The Occlusion Calculus

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Abstract

A lot of effort in Qualitative Reasoning had been spent in the RCC-8 calculus. This paper proposes a calculus named OCC (Occlusion Calculus) closely related to Galton's LOS-14 calculus, that is more expressive in a vision context. The OCC relations qualitatively describe configurations from two convex objects in the projective view from a 3D scene. To set OCC on a mathematical ground an axiomatisation of the derived relation calculus is given. Since OCC only focuses on one qualitative aspect of space it is sketched, how and when different calculi can be combined to assemble a knowledge base for a cognitive vision system on a conceptual level.

1 Introduction

This work is part of the CogViSys (http://cogvisys.iaks.uni-karlsruhe.de/mainpage.html) project. A major goal of the project is to demonstrate the usefulness of explicit knowledge representation in computer vision applications. Part of this work is the development of a query language for a knowledge base on spatio-temporal configurations. The formalism should be as domain-independent as possible by preserving the most of its expressiveness. It is envisioned to build a query language comparable to SQL for a cognitive vision system.

Relations are a common way to represent knowledge taken from a picture or a sequence of pictures explicitly. The visual content of pictures is described using relations in the language. Relations are useful to describe spatial or temporal knowledge, especially when no quantitative measure is needed or appropriate. Another interesting fact is the possibility to deduce new knowledge based on old. Whenever two observations are made a third observation has to be consistent if it is 'related'. Here is an example for this in a temporal domain: If A happened before B and B occured before C, A took place before C, since a linear order of time is commonly assumed. This sort of deduction, can be processed in compositional reasoning.

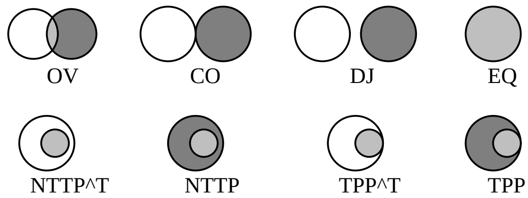

Figure 1. The RCC-8 Basis Relations

Qualitative Calculi like the region connection calculus (RCC-8) [9] or the Interval Algebra (IA) [1] and especially the CSP problems defined using these calculi to build a constraint language (e.g. [10]) are a topic of AI research for quite a time. From the picture of the RCC-8 basis relations in figure 1) the reader might gain the idea, that a calculus like RCC-8 can be used to represent relationships between objects in a vision scene. The idea is probably not bad, but a closer look at the RCC-8 calculus reveals, that due to its topological nature it cannot be used to describe visual occlusion. Visual occlusion might be a key phenomena for a cognitive vision system since humans use information gained by occlusion to derive knowledge from the observed scene. Galton [3] proposed the geometrically motivated Lines of Sights Calculus (LOS-14) and pointed out its potential usefulness for vision applications. Following this ideas [2] proposed the ROC-20 calculus generalizing the LOS-14 calculus by relations for mutual occluding obe-

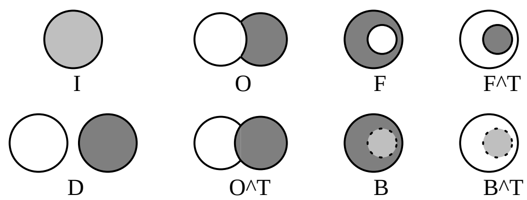

Figure 2. The OCC Basis Relations

jects. This might be useful, if the objects in a 3D scene are not convex. A spatio-temporal calculus called TeRCC-8 is investigated in [4].

The approach given in this paper focuses on a "subset" shown in figure 2 (a better definition for it will be given in section 3) of the Lines of Sights Calculus. Following Galton's ideas an axiomatic theoretic background on relation algebras is given. This shows a general method to generate new calculi from known old ones. All formal definitions needed to understand the work are given in the text, being illustrated by examples from the developed combined calculus.

The following sections are organized as follows: Section 2 gives the basic definitions for finite concrete relational algebras illustrated by two examples. A gentle introduction to the field of relations and graphs can be found in [11], where [8] is a vastly exhaustive book on relation algebras. Relation algebras should not be confused with the algebraic formulation of operators on relational databases. The generalization of binary constraint satisfaction problems to relation algebras is the topic of [7].

