Qualitative Spatial Reasoning about Line Segments

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Abstract. Representing and reasoning about orientation information is an important aspect of qualitative spatial reasoning. We present a novel approach for dealing with intrinsic orientation information by specifying qualitative relations between oriented line segments, the simplest possible spatial entities being extended and having an intrinsic direction. We identify a set of 24 atomic relations which form a relation algebra and for which we compute relational compositions based on their algebraic semantics. Reasoning over the full algebra turns out to be NP-hard. Potential applications of the calculus are motivated with a small example which shows the reasoning capabilities of the dipole calculus using constraint-based reasoning methods.

1 Introduction

Qualitative representation of space abstracts from the physical world and enables computers to make predictions about spatial relations, even when precise quantitative information is not available [2]. Different aspects of space can be represented in a qualitative way. The most important of these are topological information and orientation information about physical objects which are usually spatially extended. While it is common for representing topological information to use extended spatial regions as the basic entities, most approaches to qualitatively representing and reasoning about orientation information deal with points as the basic entities. Those orientation approaches that use extended spatial regions as the basic entities mostly approximate regions by using, for instance, minimal bounded rectangles whose sides are parallel to the axes of the global reference frame. This, however, does not account for representing intrinsic orientation information.

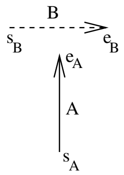

Figure 1. Orientation relation between two dipoles

In this paper we develop one of the simplest possible calculi for representing intrinsic orientation information, namely, by using oriented line segments represented by their start and end points as the basic entities. We propose calculi on different levels of granularity which all form relation algebras and as such allow for using standard constraint based reasoning mechanisms originally developed for temporal reasoning. Even on the coarsest level of granularity our calculi enable to represent polygonal lines which are particularly interesting for applications such as cognitive robotics [10] or spatial information systems [6].

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2 The Basic Representation of the Dipole Relations

The basic entities we are using are dipoles, i.e., oriented line segments formed by a pair of two points, a start point and an end point. Dipoles are denoted by $A, B, C, \ldots$, the start point by $\mathbf{s}_A$, the end point by $\mathbf{e}_A$, respectively (see Figure 1). These dipoles are used for representing spatial objects with an intrinsic orientation. Given a set of dipoles it is possible to specify many different relations of different arity, e.g., depending on the length of dipoles, the angle between different dipoles, or the dimension and nature of the underlying space. The goal of identifying different relations is to obtain a set of jointly exhaustive and pairwise disjoint *atomic* relations, i.e., between any two dipoles exactly one relation holds. If these relations form a *relation algebra* it is possible to apply standard constraint-based reasoning mechanisms which were originally developed for temporal reasoning and which have also proved valuable for spatial reasoning. In order to enable efficient reasoning, it should be tried to keep the number of different base relations relatively small.

For this reason, we will restrict for now to using two-dimensional continuous space, in particular $\mathbb{R}^2$, and distinguish the location and orientation of the different dipoles only according to whether a point lies to the left, to the right, or on the straight line through the referring dipole. Then $\mathbf{s}_B$ can either lie to the left of A (see figure 1), on the straight line through A or to the right of A, expressed as $A \ \mathrm{l} \ \mathbf{s}_B$, $A \ \mathrm{o} \ \mathbf{s}_B$ or $A \ \mathrm{r} \ \mathbf{s}_B$, respectively. Using these three relations between a dipole and a point it is possible to specify the relations between two dipoles with the following four relationships:

$$A \ \mathrm{R} \ \mathbf{s}_B \wedge A \ \mathrm{R} \ \mathbf{e}_B \wedge B \ \mathrm{R} \ \mathbf{s}_A \wedge B \ \mathrm{R} \ \mathbf{e}_A,$$

where R is one of $\{\mathrm{r}, \mathrm{l}, \mathrm{o}\}$. Since this still leads to a very large number of different atomic relations, we require in the first version of our algebra all points to be in *general position*, i.e., no more than two points are on a line (the extended version of the algebra is described in section 5). This gives us the following 14 relations that hold if the four points $\mathbf{s}_B, \mathbf{e}_B, \mathbf{s}_A, \mathbf{e}_A$ are distinct:

$$\begin{array}{lcl}
A \ \mathrm{rrrr} \ B & := & A \ \mathrm{r} \ \mathbf{s}_B \wedge A \ \mathrm{r} \ \mathbf{e}_B \wedge B \ \mathrm{r} \ \mathbf{s}_A \wedge B \ \mathrm{r} \ \mathbf{e}_A \\
A \ \mathrm{rrrl} \ B & := & A \ \mathrm{r} \ \mathbf{s}_B \wedge A \ \mathrm{r} \ \mathbf{e}_B \wedge B \ \mathrm{r} \ \mathbf{s}_A \wedge B \ \mathrm{l} \ \mathbf{e}_A \\
A \ \mathrm{rrlr} \ B & := & A \ \mathrm{r} \ \mathbf{s}_B \wedge A \ \mathrm{r} \ \mathbf{e}_B \wedge B \ \mathrm{l} \ \mathbf{s}_A \wedge B \ \mathrm{r} \ \mathbf{e}_A \\
A \ \mathrm{rrll} \ B & := & A \ \mathrm{r} \ \mathbf{s}_B \wedge A \ \mathrm{r} \ \mathbf{e}_B \wedge B \ \mathrm{l} \ \mathbf{s}_A \wedge B \ \mathrm{l} \ \mathbf{e}_A
\end{array}$$

