# Merging Interval-Based Possibilistic Belief Bases

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**Abstract.** In the last decade, several approaches were introduced in literature to merge multiple and potentially conflicting pieces of information. Within the growing field of application favourable to distributed information, *data fusion strategies* aim at providing a global and consistent point of view over a set of sources which can contradict each other. Moreover, in many situations, the pieces of information provided by these sources are uncertain.
*Possibilistic logic* is a well-known powerful framework to handle such kind of uncertainty where formulas are associated with real degrees of certainty belonging to $[0, 1]$. Recently, a more flexible representation of uncertain information was proposed, where the weights associated with formulas are in the form of intervals. This interval-based possibilistic logic extends classical possibilistic logic when all intervals are singletons, and this flexibility in representing uncertain information is handled without extra computational costs. In this paper, we propose to extend a well known approach of possibilistic merging to the notion of interval-based possibilistic knowledge bases. We provide a general semantic approach and study its syntactical counterpart. In particular, we show that convenient and intuitive properties of the interval-based possibilistic framework hold when considering the belief merging issue.

## 1 Introduction

The problem of *belief merging* [Lin96] arises when a situation requires to take into account several pieces of information obtained from distinct and often conflicting sources (or agents). This kind of situations frequently appears in many usual frameworks, such as distributed databases, multi-agent systems, or distributed information in general (e.g., semantic web), and leads to perform some combination operations on available information to extract a global and coherent point of view. Roughly speaking, merging operators introduced in literature strongly rely on the representation of available information.

In the last decade, several approaches were proposed to merge pieces of information provided without explicit priority [KPP02,EKM10,EKM12] or to the contrary to merge prioritized information [BK03,GLB06].

From a semantic point of view, these approaches are generally divided in two steps: first locally rank interpretations using some scales (depending on the considered framework, possibilistic distributions or $\kappa$-functions for instance), then aggregating these local rankings among all the bases to obtain a global total pre-order over considered interpretations (see [KPP02] for more details). The result of merging is finally obtained by considering preferred interpretations according to this global total pre-order.

In our framework, the pieces of information provided by each source may be uncertain. In this paper, these pieces of information, encoded by the means of propositional formulas, are called *beliefs*. Possibilistic logic [DLP94] is a well-known framework which allows to conveniently represent and reason with such uncertain pieces of information: uncertainty is represented by real numbers, belonging to [0,1], associated with each piece of information. Moreover, uncertainty is also represented at the semantic level by associating a possibility degree with each possible world (or interpretation). An inference mechanism was proposed in [Lan00] to derive plausible conclusions from a possibilistic knowledge base $K$, which needs $log_2(m)$ calls to the satisfiability test of a set of propositional clauses (SAT), where $m$ is the number of different degrees used in $K$.

However, in many situations, providing a precise weight to evaluate the certainty associated with a belief can be a difficult problem (e.g., when scales are provided by an expert). In [BHLR11], a flexible representation was introduced to allow the expression of an imprecision on possibilistic degrees associated with beliefs, where weights associated with formulas are in the form of intervals of $[0, 1]$. An interesting result is that handling this flexibility is done without extra computational costs with respect to the classical framework. A natural question concerns now the ability of this framework to keep such properties while considering more sophisticated issues, like the belief merging problem.

Several approaches to merge classical possibilistic belief bases were introduced in [BDPW99,BK03,QLB10]. Resulting possibilistic merging operators were analyzed in [BDKP00], where they are sorted into different classes depending on the configuration of the bases to merge. We can distinguish:

- conjunctive operators, exploiting symbolic complementarities between sources;
- disjunctives operators, which deal with conflicting but equally reliable sources;
- idempotent operators, suitable when sources to merge are not independent;
- reinforcement operators, which consider the repetition of pieces of information among sources to merge as a confirmation;
- adaptive and average operators, which adopt a disjunctive attitude in case of conflicts and a reinforcement behaviour in the other cases.

In this paper, we extend this approach to the framework of interval-based possibilistic logic. More precisely, we extend the strategies introduced in [BDKP02] by adapting possibilistic aggregation operators to deal with intervals. In particular, we show that intuitive intervals characterization principles and computational properties introduced in [BHLR11] still stand when considering the problem of belief merging. Section 3 introduces a general semantic approach, which relies on aggregation operations over intervals at the level of possibility distributions. In particular, we focus on the well-known $minimum$-based, $maximum$-based and $product$-based operations. Finally, Section 4 provides a syntactic counterpart to our semantic approach.

## 2 Background and Notations

In this paper, we consider a finite propositional language $\mathcal{L}$. We denote by $\Omega$ the finite set of interpretations of $\mathcal{L}$ and by $\omega$ an element of $\Omega$.