Section 3 shows methods to build combined relation algebras for the aimed application. In section 4 we illustrate the usefullness of the considered theorems on concrete relational algebras to construct the Occlusion Calculus. The neighbourhood structure shown in figure 9 gives hints to make this calculus more expressive if it is used in a temporal context. Concluding this paper there will be a brief outlook of directions on further research.

2 Concrete Relation Algebras

One way to represent knowledge on relationships between objects (e.g.: blobs segmented from pictures, a detected event at an instance of time, the cardinal direction of a moving object, the spatial configuration described qualitatively by up, down, left and right, the connection of visible surfaces from a rotating object in 3D space ...) is using relations. Modelling knowledge with relations is a promising approach for cognitive vision application: Relations can map infinitly many cases to a finite structure representing symmetries on a conceptual layer and qualitative knowledge, where no quantitative measure is accessible or appropriate. Like probably most other approaches, this is not the only holy stone of wisdom.

If objects are elements from a set a formal definition for a relation can be given in a merly set theoretic term by:

Definition (Binary Relation):
Let V be a set. A subset R of the cartesian product $V \times V$ is called a *binary relation on V.*

A Relation is called binary *homogen* relation, if it is based on the cartesian product of a single set $V \times V$, *heterogen* if it is based on a product of diffrent sets $V \times W$. A *ternary* relation binds three arguments and a relation on more than three arguments can be defined as well. In the following text we only make use of binary homogen relations and refer to them simply as relation.

If V is a set of regions based on a dyadic predicate $C(a,b)$ of connection between two regions, a set of relations $\mathcal{R}$ can be defined by the intersection of the inner, outer and border of the two regions. The set of relations shown in figure 1 is called the RCC-8 calculus [9]. Each binary relation can be characterised by a function:

Definition (Characteristic Function of a Relation):
Let $a, b \in V$ and R a (binary) relation on V. Then

$$\chi_R : V \times V \rightarrow \begin{cases} 1 & ; (a,b) \in R \\ 0 & ; (a,b) \notin R \end{cases}$$

is called the *characteristic function of* R.

For finite sets of elements ($|V| < \infty$) all relations defined on V can be written as a $n \times n$ matrix of boolean values from χ_R. Where a one at line a and column b stands for aRb, which is the infix notation of $R(a,b)$ holds for $(a,b) \in V \times V$. There is a duality between graphs and relations, since the matrix of a relation can be interpreted as an adjacency matrix for a graph, where the set of elements V are the edges and the set of vertices is implicitly given by the matrix. In this case knowledge can be encoded efficiently using boolean matricies.

The union, intersection and composition of relations can be defined in merely set theoretic terms by:

Definition (Union, Intersection, Negation, Composition):
Let $a, b, c \in V$ and R, S be relations von V.

$$\begin{aligned} R \sqcup S &:= \{(a,b) \in R \vee (a,b) \in S\} \\ R \sqcap S &:= \{(a,b) \in R \wedge (a,b) \in S\} \\ \overline{R} &:= \{(a,b) | (a,b) \notin R\} \\ R \circ S &:= \{(a,c) \in V \times V | \exists b \in V : \\ &\qquad (a,b) \in R \wedge (b,c) \in S\} \end{aligned}$$

For every relation R a converse relation R^T is properly defined by:

Definition (Converse Relation, Symmetric Relation):
Let $a, b \in V$ and R a relation on V. Then

$$R^T := \{(a,b) | (b,a) \in R\}$$

is called the *converse relation* of R. A relation that is selfconvers ($R = R^T$) is called *symmetric*.

The relations EQ,DJ,EC and PO in figure 1 are symmetric. TTP is converse to TTP^T, $NTTP$ is converse to

$NTTP^T$ and vice versa. The notation ^{T} for converse relations was chosen, because of its similarity to matrix transposition in linear algebra. In the finite case the boolean matrix of the converse relation can be obtained by transposition.

The next definition gives a more precise definition, when a set of relations is exhaustive to describe all possible cases for two elements taken from the set V.

