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Reasoning about Temporal Relations: A Maximal Tractable Subclass of Allen's Interval Algebra*

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Abstract

We introduce a new subclass of Allen's interval algebra we call "ORD-Horn subclass," which is a strict superset of the "pointisable subclass." We prove that reasoning in the ORD-Horn subclass is a polynomial-time problem and show that the path-consistency method is sufficient for deciding satisfiability. Further, using an extensive machine-generated case analysis, we show that the ORD-Horn subclass is a maximal tractable subclass of the full algebra (assuming P≠NP). In fact, it is the unique greatest tractable subclass amongst the subclasses that contain all basic relations.

Introduction

Temporal information is often conveyed qualitatively by specifying the relative positions of time intervals such as "...point to the figure while explaining the performance of the system ..." Further, for natural language understanding (Allen 1984; Song & Cohen 1988), general planning (Allen 1991; Allen & Koomen 1983), presentation planning in a multi-media context (Feiner *et al.* 1993), and knowledge representation (Weida & Litman 1992), the representation of qualitative temporal relations and reasoning about them is essential. Allen (1983) introduces an algebra of binary relations on intervals for representing qualitative temporal information and addresses the problem of reasoning about such information. Since the reasoning problems are NP-hard for the full algebra (Vilain & Kautz 1986), it is very unlikely that other polynomial-time algorithms will be found that solve this problem in general. Subsequent research has concentrated on designing more efficient reasoning algorithms, on identifying tractable special cases, and on isolating sources of computational complexity (Golumbic & Shamir 1992; Ladkin & Maddux 1988; Nökel 1989; Valdéz-Pérez 1987; van Beek 1989; 1990; van Beek & Cohen 1990; Vilain & Kautz 1986; Vilain, Kautz, & van Beek 1989).

We extend these previous results in three ways. Firstly, we present a new tractable subclass of Allen's interval algebra, which we call *ORD-Horn subclass*. This subclass is considerably larger than all other known tractable subclasses (it contains 10% of the full algebra) and strictly contains the *pointisable subclass* (Ladkin & Maddux 1988; van Beek 1989). Secondly, we show that the *path-consistency method* is sufficient for deciding satisfiability in this subclass. Thirdly, using an extensive machine-generated case analysis, we show that this subclass is a maximal subclass such that satisfiability is tractable (assuming P≠NP).[1]

From a practical point of view, these results imply that the path-consistency method has a much larger range of applicability than previously believed, provided we are mainly interested in satisfiability. Further, our results can be used to design backtracking algorithms for the full algebra that are more efficient than those based on other tractable subclasses.

Reasoning about Interval Relations using Allen's Interval Algebra

Allen's (1983) approach to reasoning about time is based on the notion of *time intervals* and *binary relations* on them. A **time interval** X is an ordered pair (X^-, X^+) such that $X^- < X^+$, where X^- and X^+ are interpreted as points on the real line.[2] So, if we talk about **interval interpretations** or ***I*-interpretations** in the following, we mean mappings of time intervals to pairs of distinct real numbers such that the beginning of an interval is strictly before the ending of the interval.

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[1] The programs we used and an enumeration of the ORD-Horn subclass can be obtained from the authors or by anonymous ftp from `duck.dfki.uni-sb.de` as `/pub/papers/DFKI-others/RR-93-11.programs.tar.Z`.

[2] Other underlying models of the time line are also possible, e.g., the rationals (Allen & Hayes 1985; Ladkin 1987). For our purposes these distinctions are not significant, however.