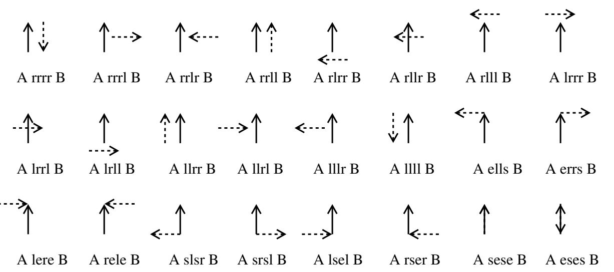


Figure 2. The 24 atomic relations of the dipole calculus

$$
\begin{aligned}
A \text{ rlrr } B &:= A \text{ r } \mathbf{s}_B \wedge A \text{ l } \mathbf{e}_B \wedge B \text{ r } \mathbf{s}_A \wedge B \text{ r } \mathbf{e}_A \\
A \text{ rllr } B &:= A \text{ r } \mathbf{s}_B \wedge A \text{ l } \mathbf{e}_B \wedge B \text{ l } \mathbf{s}_A \wedge B \text{ r } \mathbf{e}_A \\
A \text{ rlll } B &:= A \text{ r } \mathbf{s}_B \wedge A \text{ l } \mathbf{e}_B \wedge B \text{ l } \mathbf{s}_A \wedge B \text{ l } \mathbf{e}_A \\
A \text{ lrrr } B &:= A \text{ l } \mathbf{s}_B \wedge A \text{ r } \mathbf{e}_B \wedge B \text{ r } \mathbf{s}_A \wedge B \text{ r } \mathbf{e}_A \\
A \text{ lrrl } B &:= A \text{ l } \mathbf{s}_B \wedge A \text{ r } \mathbf{e}_B \wedge B \text{ r } \mathbf{s}_A \wedge B \text{ l } \mathbf{e}_A \\
A \text{ lrll } B &:= A \text{ l } \mathbf{s}_B \wedge A \text{ r } \mathbf{e}_B \wedge B \text{ l } \mathbf{s}_A \wedge B \text{ l } \mathbf{e}_A \\
A \text{ llrr } B &:= A \text{ l } \mathbf{s}_B \wedge A \text{ l } \mathbf{e}_B \wedge B \text{ r } \mathbf{s}_A \wedge B \text{ r } \mathbf{e}_A \\
A \text{ llrl } B &:= A \text{ l } \mathbf{s}_B \wedge A \text{ l } \mathbf{e}_B \wedge B \text{ r } \mathbf{s}_A \wedge B \text{ l } \mathbf{e}_A \\
A \text{ lllr } B &:= A \text{ l } \mathbf{s}_B \wedge A \text{ l } \mathbf{e}_B \wedge B \text{ l } \mathbf{s}_A \wedge B \text{ r } \mathbf{e}_A \\
A \text{ llll } B &:= A \text{ l } \mathbf{s}_B \wedge A \text{ l } \mathbf{e}_B \wedge B \text{ l } \mathbf{s}_A \wedge B \text{ l } \mathbf{e}_A
\end{aligned}
$$

The cases $A \text{ r } \mathbf{s}_B \wedge A \text{ l } \mathbf{e}_B \wedge B \text{ r } \mathbf{s}_A \wedge B \text{ l } \mathbf{e}_A$ and $A \text{ l } \mathbf{s}_B \wedge A \text{ r } \mathbf{e}_B \wedge B \text{ l } \mathbf{s}_A \wedge B \text{ r } \mathbf{e}_A$ cannot be realized on the plane. These 14 relations are similar to the relations between line segments derived by Schlieder [13]. However, in order to obtain a relation algebra, we also have to consider those relations where two dipoles share common points. Then $\mathbf{s}_B$ can be equivalent to the start point of A or to the end point of A. This is denoted as $A \text{ s } \mathbf{s}_B$ or $A \text{ e } \mathbf{s}_B$, respectively. Using these additional dipole-point relations, we obtain the following ten additional dipole-dipole relations: $\{\text{ells}, \text{errs}, \text{lere}, \text{rele}, \text{slsr}, \text{srsl}, \text{lsel}, \text{rser}, \text{sese}, \text{eses}\}$. Altogether we obtain 24 different atomic relations. These relations are jointly exhaustive and pairwise disjoint provided that all points are in general position. The relation sese is the identity relation. We use $\mathcal{D}_{24}$ to refer to the set of 24 atomic relations, and $\mathcal{DRA}_{24}$ to refer to the powerset of $\mathcal{D}_{24}$ which contains all 2^{24} possible unions of the atomic relations.