Definition (Cover):
A set of relations $\mathcal{R} = \{R_i\}_{i \in \mathcal{I}}$ is called a *cover (of $V{\times}V$)*, iff:

$$\bigsqcup_{i \in \mathcal{I}} R_i = V{\times}V.$$

If a set of relations $\mathcal{R}$ is a cover of $V \times V$ at least one relation R_i holds for any pair of objects from V. A further refinement of a cover is defined by:

Definition (Relational Partition):
A set of relations $\mathcal{R} = \{R_i\}_{i \in \mathcal{I}}$ is called a *relational partition*, iff:

1. $\mathcal{R}$ is a cover of $V{\times}V$
2. $R_k \sqcap R_l = \emptyset \quad \forall k \neq l;\ k, l \in \mathcal{I}$
   (pairwise jointly disjunct) .

Exactly one relation from a relational partition $\mathcal{R}$ holds for each pair of objects taken from the ground set V. A union of relations holds if at least one relation holds for two objects. Relations itself can be treated as elements from an algebra. Since algebras are closed on their operations and unions and intersections are operations on relations, the closure of a set of relations is defined by:

Definition (Closure of a Set):
Let $\mathcal{R}$ be a set of relations. Then the set $\mathcal{R}^*$ containing all combinations of unions and intersections from relations R_i, is called the *closure* of $\mathcal{R}$.

Relations are itself subsets from the cartesian product. If a set of relations is finite, the closure of a set of relations is a power set of the set of its smallest elements build by intersections, that are not the empty set. These relations are called *atomic*. A set of atomic relations builds a base for a finite concrete relation algebra:

Definition (Concrete (finite) Relation Algebra):
A closure $\mathcal{R}^*$ of a set of relations on V, is called an *concrete (finite) relation algebra (on V)*, iff $\mathcal{R}^*$ is closed under negation, conversion and composition. The relation algebra $\mathcal{A}$ is denoted by the tuple $\mathcal{A} = (\mathcal{R}^*, \sqcup, \sqcap, \circ, \bar{\ }, ^T)$.

Since a concrete relation algebra is defined on a set of relations $\mathcal{R}^*$, which is a closure of $\mathcal{R}$, every concrete relation algebra is isomorphic to a power set of its atoms. In the finite case the cardinality of $\mathcal{R}^*$ is a power of 2. Furthermore every finite concrete relation algebra has a finite set of atomic relations:

Definition (Basis of a Concrete Relation Algebra):
A set of atomic relations $\mathcal{R}$ is called a *basis* of a Relation algebra $\mathcal{A}$ (on V), iff $\mathcal{R}$ is a relational partition of $V{\times}V$ and it is closed under conversion.

Another nice property of finite concrete relation algebras is:

Theorem (uniqueness, exhaustiveness of basis relations):
All elements from a finite relation algebra $\mathcal{R}^*$ can be written as union from the basis elements in $\mathcal{R}$:

$$\forall S \in \mathcal{R}^* : S = R_1 \sqcup R_2 \sqcup \ldots \sqcup R_m \qquad \{R_i\}_{i \in \{1,\ldots,m\}} \in \mathcal{R}.$$

The last theorem leads to an important consequence. Since every element from a finite relation algebra can be written by basis elements all compositions can be calculated from the basis elements, according to the rule:

$$(R_1 \sqcup R_2) \circ R_3 = R_1 \circ R_3 \sqcup R_2 \circ R_3$$

Composition tables allows fast computation by table lookup. Each composition can be seen as a deduction of knowledge in the following way: If R holds for the two elements (a, b) from $V{\times}V$, and S holds for (b, c), then $R \circ S$ must hold for (a, c). Up to this point all definitions given were abstract. The following two paragraphs, will give two examples of finite concrete relation algebras.

The Point Algebra (PA): Interpretations of the point algebra are well known. An Interpretation of PA can be given by a dense linear ordered but infinite set [6] of elements where the three operators are defined on. Given two numbers from the set of real numbers $\mathbb{R}$, PA's composition table shows in figure 3, what relation aRc might hold if aRb and bRc holds (aRb is denoted on the rows, bRc on the columns): The reader might wonder why $\mathbb{N}$ with the ordinary operators $=, <, >$ denoted by $\mathbb{N}_=, \mathbb{N}_<, \mathbb{N}_>$ is not an interpretation of PA. Reconsider the definition of the composition for relations. If two succsessive numbers from $\mathbb{N}$ are chosen the