Basic Interval Relation	Symbol	Endpoint Relations
X before Y	$\prec$	$X^- < Y^-$, $X^- < Y^+$,
Y after X	$\succ$	$X^+ < Y^-$, $X^+ < Y^+$
X meets Y	m	$X^- < Y^-$, $X^- < Y^+$,
Y met-by X	$\mathsf{m}^{\smile}$	$X^+ = Y^-$, $X^+ < Y^+$
X overlaps Y	o	$X^- < Y^-$, $X^- < Y^+$,
Y overlapped-by X	$\mathsf{o}^{\smile}$	$X^+ > Y^-$, $X^+ < Y^+$
X during Y	d	$X^- > Y^-$, $X^- < Y^+$,
Y includes X	$\mathsf{d}^{\smile}$	$X^+ > Y^-$, $X^+ < Y^+$
X starts Y	s	$X^- = Y^-$, $X^- < Y^+$,
Y started-by X	$\mathsf{s}^{\smile}$	$X^+ > Y^-$, $X^+ < Y^+$
X finishes Y	f	$X^- > Y^-$, $X^- < Y^+$,
Y finished-by X	$\mathsf{f}^{\smile}$	$X^+ > Y^-$, $X^+ = Y^+$
X equals Y	$\equiv$	$X^- = Y^-$, $X^- < Y^+$, $X^+ > Y^-$, $X^+ = Y^+$

Table 1: The set **B** of the thirteen basic relations.

Given two interpreted time intervals, their relative positions can be described by *exactly one* of the elements of the set **B** of thirteen **basic interval relations** (denoted by B in the following), where each basic relation can be defined in terms of its **endpoint relations** (see Table 1). An atomic formula of the form XBY, where X and Y are intervals and $B \in \mathbf{B}$, is said to be **satisfied** by an I-interpretation iff the interpretation of the intervals satisfies the endpoint relations specified in Table 1.

In order to express indefinite information, unions of the basic interval relations are used, which are written as sets of basic relations leading to 2^{13} binary **interval relations** (denoted by R, S, T)—including the **null relation** $\emptyset$ (also denoted by $\perp$) and the **universal relation B** (also denoted by $\top$). The set of all binary interval relations $2^{\mathbf{B}}$ is denoted by $\mathcal{A}$.

An atomic formula of the form $X \{B_1, \ldots, B_n\}\, Y$ (denoted by ϕ) is called **interval formula**. Such a formula is satisfied by an I-interpretation $\Im$ iff $X\, B_i\, Y$ is satisfied by $\Im$ for some i, $1 \le i \le n$. Finite sets of interval formulas are denoted by Θ. Such a set Θ is called I-**satisfiable** iff there exists an I-interpretation $\Im$ that satisfies every formula of Θ. Further, such a satisfying I-interpretation $\Im$ is called I-**model** of Θ. If an interval formula ϕ is satisfied by every I-model of a set of interval formulas Θ, we say that ϕ is **logically implied** by Θ, written $\Theta \models_I \phi$.

Fundamental **reasoning problems** in this framework include (Golumbic & Shamir 1992; Ladkin & Maddux 1988; van Beek 1990; Vilain & Kautz 1986): Given a set of interval formulas Θ, (1) decide the *of I-satisfiability of* Θ **(ISAT)**, and (2) determine for each pair of intervals X, Y the *strongest implied relation* between them **(ISI)**.

In the following, we often consider **restricted reasoning problems** where the relations used in interval formulas in Θ are only from a subclass $\mathcal{S}$ of all interval relations. In this case we say that Θ **is a set of formulas over** $\mathcal{S}$, and we use a parameter in the problem description to denote the subclass considered, e.g., ISAT($\mathcal{S}$). As is well-known, ISAT and ISI are equivalent with respect to polynomial Turing-reductions (Vilain & Kautz 1986) and this equivalence also extends to the restricted problems ISAT($\mathcal{S}$) and ISI($\mathcal{S}$), provided $\mathcal{S}$ contains all basic relations.

The most prominent method to solve these problems (approximately for all interval relations or exactly for subclasses) is *constraint propagation* (Allen 1983; Ladkin & Maddux 1988; Nökel 1989; van Beek 1989; van Beek & Cohen 1990; Vilain & Kautz 1986) using a slightly simplified form of the *path-consistency algorithm* (Mackworth 1977; Montanari 1974). In the following, we briefly characterize this method without going into details, though. In order to do so, we first have to introduce Allen's interval algebra.