The relations which are introduced above in an informal way can be defined in an algebraic way. Every dipole D on the plane $\mathbb{R}^2$ is an ordered pair of two points $\mathbf{s}_D$ and $\mathbf{e}_D$, each of them is represented by its Cartesian coordinates x and y, with $x, y \in \mathbb{R}$ and $\mathbf{s}_D \neq \mathbf{e}_D$.

$$D = (\mathbf{s}_D, \mathbf{e}_D), \qquad \mathbf{s}_D = ((\mathbf{s}_D)_x, (\mathbf{s}_D)_y)$$

The basic relations are then described as polynomial equations with the coordinates as variables. The set of solutions for a system of equations describes all the possible coordinates for these points. As an example, we will have a more detailed look at the relation $A \text{ rrrr } B$. We need to find an equation, which is solvable iff a point lies to the right of a given line. Then, we can use this equation to express the premises of the relation: $A \text{ r } \mathbf{s}_B, A \text{ r } \mathbf{e}_B, B \text{ r } \mathbf{s}_A, B \text{ r } \mathbf{e}_A$. The equation for "right of" is constructed as follows ($A \text{ r } \mathbf{s}_B$ serves as example):

With $\vec{A} = \binom{(\mathbf{e}_A)_x - (\mathbf{s}_A)_x}{(\mathbf{e}_A)_y - (\mathbf{s}_A)_y}$, hence $\vec{A'} = \binom{(\mathbf{e}_A)_y - (\mathbf{s}_A)_y}{(\mathbf{s}_A)_x - (\mathbf{e}_A)_x}$ and $\vec{P} = \binom{(\mathbf{s}_B)_x - (\mathbf{s}_A)_x}{(\mathbf{s}_B)_y - (\mathbf{e}_A)_y}$. Whenever $\mathbf{s}_B$ lies on the right of the line $\overline{\mathbf{s}_A \mathbf{e}_A}$, the inequation

$$^t\vec{A'} \cdot \vec{P} > 0 \tag{1}$$

holds. To change this into a equation, we introduce a new variable v.

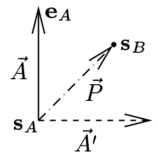


Figure 3. Constructing equations with the coordinates as variables

As v^2 can only take nonnegative values, the resulting equation

$$^t\vec{A'} \cdot \vec{P} - v^2 = 0 \qquad \text{with } v \in \mathbb{R}\backslash\{0\} \tag{2}$$

will have a solution iff the point $\mathbf{s}_B$ lies to the right of the line $\overline{\mathbf{s}_A \mathbf{e}_A}$. The equation 1 is modified in a similar way for the premise "left of" (l):

$$A \text{ l } \mathbf{s}_B \quad : \quad {}^t\vec{A'} \cdot \vec{P} + v^2 = 0 \qquad \text{with } v \in \mathbb{R}\backslash\{0\} \tag{3}$$

Note that the equations will only have a solution, when $\mathbf{s}_A \neq \mathbf{e}_A$. Constructing the dipole-point relations s and e is done by using the same variables for the identical points.

For the following substitutions $x_1 = (\mathbf{s}_A)_x$, $x_2 = (\mathbf{s}_A)_y$, $x_3 = (\mathbf{e}_A)_x$, $x_4 = (\mathbf{e}_A)_y$, $x_5 = (\mathbf{s}_B)_x$, $x_6 = (\mathbf{s}_B)_y$, $x_7 = (\mathbf{e}_B)_x$, $x_8 = (\mathbf{e}_B)_y$, and new introduced variables $v_1, \ldots, v_4$, the complete set of equations describing relation $A \text{ rrrr } B$ reads as:

$$
\begin{aligned}
-x_1x_4 + x_1x_6 + x_2x_3 - x_2x_5 - x_3x_6 + x_4x_5 - v_1^2 &= 0 \\
-x_1x_4 + x_1x_8 + x_2x_3 - x_2x_7 - x_3x_8 + x_4x_7 - v_2^2 &= 0 \\
-x_1x_6 + x_1x_8 + x_2x_5 - x_2x_7 - x_5x_8 + x_6x_7 - v_3^2 &= 0 \\
-x_3x_6 + x_3x_8 + x_4x_5 - x_4x_7 - x_5x_8 + x_6x_7 - v_4^2 &= 0
\end{aligned}
$$

with $x_1, \ldots, x_8 \in \mathbb{R}$, $v_1, \ldots, v_4 \in \mathbb{R}\backslash\{0\}$. The other relations are constructed in an analogous way.