	$=$	$<$	$>$
$=$	$=$	$<$	$>$
$<$	$<$	$<$	$<,=,>$
$>$	$>$	$<,=,>$	$>$

Figure 3. The PA Composition Table

composition is not closed, since

$$\begin{aligned} & \text{choose} \quad a = 1\,;\, c = 2\,;\, R = \mathbb{N}_< \,;\, S = \mathbb{N}_> \\ \Rightarrow \quad & R \circ S = \{(a,c) | \exists b \in \mathbb{N} : (1,b) \in \mathbb{N}_< \wedge (b,2) \in \mathbb{N}_> \} \\ \Rightarrow \quad & R \circ S = \emptyset \notin \{\mathbb{N}_=, \mathbb{N}_<, \mathbb{N}_> \} \end{aligned}$$

leads to contradiction.

The Containment Algebra (CA): Like RCC-8, RCC-5 relations express the relation between two regions. If borders are not taken into account, the set of RCC-8 basis relations can be merged to 5 relations. These resulting basis relations are shown in figure 4. The composition of all RCC-5

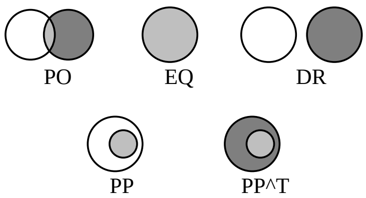

Figure 4. The RCC-5 Basis Relations

basis relations is shown in figure 5. RCC-5 is an example of a finite concrete relation algebra; it is closed under composition, converse, negation, and disjunction. The algebra is purely defined on containment, therefore it is sometimes named containment algebra (CA). An important subset of

	EQ	DR	PO	PP	PP^T
EQ	EQ	DR	PO	PP	PP^T
DR	DR	L	DR, PO, PP	DR, PO, PP	DR
PO	PO	DR, PO, PP^T	L	PO, PP	DR, PO, PP^T
PP	PP	DR	DR, PO, PP	PP	L
PP^T	PP^T	DR, PO, PP^T	PO, PP^T	EQ, PP, PP^T	PP^T

Figure 5. The RCC-5 Composition Table

the RCC-5 algebra is:

$$\begin{aligned} I &:= EQ \\ D &:= DR \\ O &:= EQ \sqcup PO \sqcup PP \sqcup PP^T \\ P &:= EQ \sqcup PP \\ P^T &:= EQ \sqcup PP^T \end{aligned}$$

The given subset is a cover but no basis, since O, P and P^T all contain EQ and the intersection of them is not the empty set. Let L denote the *universal Relation* ($L = \bigsqcup_{i \in \mathcal{I}} R_i$). Then a composition of the set can be calculated from basis relations $\{EQ, DR, PO, PP, PP^T\}$. It is shown in the table 6.

	I	D	O	P	P^T
I	EQ	DR	EQ, PO, PP, PP^T	EQ, PP	EQ, PP^T
D	DR	L	DR, PO, PP	DR, PO, PP	DR
O	EQ, PO, PP, PP^T	DR, PO, PP^T	L	EQ, PO, PP, PP^T	L
P	EQ, PP	DR	L	EQ, PP	L
P^T	EQ, PP^T	DR, PO, PP^T	EQ, PO, PP, PP^T	EQ, PP, PP^T	EQ, PP^T

Figure 6. Composition Table for the set $\{I, D, O, P, P^T\}$

3 Combining Concrete Relation Algebras

The point algebra has 3 relations that can be used to represent knowledge on a partially ordered dense set like $\mathbb{R}$ with $\mathbb{R}_=, \mathbb{R}_<, \mathbb{R}_>$. CA (RCC-5) can be used to represent containment. Both algebras are very general and can be used with different operational semantics on different sets of object. Further more finite concrete relation algebras can be combined to form new finite concrete relation algebras. The direct product of two finite concrete relation algebras builds a finite concrete relation algebra on tuples. Formally this can be denoted by:

Theorem (Direct Product of Concrete Relational Algebras): Let $\mathcal{R}$ be a basis for a relation algebra $\mathcal{A}$ on V and $\mathcal{S}$ be a basis for a relation algebra $\mathcal{B}$ on W, where $V \cap W = \emptyset$, then

$$\mathcal{A}\times\mathcal{B} := \left((\bigsqcup_{i\in\mathcal{I},j\in\mathcal{J}} (\mathcal{R}_i,\mathcal{S}_j))^*, \sqcup, \sqcap, \bar{}, \circ, ^T \right)$$

is a relation algebra on $V\times W$, where composition, intersection and union are declared componentwise for the tuples $(\mathcal{R}_i,\mathcal{S}_j)$.

