatisfiable.

Sometimes things can be even worse. It can happen that even if only base relations are used between all pairs of variables and the constraint system is 3-consistent, the constraint system may not be satisfiable. An example for this case is the *pentagonal relation algebra* (Maddux, 1991; Hirsch, 1997).

Often however, it is possible to identify sub-algebras of the full constraint algebra, for which we can come up with the guarantee that a 3-consistent constraint system that does not contain the empty relation is satisfiable. For Allen's interval algebra, e.g., this is the case. There exists a unique largest sub-algebra containing all base relations for which the path-consistency method decides satisfiability (Nebel and Bürckert, 1995). Similarly, for the topological constraint algebra RCC8, largest such sub-algebra have been identified (Renz and Nebel, 1999; Renz, 1999). Golumbic and Shamir (1993) also analyzed subsets of relations of Allen's interval algebra that are not sub-algebras and came up with some interesting results.

All these results are useful, when only limited expressivity is needed in an application. However, even when the full constraint algebra is needed, the results are useful. They can be used to speed up the backtracking algorithm which is used to determine satisfiability in the general case (Nebel, 1997; Renz and Nebel, 2001).

3 Conclusions and Discussion

One approach to qualitative spatial representation and reasoning is to use constraint calculi, which are based on the mathematical notion of proper relation algebras or constraint algebras over systems of binary relations. Inference in these systems is performed by a special constraint-propagation technique, namely by the path-consistency method. Starting with Allen's interval calculus, a large number of such calculi have been developed and analyzed in the past. From a formal point of view, it is quite straightforward to develop a constraint calculus for a particular application. The difficult problems are the computation of a composition table, the verification that all the closure properties are met, and the determination of the computational complexity. Finally, and most difficult, is the problem to identify fragments for which the path-consistency method is sufficient to decide satisfiability. However, even ignoring this issue, the path-consistency method can always be relied on to generate sound inferences.

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