3 Constraint Reasoning with the Dipole Calculus

For reasoning about the dipole relations we apply constraint-based reasoning techniques which were originally introduced for temporal reasoning [1] and which also proved valuable for spatial reasoning [12]. In order to apply these techniques to a set of relations, these relations must form a relation algebra [8], i.e., they must be closed under composition ($\circ$), intersection ($\cap$), complement ($\bar{\cdot}$), and converse ($\smile$) and there must be an empty relation, a universal relation, and an identity relation. While the converse (see Table 1), the complement, and the intersection of relations can be computed from the set-theoretic definitions of the relations, the composition of relations must be computed based on the semantics of the relations. The compositions are usually computed only for the atomic relations which are then stored in a composition table. The composition of compound relations can be obtained as the union of the compositions of the corresponding atomic relations.

We computed the compositions of the atomic relations using the algebraic semantics of the relations. For this we apply the method of "Gröbner Bases" using a geometric theorem prover [3]. A possible composition table entry $R_x \circ R_y \mapsto R_z$ is represented (for every combination of R_x, R_y, and R_z) by a set of equations. This set results from the union of three sets, one for each relation as shown in the previous section. $R_x(A, B) \wedge R_y(B, C) \wedge R_z(A, C)$ is a contradiction if and only if the set of equations has no solution. This can happen because of a equation with no solution (e.g. $x_i^2 = -1$) or a violation of the condition $v_1, \ldots, v_n \in \mathbb{R}\backslash\{0\}$ (e.g. $v_i^2 + v_j^2 = 0$). By computing the Gröbner Base, equations are generated which do not change the systems solution. These generated equations allow the prover to detect, if there cannot be a solution. For all combinations of R_x, R_y, and R_z where no contradiction was detected, we have to construct a possible configuration of points in the plane. Instead of generating this configuration from the equations (which can be quite complicated), we simply search for a valid configuration of points on a grid.

R	rrrr	rrrl	rrlr	rrll	rlrr	rllr	rlll	lrrr
$R^{\smile}$	rrrr	rlrr	lrrr	llrr	rrrl	lrrl	llrl	rrlr
R'	llll	lllr	llrl	llrr	lrll	lrrl	lrrr	rlll

R	lrrl	lrll	llrr	llrl	lllr	llll	ells	errs
$R^{\smile}$	rllr	lllr	rrll	rlll	lrll	llll	lsel	rser
R'	rllr	rlrr	rrll	rrlr	rrrl	rrrr	errs	ells

R	lere	rele	slsr	srsl	lsel	rser	sese	eses
$R^{\smile}$	rele	lere	srsl	slsr	ells	errs	sese	eses
R'	rele	lere	srsl	slsr	rser	lsel	sese	eses

Table 1. Converse and reflection table of the dipole calculus

The composition table for the atomic relations is given in Table 2 [4]. We use $*$ to mark places which can be filled with r or l. In order to reduce the size of the table, trivial cases (sese,eses) for the columns are omitted. Symmetric cases can be derived using the converse operation and a reflection operation (reflection on an axis, denoted R', see also Table 1). The missing entries can be calculated using the following equation:

$$R_1 \circ R_2 = (R_2^{\smile} \circ R_1^{\smile})^{\smile} = (R_1' \circ R_2')' \qquad (4)$$

Dipole constraints are written as xRy where x, y are variables for dipoles and R is a $\mathcal{DRA}_{24}$ relation. Given a set Θ of dipole constraints, an important reasoning problem is deciding whether Θ is *consistent*, i.e., whether there is an assignment of all variables of Θ with dipoles such that all constraints are satisfied (a *solution*). We call this problem DSAT. DSAT is a Constraint Satisfaction Problem (CSP) [9] and can be solved using the standard methods developed for CSP's with infinite domains (see, e.g. [8]).

A partial method for determining inconsistency of a set of constraints Θ is the *path-consistency method* which enforces path-consistency on Θ [9]. A set of constraints is path-consistent if and only if for any two variables, there exists an instantiation of any third variable such that the three values taken together are consistent. It is necessary but not sufficient for the consistency of a set of constraints that path-consistency can be enforced. A naive way to enforce path-consistency is to strengthen relations by successively applying the following operation until a fixed point is reached:

$$\forall i, j, k : \quad R_{ij} \leftarrow R_{ij} \cap (R_{ik} \circ R_{kj})$$

where i, j, k are nodes and R_{ij} is the relation between i and j. The resulting set of constraints is equivalent to the original set, i.e., it has the same set of solutions. If the empty relation occurs while performing this operation Θ is inconsistent, otherwise the resulting set is path-consistent. In Section 6 we use the path-consistency method to solve a small navigation problem with the dipole calculus.