The knowledge of basis elements is sufficient, to construct the direct product. The new relations from the product algebra can be calculated from the old ones, since the new base is build from the old ones. A composition table can be calculated from the known algebras as well, using the rule

$$(R_i,S_k)\circ(R_j,S_l) = (R_i\circ R_j, S_k\circ S_l)$$

RCC-8 and RCC-5 are closely related algebras. With the help of the following two definitions we can give a precise description of the kind of relationship:

Definition (Coarser and Finer Sets of Relations):
Let $\mathcal{R}$ and $\mathcal{S}$ be two sets of relations on V. Then $\mathcal{R}$ is called a finer set of relations than $\mathcal{S}$, iff:

$$\forall R_i \in \mathcal{R}\ \exists R_j \in \mathcal{S} : R_i \subset R_j.$$

$\mathcal{R}$ is called a coarser set of relations than $\mathcal{S}$, iff:

$$\forall R_i \in \mathcal{R}\ \exists R_j \in \mathcal{S} : R_i \supset R_j.$$

Since the basis from RCC-5 is a coarser set of relations, than the basis of RCC-8 and both structures are closed to all operation of a relation algebra the following definition is motivated to have the common name for it:

Definition (Subalgebra):
Every subset of the set of relations from a finite relation algebra $\mathcal{A}$ that is closed under union, intersection, negation and conversion is a called a *subalgebra* of $\mathcal{A}$

For finite concrete relation algebras there is a simple way to find subalgebras, since some of them can be constructed by a coarser basis:

Theorem (Coarser Relation Algebra):
Let $\mathcal{R}^*$ a finite relation algebra with basis $\mathcal{R} = \{R_i\}_{i\in I}$. Then

$$\left(\mathcal{R} \setminus \{R_i, R_j, R_i^T, R_j^T\} \cup \{R_i \sqcup R_j, R_i^T \sqcup R_j^T\}\right)^*$$

is a finite relation algebra.

More generally every picture from a homomorphic function on R^* is a finite concrete relation algebra as well, therefore we got two ways to express the same.

Theorem:
Every coarser relation algebra of a finite concrete relation algebra is a subalgebra.

4 Constructing the Occlusion Calculus

Up to this point the relation algebras were given without a declarative semantics of their operators useful for vision applications. The following approach shows its usefulness if the semantics of the operators are given. A spatial configuration of two convex objects in 3D space as seen by an observer from his fixed point of view can be described qualitatively by relations. If V is the set of pairwise non intersecting convex objects in a 3D metrical space, the set of relations $\mathcal{R}$ which form a basis can be defined by the containment of the projective pictures as seen from the observer. This is an interpretation of RCC-5. Let $\text{pic}(O_i)$ be the 2D picture of Object O_i projected to $\mathbb{Z}^2$. The Interpretation for the RCC-5 relations is:

$$\begin{aligned}
EQ &:\Leftrightarrow \text{pic}(O_i) = \text{pic}(O_j)\\
DR &:\Leftrightarrow \text{pic}(O_i) \cap \text{pic}(O_j) = \emptyset\\
PO &:\Leftrightarrow \text{pic}(O_i) \cap \text{pic}(O_j) \notin \{\emptyset, \text{pic}(O_i), \text{pic}(O_j)\}\\
PP &:\Leftrightarrow \text{pic}(O_i) \subset \text{pic}(O_j)\\
PP^T &:\Leftrightarrow \text{pic}(O_i) \supset \text{pic}(O_j)
\end{aligned}$$

The operators on the Point Algebra are used according to the fact, which object occludes the other (possibly the depth information is gained by stereo vision or by strong models of the objects shapes, which allows to determine depth ordering for each pair of objects). Cases of intersections from pictures $\{PO, PP, PP^T\}$ of objects at the same 'depth' are not possible by the precondition of non intersecting in 3D space. In this case the empty set is in the product. We get an algebra of relations, that describes the phenomena of occlusion. The approach chosen is very similar to [3], but there are slight differences in the calculation and the resulting composition table. Figure 7 show the construction table of the new basis relations of OCC from RCC-5 and PA's basis relations. The OCC basis relations are shown in 2, the

	EQ	DR	PO	PP	PP^T
$=$	I	D	$\emptyset$	$\emptyset$	$\emptyset$
$<$	B^T	D	O	F	B^T
$>$	B	D	O^T	B	F^T

Figure 7. The combined relational partition for OCC

resulting composition table is shown in figure 8.