### 2.1 Possibilistic logic

**Possibility distributions** A possibility distribution, denoted by $\pi$, is a function from $\Omega$ to $[0,1]$. $\pi(\omega)$ represents the degree of compatibility (or consistency) of the interpretation $\omega$ with the available knowledge. $\pi(\omega) = 1$ means that $\omega$ is fully consistent with the available knowledge, while $\pi(\omega) = 0$ means that $\omega$ is impossible. $\pi(\omega) > \pi(\omega')$ simply means that $\omega$ is more compatible than $\omega'$. A possibility distribution $\pi$ is said to be normalized if there exists an interpretation $\omega$ such that $\pi(\omega) = 1$. Otherwise, the distribution is inconsistent and is called subnormalized.

A possibility distribution allows to define two functions from $\mathcal{L}$ to $[0,1]$ called possibility and necessity measures, denoted by $\Pi$ and $N$, and defined by:

$$\Pi(\varphi) = \max\{\pi(\omega) : \omega \in \Omega,\ \omega \models \varphi\} \qquad \text{and} \qquad N(\varphi) = 1 - \Pi(\neg\varphi)$$

$\Pi(\varphi)$ measures to what extent the formula $\varphi$ is compatible with the available knowledge while $N(\varphi)$ measures to what extent it is entailed.

Given a possibility distribution $\pi$ encoding some available knowledge, a formula $\varphi$ is said to be a consequence of $\pi$, denoted by $\pi \models_\pi \varphi$, iff $\Pi(\varphi) > \Pi(\neg\varphi)$.

**Possibilistic knowledge bases** A possibilistic formula is a tuple $\langle\varphi, \alpha\rangle$ where $\varphi$ is an element of $\mathcal{L}$ and $\alpha \in (0,1]$ is a valuation of $\varphi$ representing $N(\varphi)$. Note that no formula can be of type $\langle\varphi, 0\rangle$ as it brings no information. Then, a possibilistic base $K = \{\langle\varphi_i, \alpha_i\rangle, 1 \leq i \leq n\}$ is a set of possibilistic formulas.

An important notion that plays a central role in the inference process in the one of strict $\alpha$-cut. A strict $\alpha$-cut, denoted by $K_\alpha$, is a set of propositional formulas defined by $K_\alpha = \{\varphi : \langle\varphi, \beta\rangle \in K \text{ and } \beta > \alpha\}$. The strict $\alpha$-cut is useful to measure the inconsistency degree of $K$ defined by $Inc(K) = \max\{\alpha : K_\alpha \text{ is inconsistent }\}$.

If $Inc(K) = 0$ then $K$ is said to be completely consistent. If a possibilistic base is partially inconsistent, then $Inc(K)$ can be seen as a threshold below which every formula is considered as not enough entrenched to be taken into account in the inference process. More precisely, we define the notion of core of a knowledge base as the set of formulas with a necessity value greater than $Inc(K)$, i.e.,

$$\mathcal{C}ore(K) = K_{Inc(K)} = \{\varphi : \langle\varphi, \alpha\rangle \in K \text{ and } \alpha > Inc(K)\}$$

A formula $\varphi$ is a consequence of a possibilistic base $K$, denoted by $K \vdash_\pi \varphi$, iff $\mathcal{C}ore(K) \vdash \varphi$.

Given a possibilistic base $K$, we can generate a unique possibility distribution where interpretations $\omega$ satisfying all propositional formulas in $K$ have the highest possible degree $\pi(\omega) = 1$ (since they are fully consistent), whereas the others are pre-ordered w.r.t. highest formulas they falsify. More formally:

$$\forall \omega \in \Omega, \pi_K(\omega) = \begin{cases} 1 \text{ if } \forall \langle \varphi, \alpha \rangle \in K,\ \omega \models \varphi \\ 1 - max\{\alpha_i : \langle \varphi_i, \alpha_i \rangle \in K, \omega \not\models \varphi_i\} \text{ otherwise.} \end{cases}$$

The following completeness and soundness result holds:

$$K \vdash_\pi \varphi \text{ iff } \pi_K \models_\pi \varphi.$$

### 2.2 Merging possibilistic belief bases

Let us consider a multi-set of possibilistic belief bases $E = \{K_1, \ldots, K_n\}$ and their associated possibilistic distributions $\pi_1, \ldots, \pi_n$, each of these bases representing the local point of view associated with a single source. The aim of belief merging is to compute an unique possibilistic distribution, denoted $\pi_\oplus$, representing a global and consistent point of view among pieces of information provided by sources, even if some of these sources contradict each others. The most common approach to merge possibilistic knowledge bases is the one presented in [BDKP00,BDKP02]. These strategies are close to the ones introduces in [KPP02] in the case of merging classical non prioritized propositional belief bases.

Generally speaking, most common belief merging operators, denoted $\Delta$, are divided in two steps. First, all interpretations are rank ordered with respect to individual sources. In the framework of possibilistic logic, this step is performed quite straightforwardly since each possibilistic belief base induces an unique possibilistic distributions over interpretations. Then, ranks individually computed are aggregated among all belief bases to merge, using an aggregation operator denoted $\oplus$, to associate a global rank to each considered interpretations: these ranks allow to induce a global order, denoted in this paper $<_{\pi_\oplus}$, where preferred interpretations are usually considered as models of the result of merging, denoted $\Delta_\oplus(E)$. This distribution finally induces a possibilistic belief base, denoted $K_\oplus$, representing the result of merging. Obviously, several aggregation operators are possible, depending on expected properties for the result of merging.