4 Computational Properties of the Dipole Calculus

Although we restricted the possible binary relations between dipoles to 24 atomic relations, $\mathcal{DRA}_{24}$ is very expressive. For instance, it is possible to express directed and undirected graphs and their properties such as planarity or (convex) cycles. Hence, it is not surprising that $\mathsf{DSAT}(\mathcal{DRA}_{24})$ is NP-hard which can be shown by reduction of the BETWEENNESS problem (Instance: Finite set A, collection C of ordered triples (a, b, c) of distinct elements from A, Question: Is there a one-to-one function f from A to $1, 2, ..., |A|$ such that for each (a, b, c) in C, $f(a) < f(b) < f(c)$ or $f(c) < f(b) < f(a)$. [5])

Theorem 1 $\mathsf{DSAT}(\mathcal{DRA}_{24})$ *is* NP-*hard*

Proof. Reduction from BETWEENNESS. Given a finite set A and a collection C of ordered triples (a, b, c) of distinct elements from A. For every element a of A introduce two dipoles a_1 and a_2 such that $a_1\{ells, errs\}a_2$ holds. For every pair a, b of distinct elements of A we require that $a_1\{slsr, srsl\}b_1$, $a_1\{lere, rele\}b_1$, and $a_i\{rllr, lrrl\}b_i$ (for $i = 1, 2$) holds. The latter constraint guarantees that the graph formed by the dipoles $a_1, a_2, b_1, b_2, \dots$ is planar (see Figure 4).

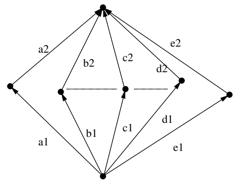
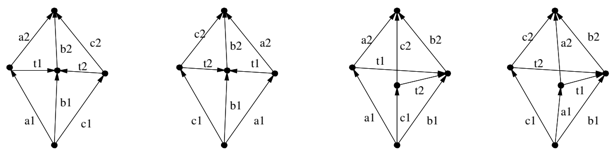


Figure 4. Reduction of a set A to a graph of dipoles



(a) (b) (c) (d)

Figure 5. Reduction of a triple (a, b, c) to dipole constraints. If b is between a and c, the constraints are satisfied (see (a),(b)), if b is between a and c, then either $t1$ overlaps $c1$ or $c2$ or $t2$ overlaps $a1$ or $a2$ which contradicts the constraints (see (c),(d)).

For every ordered triple $t = (a, b, c)$ we introduce the two dipoles t_1, t_2 and the constraints $a_1\{ells, errs\}t_1$, $b_1\{lere, rele\}t_1$, $b_1\{lere, rele\}t_2$, $c_1\{ells, errs\}t_2$, $a_i\{rllr, lrrl\}t_2$, and $b_i\{rllr, lrrl\}t_1$. As it can be seen in Figure 5, these constraints guarantee that the set of dipole constraints Θ is consistent iff there a one-to-one function f from A to $1, 2, ..., |A|$ such that for each (a, b, c) in C, $f(a) < f(b) < f(c)$ or $f(c) < f(b) < f(a)$. □

We have so far neither been able to prove that $\mathsf{DSAT}(\mathcal{DRA}_{24})$ is a member of NP nor whether reasoning over the atomic relations is tractable. However, it follows from the algebraic semantics of the relations that $\mathsf{DSAT}(\mathcal{DRA}_{24})$ is a member of PSPACE. This is because all relations can be expressed as equalities over polynomials with integer coefficients. Systems of such equalities can be solved using polynomial space [11].

5 An Extended Version of the Dipole Calculus

In certain domains we might want to represent spatial arrangements in which more than two start or end points of dipoles are on a straight line. Then we need three more dipole-point relations. The additional relations describe the cases when the point is straight behind the dipole (b), in the interior of the dipole (i) or straight in front of the dipole (f). The corresponding regions are shown on Figure 6. Such a set of relations was proposed by Freksa [4].

[4] An electronic version of the table can be obtained at `http://www.informatik.uni-hamburg.de/WSV/DRA`

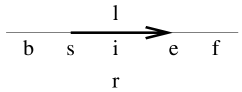


Figure 6. Extended dipole-point relations

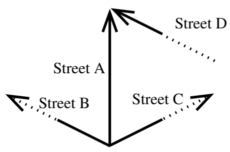


Figure 7. Navigating in an arrangement of one-way streets

Using the same notation scheme like the one for the coarse Dipole Relation Algebra $\mathcal{DRA}_{24}$ we get the following 69 atomic relations: { rrrr, rrrl, rrlr, rrll, rlrr, rllr, rlll, lrrr, lrrl, lrll, llrr, llrl, lllr, llll, ells, errs, lere, rele, slsr, srsl, lsel, rser, sese, eses, lllb, llfl, llbr, llrf, lirl, lfrr, lril, lrri, blrr, irrl, frrr, rbrr, lbll, flll, brll, rfll, rlli, rrlf, illr, rilr, rrbl, rlir, rrfr, rrrb, ffbb, efbs, ifbi, bfii, sfsi, beie, ffbb, bsef, biif, iibf, sisf, iebe, ffff, fefe, fifi, fbii, fsei, ebis, iifb, eifs, iseb }. The derived fine grain Dipole Relation Algebra is called $\mathcal{DRA}_{69}$. $\mathcal{DRA}_{69}$ contains Allen's interval relations [1] as a special case:

$=$	$\mapsto$	sese			
$<$	$\mapsto$	ffbb	$>$	$\mapsto$	bbff
m	$\mapsto$	efbs	mi	$\mapsto$	bsef
o	$\mapsto$	ifbi	oi	$\mapsto$	biif
d	$\mapsto$	bfii	di	$\mapsto$	iibf
s	$\mapsto$	sfsi	si	$\mapsto$	sisf
f	$\mapsto$	beie	fi	$\mapsto$	iebe

In these cases the dipoles are on the same straight line and have the same direction. $\mathcal{DRA}_{69}$ also contains 10 additional relations which correspond to the case with dipoles on a line and opposite directions (only 10 out of 13 because there are more self-converse cases). The composition table for $\mathcal{DRA}_{69}$ can be obtained at `http://www.informatik.uni-hamburg.de/WSV/DRA`.