Allen's interval algebra (1983) consists of the set $\mathcal{A} = 2^{\mathbf{B}}$ of all binary interval relations and the operations unary **converse** (denoted by $\cdot^{\smile}$), binary **intersection** (denoted by $\cap$), and binary **composition** (denoted by $\circ$), which are defined as follows:

$$\begin{array}{lrcl} \forall X,Y\colon & X R^{\smile} Y & \leftrightarrow & YRX \\ \forall X,Y\colon & X\,(R \cap S)\,Y & \leftrightarrow & XRY \wedge XSY \\ \forall X,Y\colon & X\,(R \circ S)\,Y & \leftrightarrow & \exists Z\colon (XRZ \wedge ZSY). \end{array}$$

Assume an operator Γ that maps finite sets of interval formulas to finite sets of interval formulas in the following way:

$$\begin{array}{rcl} \Gamma(\Theta) & = & \Theta \cup \{X \top Y \mid X, Y \text{ appear in } \Theta\} \\ & & \cup \{XRY \mid (Y\ R^{\smile}\ X) \in \Theta\} \\ & & \cup \{X\,(R \cap S)\,Y \mid (XRY), (XSY) \in \Theta\} \\ & & \cup \{X\,(R \circ S)\,Y \mid (XRZ), (ZSY) \in \Theta\}. \end{array}$$

Since there are only finitely many different interval formulas for a finite set of intervals and Γ is monotone, it follows that for each Θ there exists a natural number n such that $\Gamma^n(\Theta) = \Gamma^{n+1}(\Theta)$. $\Gamma^n(\Theta)$ is called the **closure** of Θ, written $\overline{\Theta}$. Considering the formulas of the form $(X R_i Y) \in \overline{\Theta}$ for given X, Y, it is evident that the R_i's are closed under intersection, and hence there exists $(XSY) \in \overline{\Theta}$ such that S is the *strongest relation* amongst the R_i's, i.e., $S \subseteq R_i$, for every i. The subset of a closure $\overline{\Theta}$ containing for each pair of intervals only the strongest relations is called the **reduced closure** of Θ and is denoted by $\widehat{\Theta}$.

As can be easily shown, every reduced closure of a set Θ is **path consistent** (Mackworth 1977), which means that for every three intervals X, Y, Z and for every interpretation $\Im$ that satisfies $(XRY) \in \widehat{\Theta}$, there exists an interpretation $\Im'$ that agrees with $\Im$ on X and Y and in addition satisfies $(XSZ), (ZS'Y) \in \widehat{\Theta}$. Under the assumption that $(XRY) \in \Theta$ implies $(Y\ R^{\smile}\ X) \in \Theta$, it is also easy to show that path consistency of Θ implies that $\Theta = \widehat{\Theta}$. For this reason, we will use the term **path-consistent set** as a synonym for a set that is the reduced closure of itself. Finally, computing $\widehat{\Theta}$ is polynomial in the size of Θ (Mackworth & Freuder 1985; Montanari 1974).

The ORD-Horn Subclass

Previous results on the tractability of ISAT($\mathcal{S}$) (and hence ISI($\mathcal{S}$)) for some subclass $\mathcal{S} \subseteq \mathcal{A}$ made use of the *expressibility* of interval formulas over $\mathcal{S}$ as certain logical formulas involving endpoint relations.