5 Conclusion and further work

Calculi like RCC-8, RCC-5, IA, PA, OCC and TeRCC-8 are far too general to be useful for vision applications just on itself. A major benefit and at the same time the major

	I	D	O	O^T	F	F^T	B	B^T
I	I	D	O	O^T	F	F^T	B	B^T
D	D	L	D, O, O^T, F, B	D, O, O^T, F, B	D, O, O^T, F, B	D	D, O, O^T, F, B	D
O	O	D, O, O^T, F^T, B^T	D, O, F, B^T	L	O, F	D, O, O^T, F^T, B^T	O, O^T, F, B	D, O, B^T
O^T	O^T	D, O, O^T, F^T, B^T	L	D, O^T, F^T, B	O, O^T, F, B	D, O^T, F^T	O^T, B	D, O, O^T, F^T, B^T
F	F	D	D, O, F	D, O, O^T, F, B	F	L	F, B	D, O, F, B^T
F^T	F^T	D, O, O^T, F^T, B^T	O, O^T, F^T, B^T	O^T, F^T	$O, O^T, F, B, I, F^T, B^T$	F^T	O^T, F^T, B	F^T, B^T
B	B	D	D, O, O^T, F, B	D, O^T, B	F, B	D, O^T, F^T, B	B	L
B^T	B^T	D, O, O^T, F^T, B^T	O, B^T	O, O^T, F^T, B^T	$O F, B^T$	F^T, B^T	$O, O^T, F, F^T, B, B^T, I$	B^T

Figure 8. The OCC Basis Relations

drawback is the minimum amount of knowledge on one or two qualitative aspects of space, time, direction or speed of the modelled objects. OCC exploits only a very basic assumptions of a 3D scene: relative containment of the projected pictures from the objects in the scene and their relative distance towards each other. Therefor it can be used to generate correct hypothesis qualitatively from vision scenes independent of the point of view, exact shape of objects, and their quantitative distances relative to each other.

Only combinations from multiple hypothesis refined with combinations of such algebras or other calculi can bridge the gap to an overall explanation of a vision scene. Since it is a high goal to assemble a domain-independent knowledge base for a cognitive vision system, the focus of this work was, to find the building blocks of knowledge representation for vision scenes. The expressiveness of the derived calculus is very poor, but the way it is constructed shows, that it is a building block to represent constraints in a wide variety of vision domains.

In a next step a spatio-temporal calculus will be developed, to make use of additional constraints, already in sight by the neighbourhood diagram shown in figure 9. Applying the assumption of continous movement, not all transitions between these qualitative spatial configurations are possible. The arcs in the neighbourhood diagram show the possible transitions, that can be refined if size constraints are also given.

The nice thing about these calculi is, that they will sort up quite general hypothesis in pictures provable correct on

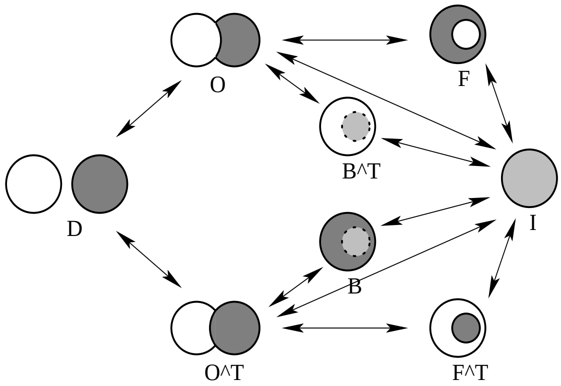

Figure 9. The OCC Neighbourhood Diagram

the underlying model. The probably bad thing, might be their inappropriateness to false positive. An extension by a stochastical interpretation of relations similar to [5] might help to fix this potential problem.

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