6 A Sample Application of the Dipole Calculus

The dipole calculus can be used in navigation domains. A small example shows a scenario in which a car navigates through a network of one-way streets (see Figure 7). The car starts from street A and wants to reach a goal within street D. Because of the direction of D it cannot enter D directly from A. Therefore the car has to enter B or C to reach D. It is unknown whether B of C meets D. Only the position of the streets with respect to A is known. We now can use $\mathcal{DRA}_{24}$ to express our initial knowledge:

$$A\{\mathrm{slsr}\}B, \quad A\{\mathrm{srsl}\}C, \quad A\{\mathrm{rele}\}D \tag{5}$$

The question is whether street B or street C can be used to drive into street D:

$$B\{\mathrm{ells}, \mathrm{errs}\}D \tag{6}$$

$$C\{\mathrm{ells}, \mathrm{errs}\}D \tag{7}$$

To decide this question we build two sets of constraints, Θ_1 contains the constraints (5), (6) (corresponding to the assumption B meets D) and Θ_2 contains the constraints (5), (7). By applying the path-consistency method to both sets it turns out that Θ_1 contains a contradiction while path-consistency can be enforced to Θ_2. This gives us the following solution to the navigation problem: street B cannot meet street D, street C has a chance to meet street D. Thus, we have a good reason to turn into street C instead of street B.

7 Related Work

Schlieder [13] suggested a calculus for reasoning about oriented line segments which is based on clockwise and counter-clockwise orientation of triples of points. Schlieder's calculus does, however, not form a relation algebra (e.g., it does not contain an identity relation) and as such does not allow using constraint based reasoning methods. Instead, Schlieder uses inferences based on conceptual neighborhood structures. The double-cross calculus by Freksa [4] describes relations between triples of points, which can be regarded as relationships between a dipole and an isolated point. In contrast to Freksa's ternary relations, our dipole relations are binary relations which makes reasoning much easier. Also, Freksa distinguishes more possible relations between a dipole and a point than we do. Isli and Cohn [7] developed a ternary algebra for reasoning about orientations. Their algebra has a tractable subset containing the base relations.

8 Conclusion and Perspective

We presented a calculus for representing and reasoning about qualitative intrinsic orientation information. We chose oriented line segments as the basic entities since they are the simplest spatial entities that show two important features of physical objects: they have an intrinsic orientation and they are extended. We identified a system of 24 atomic relations between dipoles and computed the composition table based on their algebraic semantics, which allows for applying constraint-based reasoning methods. We further proved that reasoning over these relations is NP-hard and in PSPACE. It is a matter of further studies whether the calculus is in NP and whether reasoning over the atomic relations is even tractable. Potential applications of the calculus are demonstrated with a small navigation example.