As usual, by a **clause** we mean a disjunction of literals, where a **literal** in turn is an atomic formula or a negated atomic formula. As **atomic formulas** we allow $a \leq b$ and $a = b$, where a and b denote endpoints of intervals. The negation of $a = b$ is also written as $a \neq b$. Finite sets of such clause will be denoted by Ω. In the following, we consider a slightly restricted form of clauses, which we call **ORD clauses**. These clauses do not contain negations of atoms of the form $(a \leq b)$, i.e., they only contain literals of the form:

$$a = b,\ a \leq b,\ a \neq b.$$

The **ORD-clause form** of an interval formula ϕ, written $\pi(\phi)$, is the set of ORD clauses over endpoint relations that is equivalent to ϕ, i.e., every interval model of ϕ can be transformed into a model of the ORD-clause form over the reals and *vice versa* using the obvious transformation. Consider, for instance, $\pi(X\ \{\mathsf{d},\mathsf{o},\mathsf{s}\}\ Y)$:

$$\begin{array}{ll} \{(X^- \leq X^+), & (X^- \neq X^+), \\ (Y^- \leq Y^+), & (Y^- \neq Y^+), \\ (X^- \leq Y^+), & (X^- \neq Y^+), \\ (Y^- \leq X^+), & (X^+ \neq Y^-), \\ (X^+ \leq Y^+), & (X^+ \neq Y^+)\}. \end{array}$$

The function $\pi(\cdot)$ is extended to finite sets of interval formulas in the obvious way, i.e., for identical intervals in Θ, identical endpoints are used in $\pi(\Theta)$. Similarly to the notions of I-satisfiability, we define **R-satisfiability** of Ω to be the satisfiability of Ω over the real numbers.

Proposition 1 *Θ is I-satisfiable iff $\pi(\Theta)$ is R-satisfiable.*

Not all relations permit a ORD-clause form that is as concise as the the one shown above, which contains only *unit clauses*. However, in particular those relations that allow for such a clause form have interesting computational properties. For instance, the **continuous endpoint subclass** (which is denoted by $\mathcal{C}$) can be defined as the subclass of interval relations that (1) permit a clause form that contains only unit clauses, and (2) for each unit clause $a \neq b$, the clause form contains also a unit clause of the form $a \leq b$ or $b \leq a$.

As demonstrated above, the relation $\{\mathsf{d},\mathsf{o},\mathsf{s}\}$ is a member of the continuous endpoint subclass. This subclass has the favorable property that the path-consistency method solves ISI($\mathcal{C}$) (van Beek 1989; van Beek & Cohen 1990; Vilain, Kautz, & van Beek 1989). A slight generalization of the continuous endpoint subclass is the **pointisable subclass** (denoted by $\mathcal{P}$) that is defined in the same way as $\mathcal{C}$, but without condition (2). Path-consistency is not sufficient for solving ISI($\mathcal{P}$) (van Beek 1989) but still sufficient for deciding satisfiability (Ladkin & Maddux 1988; Vilain & Kautz 1986).

We generalize this approach by being more liberal concerning the clause form. We consider the subclass of Allen's interval algebra such that the relations permit an ORD-clause form containing only clauses with *at most one positive literal*, which we call **ORD-Horn clauses**. The subclass defined in this way is called **ORD-Horn subclass**, and we use the symbol $\mathcal{H}$ to refer to it. The relation $\{\mathsf{o},\mathsf{s},\mathsf{f}^\smile\}$ is, for instance, an element of $\mathcal{H}$ because $\pi(X\ \{\mathsf{o},\mathsf{s},\mathsf{f}^\smile\}\ Y)$ can be expressed as follows:

$$\begin{array}{lll} \{(X^- \leq X^+), & (X^- \neq X^+), & \\ (Y^- \leq Y^+), & (Y^- \neq Y^+), & \\ (X^- \leq Y^-), & (X^- \leq Y^+), & (X^- \neq Y^+), \\ (Y^- \leq X^+), & (X^+ \neq Y^-), & (X^+ \leq Y^+), \\ (X^- \neq Y^- \vee X^+ \neq Y^+)\}. & & \end{array}$$

By definition, the ORD-Horn subclass contains the pointisable subclass. Further, by the above example, this inclusion is strict.