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Rel BC / Rel AB	llll	lllr	llrl	llrr	lrrl	lrrr	rlrr	srsl	errs	rele
llll	sese lsel rser lere rele ells errs slsr srsl ll** r*l* *r*l **rr	lsel lere errs srsl l**l *r*l l*r* *rr*	rser rele ells slsr *ll* r*l* *l*r r**r	eses lsel rser lere rele ells errs slsr srsl ll** r*l* *r*l **rr	ells slsr *ll* l*ll *l*r l*rr	ells errs slsr srsl *ll* *r*l **rr	lsel rser lere rele l**l r*l* **rr	lrrr lrll llrr lllr llll	lrrr lrll llrr lllr llll	rlrr rlll llrr llrl llll
lllr	lsel rser slsr srsl ll** *r*l r**r	lsel srsl l**l lll* *r*l	rser slsr llr* *l*r r**r	lsel rser slsr srsl ll** *r*l r**r	lere slsr l*r* *l*r	lere slsr l*r* *l*r	eses lsel rser ells errs *ll* l**l *rr* r**r	lere lrrr lrrl llrr llrl lllr	lere lrrr lrrl llrr llrl lllr	ells rllr rlll llrl lllr llll
llrl	lere rele ells errs ll** r*l* *rr*	lere errs l*r* *rr* ll*r	rele ells *ll* ll*l r*l*	lere rele ells errs ll** r*l* *rr*	lsel ells *ll* l**l	eses lsel rser ells errs *ll* l**l *rr* r**r	lere slsr l*r* *l*r	lsel lrrl lrll llrl lllr llll	lsel lrrl lrll llrl lllr llll	slsr rlrr rllr llrr llrl lllr
llrr	**ll ll** rr** **rr	lll* *rll l*rr *rrr	ll*l r*ll *lrr r*rr	**ll ll** rr** **rr	lsel lere *lll l**l l*r* *lrr	lsel rser lere rele l**l r*l* **rr	ells errs slsr srsl *ll* *r*l **rr	lsel lere lrrr lrrl lrll llrr llrl llll	lsel lere lrrr lrrl lrll llrr llrl llll	ells slsr rlrr rllr rlll llrr lllr llll
lrll	rser rele ells slsr *ll* r*l* *l*r r**r	sese lsel rser lere rele ells errs slsr srsl ll** r*l* *r*l **rr	eses lsel rser lere rele ells errs slsr srsl ll** r*l* *r*l **rr	lsel lere errs srsl l**l *r*l l*r* *rr*	ells errs slsr srsl *ll* *r*l **rr	errs srsl *r*l *rr* r*rr	lsel lere l**l l*r* *lrr	lrll llrr lllr llll	rrrr rrrl rrll rlrr	rrrr rrlr rrll lrll
lrrl	rele ells *ll* ll*l r*l* rr*l	lere rele ells errs ll** r*l* *rr*	lere rele ells errs ll** r*l* *rr*	lere errs l*r* *rr* ll*r rr*r	eses lsel rser ells errs *ll* l**l *rr* r**r	rser errs *rr* r**r	lere slsr l*r* *l*r	lsel lrrl lrll llrl lllr llll	rser rrrr rrrl rrlr rlrr rllr	srsl rrrl rrlr rrll lrrl lrll
lrrr	ll*l r*ll llr* rrr*	lll* r*ll rrr* l*rr	ll*l r*ll rr*r l*rr	lll* rrl* l*rr *rrr	lsel rser lere rele l**l r*l* **rr	rser rele r*l* r**r *rrr	ells slsr *ll* *l*r l*rr	lsel lere lrrr lrrl lrll llrr llrl llll	rser rele rrrr rrlr rrll rlrr rllr rlll	errs srsl rrrr rrrl rrll lrrr lrrl lrll
rlll	lsel lere errs srsl l**l *r*l l*r* *rr*	eses lsel rser lere rele ells errs slsr srsl ll** r*l* *r*l **rr	sese lsel rser lere rele ells errs slsr srsl ll** r*l* *r*l **rr	rser rele ells slsr *ll* r*l* *l*r r**r	ells errs slsr srsl *ll* *r*l **rr	ells slsr *ll* *l*r l*rr	rser rele r*l* r**r *rrr	rrrr rrrl rrll rlll	lrrr llrr lllr llll	rlll llrr llrl llll
rllr	lsel srsl l**l lll* *r*l rrl*	lsel rser slsr srsl ll** *r*l r**r	lsel rser slsr srsl ll** *r*l r**r	rser slsr llr* rrr* *l*r r**r	sese lere rele slsr srsl r*l* *r*l l*r* *l*r	lere slsr l*r* *l*r	rser errs *rr* r**r	rele rrrl rrlr rrll rllr rlll	lere lrrr lrrl llrr llrl lllr	ells rllr rlll llrl lllr llll