Consider now the theory *ORD* that axiomatizes "$=$" as an equivalence relation and "$\leq$" as a partial ordering over the equivalence classes:

$$\begin{array}{llll} \forall x,y,z\colon & x \leq y \wedge y \leq z & \rightarrow & x \leq z \\ \forall x\colon & x \leq x & & \\ \forall x,y\colon & x \leq y \wedge y \leq x & \rightarrow & x = y \\ \forall x,y\colon & x = y & \rightarrow & x \leq y \\ \forall x,y\colon & x = y & \rightarrow & y \leq x. \end{array}$$

Although this theory is much weaker than the theory of the reals, R-satisfiability of a set Ω of ORD clauses is nevertheless equivalent to the satisfiability of $\Omega \cup ORD$ over arbitrary interpretations.

Proposition 2 *A set of ORD clauses Ω is R-satisfiable iff $\Omega \cup ORD$ is satisfiable.*[3]

Proof Sketch. Any linearization of a partial order that satisfies all atoms appearing in ORD clauses also satisfies these atoms. Hence, a model of $\Omega \cup ORD$ can be used to generate an R-model for Ω. The other direction is trivial. ■

In the following, ORD_Ω shall denote the axioms of *ORD* instantiated to all endpoints mentioned in Ω. As a specialization of the Herbrand theorem, we obtain the next proposition.

Proposition 3 *$\Omega \cup ORD$ is satisfiable iff $\Omega \cup ORD_\Omega$ is satisfiable.*

[3] Full proofs are given in the long paper (Nebel & Bürckert 1993), which can be obtained by anonymous ftp from `duck.dfki.uni-sb.de`.

From the fact that ORD_Ω and Ω are propositional Horn formulas, polynomiality of ISAT($\mathcal{H}$) is immediate.

Theorem 4 ISAT($\mathcal{H}$) *is polynomial.*

The Applicability of Path-Consistency

Enumerating the ORD-Horn subclass reveals that there are 868 relations (including the null relation $\bot$) in Allen's interval algebra that can be expressed using ORD-Horn clauses. Since the full algebra contains $2^{13} = 8192$ relations, $\mathcal{H}$ covers more than 10% of the full algebra. Comparing this with the continuous endpoint subclass $\mathcal{C}$, which contains 83 relations, and the pointisable subclass $\mathcal{P}$, which contains 188 relations,[4] having shown tractability for $\mathcal{H}$ is a clear improvement over previous results. However, there remains the question of whether the "traditional" method of reasoning in Allen's interval algebra, i.e., constraint propagation, gives reasonable results. As it turns out, this is indeed the case.

Theorem 5 *Let $\widehat{\Theta}$ be a path-consistent set of interval formulas over $\mathcal{H}$. Then $\widehat{\Theta}$ is I-satisfiable iff $(X \bot Y) \notin \widehat{\Theta}$.*

Proof Sketch. A case analysis over the possible non-unit clauses in $\pi(\widehat{\Theta}) \cup ORD_{\pi(\widehat{\Theta})}$ reveals that no new units can be derived by *positive unit resolution*, if the ORD-clause form of the interval formulas satisfies the requirement that it contains all implied atoms and the clauses are minimal. By refutation completeness of positive unit resolution (Henschen & Wos 1974), the claim follows. ∎

The only remaining part we have to show is that transforming Θ over $\mathcal{H}$ into its equivalent path-consistent form $\widehat{\Theta}$ does not result in a set that contains relations not in $\mathcal{H}$. In order to show this we prove that $\mathcal{H}$ is closed under converse, intersection, and composition, i.e., $\mathcal{H}$ (together with these operations) defines a **subalgebra** of Allen's interval algebra.

Theorem 6 *$\mathcal{H}$ is closed under converse, intersection, and composition.*

Proof Sketch. The main problem is to show that the composition of two relations has an ORD-Horn form. We show that by proving that any minimal clause C implied by $\pi(\{XRY, YSZ\})$ is either ORD-Horn or there exists a set of ORD-Horn clauses that are implied by $\pi(\{XRY, YSZ\})$ and imply C. ∎

From that it follows straightforwardly that ISAT($\mathcal{H}$) is decided by the path-consistency method.

Theorem 7 *If Θ is a set over $\mathcal{H}$, then Θ is satisfiable iff $(X \bot Y) \notin \widehat{\Theta}$ for all intervals X, Y.*

[4]An enumeration of $\mathcal{C}$ and $\mathcal{P}$ is given by van Beek and Cohen (1990).

The Borderline between Tractable and NP-complete Subclasses

Having identified the tractable fragment $\mathcal{H}$ that contains the previously identified tractable fragment $\mathcal{P}$ and that is considerably larger than $\mathcal{P}$ is satisfying in itself. However, such a result also raises the question for the the boundary between polynomiality and NP-completeness in Allen's interval algebra.

While the introduction of the algebraic structure on the set of expressible interval relations may have seem to be only motivated by the particular approximation algorithm employed, this structure is also useful when we explore the computational properties of restricted problems. For any arbitrary subset $\mathcal{S} \subseteq \mathcal{A}$, $\overline{\mathcal{S}}$ shall denote the **closure** of $\mathcal{S}$ under converse, intersection, and composition. In other words, $\overline{\mathcal{S}}$ is the carrier of the **least subalgebra generated by** $\mathcal{S}$. Apparently, it is possible to translate any set of interval formulas Θ over $\overline{\mathcal{S}}$ into a set Θ' over $\mathcal{S}$ in polynomial time in a way such that I-satisfiability is preserved.

Theorem 8 ISAT($\overline{\mathcal{S}}$) *can be polynomially transformed to* ISAT($\mathcal{S}$).

In other words, once we have proven that satisfiability is polynomial for some set $\mathcal{S} \subseteq \mathcal{A}$, this result extends to the least subalgebra generated by $\mathcal{S}$. Conversely, NP-hardness for a subalgebra is "inherited" by all subsets that generate this subalgebra.

It still takes some effort to prove that a given fragment $\mathcal{S}$ is a *maximal* tractable subclass of Allen's interval algebra. Firstly, one has to show that $\mathcal{S} = \overline{\mathcal{S}}$. For the ORD-Horn subclass, this has been done in Theorem 6. Secondly, one has to show that ISAT($\mathcal{T}$) is NP-complete for all *minimal* subalgebras $\mathcal{T}$ that strictly contain $\mathcal{S}$. This, however, means that these subalgebras have to be identified. Certainly, such a case analysis cannot be done manually. In fact, we used a program to identify the minimal subalgebras strictly containing $\mathcal{H}$. An analysis of the clause form of the relations appearing in these subalgebras leads us to consider the following two relations:

$$\begin{aligned} \mathrm{N}_1 &= \{\mathsf{d}, \mathsf{d}^\smile, \mathsf{o}^\smile, \mathsf{s}^\smile, \mathsf{f}\} \\ \mathrm{N}_2 &= \{\mathsf{d}^\smile, \mathsf{o}, \mathsf{o}^\smile, \mathsf{s}^\smile, \mathsf{f}^\smile\}. \end{aligned}$$

One of these two relations can be found in all minimal subalgebras strictly containing $\mathcal{H}$, as can be shown using a machine-assisted case analysis.

Lemma 9 *Let $\mathcal{S} \subseteq \mathcal{A}$ be any set of interval relations that strictly contains $\mathcal{H}$. Then N_1 or N_2 is an element of $\overline{\mathcal{S}}$.*

For reasons of simplicity, we will not use the ORD clause form in the following, but a clause form that also contains literals over the relations $\geq, <, >$. Then the clause form for the relations mentioned in the lemma can be given as follows:

$$\begin{aligned} \pi(X\, \mathrm{N}_1\, Y) = \{ & (X^- < X^+), \quad (Y^- < Y^+), \\ & (X^- < Y^+), \quad (X^+ > Y^-), \\ & ((X^- > Y^-) \vee (X^+ > Y^+))\}, \end{aligned}$$

$$\pi(X\,\mathrm{N}_2\,Y) = \{(X^- < X^+),\ (Y^- < Y^+),\ (X^- < Y^+),\ (X^+ > Y^-),\ ((X^- < Y^-) \vee (X^+ > Y^+))\}.$$