rel BC / rel AB	llll	lllr	llrl	llrr	lrrl	lrrr	rlrr	srsl	errs	rele
rlrr	lll* *rll ll*r rr*r	lll* *rll rrr* *lrr	ll*l *rll rr*r *lrr	ll*l rr*l *lrr r*rr	lsel rser lere rele l**l r*l* **rr	lsel lere l**l l*r* *lrr	errs srsl *r*l *rr* r*rr	rser rele rrrr rrlr rrll rlrr rllr rlll	lsel lere lrrr lrrl lrll llrr llrl llll	ells slsr rlrr rllr rlll llrr lllr llll
rrll	eses lsel rser lere rele ells errs slsr srsl ll** r*l* *r*l **rr	rser rele ells slsr *ll* r*l* *l*r r**r	lsel lere errs srsl l**l *r*l l*r* *rr*	sese lsel rser lere rele ells errs slsr srsl ll** r*l* *r*l **rr	errs srsl r*ll *r*l *rr* r*rr	ells errs slsr srsl *ll* *r*l **rr	lsel rser lere rele l**l r*l* **rr	rrrr rrrl rrll rlrr rlll	rrrr rrrl rrll rlrr rlll	rrrr rrlr rrll lrrr lrll
rrlr	lsel rser slsr srsl ll** *r*l r**r	rser slsr rrr* *l*r r**r	lsel srsl l**l *r*l rrl*	lsel rser slsr srsl ll** *r*l r**r	rele srsl r*l* *r*l	sese lere rele slsr srsl r*l* *r*l l*r* *l*r	rser errs *rr* r**r	rele rrrl rrlr rrll rllr rlll	rele rrrl rrlr rrll rllr rlll	errs rrrr rrrl rrlr lrrr lrrl
rrrl	lere rele ells errs ll** r*l* *rr*	rele ells *ll* r*l* rr*l	lere errs l*r* *rr* rr*r	lere rele ells errs ll** r*l* *rr*	rser errs *rr* r**r	rser errs *rr* r**r	sese lere rele slsr srsl r*l* *r*l l*r* *l*r	rser rrrr rrrl rrlr rlrr rllr	rser rrrr rrrl rrlr rlrr rllr	srsl rrrl rrlr rrll lrrl lrll
rrrr	**ll ll** rr** **rr	*lll rr*l *lrr r*rr	l*ll rrl* l*rr *rrr	**ll ll** rr** **rr	rser rele r*l* *rll r**r *rrr	lsel rser lere rele l**l r*l* **rr	ells errs slsr srsl *ll* *r*l **rr	rser rele rrrr rrlr rrll rlrr rllr rlll	rser rele rrrr rrlr rrll rlrr rllr rlll	errs srsl rrrr rrrl rrll lrrr lrrl lrll
srsl	rele ells rrlr rrll rllr rlll lllr llll	rele ells rrlr rrll rllr rlll lllr llll	lere errs rrrr rrrl lrrr lrrl llrr llrl	lere errs rrrr rrrl lrrr lrrl llrr llrl	errs rrrr rrrl rlrr lrrr lrrl	errs rrrr rrrl rlrr lrrr lrrl	lere rlrr lrrr lrrl llrr llrl	slsr srsl	rrrr rrrl rlrr	rrlr rrll lrll
slsr	rrlr rrll lrll lllr llll	rrll lrll lllr llll	rrrr rlrr llrr llrl	rrrr rrrl rlrr llrr llrl	lere rlrr lrrr lrrl llrr llrl	lere rlrr lrrr lrrl llrr llrl	errs rrrr rrrl rlrr lrrr lrrl	sese slsr srsl	lere lrrr lrrl llrr llrl	ells rllr rlll lllr llll
errs	rrrl rrll rlll llrl llll	rrrl rrll rlll llll	rrrr rrlr lrrr llrr	rrrr rrlr lrrr llrr lllr	rser rrrr rrlr rlrr rllr lrrr	rser rrrr rrlr rlrr rllr lrrr	slsr rlrr rllr lrrr llrr lllr	eses ells errs	rser rrrr rrlr rlrr rllr	srsl rrrl rrll lrrl lrll
ells	lsel srsl rrrl rrll lrrl lrll llrl llll	lsel srsl rrrl rrll lrrl lrll llrl llll	rser slsr rrrr rrlr rlrr rllr llrr lllr	rser slsr rrrr rrlr rlrr rllr llrr lllr	slsr rlrr rllr lrrr llrr lllr	slsr rlrr rllr lrrr llrr lllr	rser rrrr rrlr rlrr rllr lrrr	ells errs	lrrr llrr lllr	rlll llrl llll
rele	lsel srsl rrrl rrll lrrl lrll llrl llll	rser slsr rrrr rrlr rlrr rllr llrr lllr	lsel srsl rrrl rrll lrrl lrll llrl llll	rser slsr rrrr rrlr rlrr rllr llrr lllr	srsl rrrl rrll rlll lrrl lrll	slsr rlrr rllr lrrr llrr lllr	rser rrrr rrlr rlrr rllr lrrr	rrrl rrll rlll	ells errs	lere rele
lere	rrrl rrll rlll llrl llll	rrrr lrrr llrr lllr	rrll rlll llrl llll	rrrr rrlr lrrr llrr lllr	lsel rlll lrrl lrll llrl llll	rser rrrr rrlr rlrr rllr lrrr	slsr rlrr rllr lrrr llrr lllr	lsel lrrl lrll llrl llll	eses ells errs	sese lere rele
rser	rrlr rrll lrll lllr llll	rrrr rrrl rlrr llrr	rrlr rrll lrll llll	rrrr rrrl rlrr llrr llrl	rele rrlr rrll rllr rlll lrll	lere rlrr lrrr lrrl llrr llrl	errs rrrr rrrl rlrr lrrr lrrl	rele rrlr rrll rllr rlll	sese slsr srsl	eses lsel rser
lsel	rele ells rrlr rrll rllr rlll lllr llll	lere errs rrrr rrrl lrrr lrrl llrr llrl	rele ells rrlr rrll rllr rlll lllr llll	lere errs rrrr rrrl lrrr lrrl llrr llrl	ells rllr rlll lrll lllr llll	errs rrrr rrrl rlrr lrrr lrrl	lere rlrr lrrr lrrl llrr llrl	lrll lllr llll	slsr srsl	lsel rser
sese	llll	lllr	llrl	llrr	lrrl	lrrr	rlrr	srsl	errs	rele
eses	rrll	rrrl	rrlr	rrrr	rllr	rlrr	lrrr	ells	slsr	lsel

Table 2. Composition table of the atomic relations of $\mathcal{DRA}_{24}$