We will show that each of these relations together with the two relations

$$\mathrm{B}_1 = \{\prec, \mathsf{d}^\smile, \mathsf{o}, \mathsf{m}, \mathsf{f}^\smile\}$$
$$\mathrm{B}_2 = \{\prec, \mathsf{d}, \mathsf{o}, \mathsf{m}, \mathsf{s}\},$$

which are elements of $\mathcal{C}$, are enough for making the interval satisfiability problem NP-complete. The clause form of these relations looks as follows:

$$\pi(X\,\mathrm{B}_1\,Y) = \{(X^- < X^+),\ (Y^- < Y^+),\ (X^- < Y^-),\ (X^- < Y^+)\}$$
$$\pi(X\,\mathrm{B}_2\,Y) = \{(X^- < X^+),\ (Y^- < Y^+),\ (X^+ < Y^+),\ (X^- < Y^+)\}$$

Lemma 10 ISAT($\mathcal{S}$) *is NP-complete if*

1. $\mathcal{N}_1 = \{\mathrm{B}_1, \mathrm{B}_2, \mathrm{N}_1\} \subseteq \mathcal{S}$, *or*
2. $\mathcal{N}_2 = \{\mathrm{B}_1, \mathrm{B}_2, \mathrm{N}_2\} \subseteq \mathcal{S}$.

Proof Sketch. Since ISAT($\mathcal{A}$) $\in$ NP, membership in NP follows.

For the NP-hardness part we will show that 3SAT can be polynomially transformed to ISAT($\mathcal{N}_k$). We will first prove the claim for $\mathcal{N}_1$. Let $D = \{C_i\}$ be a set of clauses, where $C_i = l_{i,1} \vee l_{i,2} \vee l_{i,3}$ and the $l_{i,j}$'s are literal occurrences. We will construct a set of interval formulas Θ over $\mathcal{N}_1$ such that Θ is I-satisfiable iff D is satisfiable.

For each literal occurrence $l_{i,j}$ a pair of intervals $X_{i,j}$ and $Y_{i,j}$ is introduced, and the following first group of interval formulas is put into Θ:

$$(X_{i,j}\,\mathrm{N}_1\,Y_{i,j}).$$

This implies that $\pi(\Theta)$ contains among other things the following clauses $(X^-_{i,j} > Y^-_{i,j} \vee X^+_{i,j} > Y^+_{i,j})$.

Additionally, we add a second group of formulas for each clause C_i:

$$(X_{i,2}\,\mathrm{B}_1\,Y_{i,1}), (X_{i,3}\,\mathrm{B}_1\,Y_{i,2}), (X_{i,1}\,\mathrm{B}_1\,Y_{i,3}),$$

which leads to the inclusion of the clauses $(Y^-_{i,1} > X^-_{i,2})$, $(Y^-_{i,2} > X^-_{i,3})$, $(Y^-_{i,3} > X^-_{i,1})$ in $\pi(\Theta)$.

This construction leads to the situation that there is no model of Θ that satisfies for given i all disjuncts of the form $(X^-_{i,j} > Y^-_{i,j})$ in the clause form of $\pi(X_{i,j}\mathrm{N}_1Y_{i,j})$. If the jth disjunct $(X^-_{i,j} > Y^-_{i,j})$ is unsatisfied in an I-model of Θ, we will interpret this as the satisfaction of the literal occurrence $l_{i,j}$ in C_i of D.

In order to guarantee that if a literal occurrence $l_{i,j}$ is interpreted as satisfied, then all complementary literal occurrences in D are interpreted as unsatisfied, the following third group of interval formulas for complementary literal occurrences $l_{i,j}$ and $l_{g,h}$ are added to Θ:

$$(X_{g,h}\,\mathrm{B}_2\,Y_{i,j}), \quad (X_{i,j}\,\mathrm{B}_2\,Y_{g,h}),$$

which leads to the inclusion of $(Y^+_{i,j} > X^+_{g,h})$, $(Y^+_{g,h} > X^+_{i,j})$. This construction guarantees that Θ is I-satisfiable iff D is satisfiable.

The transformation for $\mathcal{N}_2$ is similar. ■

Based on this result, it follows straightforwardly that $\mathcal{H}$ is indeed a maximal tractable subclass of $\mathcal{A}$.

Theorem 11 *If $\mathcal{S}$ strictly contains $\mathcal{H}$, then* ISAT($\mathcal{S}$) *is NP-complete.*

The next question is whether there are other maximal tractable subclasses that are incomparable with $\mathcal{H}$. One example of an incomparable tractable subclass is $\mathcal{U} = \{\{\prec, \succ\}, \top\}$. Since $\{\prec, \succ\}$ has no ORD-Horn clause form, this subclass is incomparable with $\mathcal{H}$, and since all sets of interval formulas over $\mathcal{U}$ are trivially satisfiable (by making all intervals disjoint), ISAT($\mathcal{U}$) can be decided in constant time. The subclass $\mathcal{U}$ is, of course, not a very *interesting* fragment. Provided we are interested in temporal reasoning in the framework as described by Allen (1983), one necessary requirement is that *all basic relations* are contained in the subclass. A machine-assisted exploration of the space of subalgebras leads us to the following machine-verifiable lemma.

Lemma 12 *If $\mathcal{S}$ is a subclass that contains the thirteen basic relations, then $\overline{\mathcal{S}} \subseteq \mathcal{H}$, or* N_1 *or* N_2 *is an element of $\overline{\mathcal{S}}$.*

Using the fact that $\mathrm{B}_1, \mathrm{B}_2$ are elements of the least subalgebra generated by the set of basic relations and employing Lemma 10 again, we obtain the quite satisfying result that $\mathcal{H}$ is in fact the unique greatest tractable subclass amongst the subclasses containing all basic relations.

Theorem 13 *Let $\mathcal{S}$ be any subclass of $\mathcal{A}$ that contains all basic relations. Then either $\mathcal{S} \subseteq \mathcal{H}$ and* ISAT($\mathcal{S}$) *is polynomial or* ISAT($\mathcal{S}$) *is NP-complete.*

Conclusion

We have identified a new tractable subclass of Allen's interval algebra, which we call *ORD-Horn subclass* and which contains the previously identified *continuous endpoint* and *pointisable* subclasses. Enumerating the ORD-Horn subclass reveals that this subclass contains 868 elements out of 8192 elements in the full algebra, i.e., more than 10% of the full algebra. Comparing this with the continuous endpoint subclass that covers approximately 1% and with the pointisable subclass that covers 2%, our result is a clear improvement in quantitative terms.

Furthermore, we showed that the "traditional" method of reasoning in Allen's interval algebra, namely, the *path-consistency method*, is sufficient for deciding satisfiability in the ORD-Horn subclass. In other words, our results indicate that the path-consistency method has a much larger range of applicability for reasoning in Allen's interval algebra than previously believed—if we are mainly interested in satisfiability.

Provided that a restriction to the subclass $\mathcal{H}$ is not possible in an application, our results may be employed in designing faster backtracking algorithms for the full algebra (Valdéz-Pérez 1987; van Beek 1990). Since our subclass contains significantly more relations than other tractable subclasses, the branching factor in a backtrack search can be considerably decreased if the ORD-Horn subclass is used.

Finally, we showed that it is impossible to improve on our results. Using a machine-generated case analysis, we showed that the ORD-Horn subclass is a *maximal* tractable subclass of Allen's interval algebra and, in fact, even the *unique greatest* tractable subclass in the set of subclasses that contain all basic relations. In other words, the ORD-Horn subclass presents an optimal tradeoff between expressiveness and tractability (Levesque & Brachman 1987) for reasoning in Allen's interval algebra.

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