In the context of possibilistic logic, several aggregation function were discussed in [BDKP02] to compute the value of $\pi_\oplus(\omega)$ from the $\nu_E(\omega) = \langle \pi_1(\omega), \ldots, \pi_n(\omega) \rangle$ vector. These operators were divided into several categories: conjunctive (adequate when the sources are consistent), disjunctive (adequate when the sources are conflicting), idempotent (ignoring the redundancies) and reinforcement (seeing redundancies as confirmation).

Generally speaking, any function $\oplus$ which respects the following conditions can be considered a possibilistic aggregating function:

1. $\oplus(1, \ldots, 1) = 1$
2. If $\forall 1 \leq i \leq n, a_i \geq b_i$ then $\oplus(a_1, \ldots, a_n) \geq \oplus(b_1, \ldots, b_n)$

Note that clearly, many aggregation operators are possible to combine initial distributions, offering different behaviours in computing the result of merging. The most common operators used in the context of the possibilistic merging are the following ones:

- the minimum: $\oplus_{min}(\pi_1, \ldots, \pi_n) = min(\pi_1, \ldots, \pi_n)$;
- the product: $\oplus_{prd}(\pi_1, \ldots, \pi_n) = \sqrt[n]{\Pi \pi_i}$;
- the maximum: $\oplus_{max}(\pi_1, \ldots, \pi_n) = max(\pi_1, \ldots, \pi_n)$;
- the dual product: $\oplus_{dpr}(\pi_1, \ldots, \pi_n) = 1 - \sqrt[n]{\Pi(1 - \pi_i)}$;
- the probabilistic sum: $\oplus_{prs}(\pi_1, \ldots, \pi_n) = 1 - \Pi(1 - \pi_i)$;
- the averaging: $\oplus_{ave}(\pi_1, \ldots, \pi_n) = \Sigma \pi_i / n$.

In particular, this paper focuses on the *minimum*-based, the *maximum*-based and the *product*-based merging operators.

Moreover, a syntactic counterpart of possibilistic merging operators was introduced in [BDPW99,BDKP02]. Namely, authors show that the result of merging can be characterized by the belief base $\mathcal{B}_\oplus$ defined as follows:

$$\mathcal{B}_\oplus = \{(D_j, 1 - \oplus(x_1, \ldots, x_n)) \;:\; j = 1, \ldots, n\}$$

where $D_j$ are disjunctions of size $j$ between formulas $\phi_i$ obtained from each $B_i$ and $x_i = 1 - \alpha_i$ if $\phi_i \in D_j$, $x_i = 1$ otherwise.

Desterecke et al. [DDC09] proposed another way of merging possibilistic bases on the ground of maximal coherent subsets which is closer to what is usually done in propositional logic which will be studied here.

### 2.3 Interval-based possibilistic logic

Interval-based possibilistic logic was introduced in [BHLR11]. This framework can be described as a generalization of possibilistic logic, where uncertainty associated to beliefs is represented by the means of an interval of $I = [\alpha, \beta] \subseteq [0, 1]$ instead of a number. The intuitive meaning behind this interval is that the real value of uncertainty is unknown and belong to the interval.

The set of intervals of $[0, 1]$ is denoted by $\mathcal{I}$. An interval based possibility distribution, denoted by $\pi_\mathcal{I}$, is then also described by the means of intervals of $\mathcal{I}$. This induces a partial pre-ordering among the set of interpretations of $\Omega$. More precisely, an interpretation $\omega$ is said to be preferred to $\omega'$, denoted by $\omega \lhd \omega'$, iff $\beta < \alpha'$ where $\pi_\mathcal{I}(\omega) = [\alpha, \beta]$ and $\pi_\mathcal{I}(\omega') = [\alpha', \beta']$.

A first approach to compute possibility and necessity measures is to use the notion of compatible possibility distribution. Formally, a classical possibility distribution $\pi$ is said to be compatible with $\pi_\mathcal{I}$ iff $\forall \omega \in \Omega$, $\pi(\omega) \in \pi_\mathcal{I}(\omega)$. The non finite set of all compatible possibility distributions obtained from $\pi_\mathcal{I}$ is denoted by $\mathcal{C}mp(\pi_\mathcal{I})$. Possibility and necessity measures are then defined as follows:

$$\Pi_\mathcal{I}(\varphi) = [\min_{\pi \in \mathcal{C}mp(\pi_\mathcal{I})} \Pi(\varphi), \max_{\pi \in \mathcal{C}mp(\pi_\mathcal{I})} \Pi(\varphi)]$$
$$N_\mathcal{I}(\varphi) = [\min_{\pi \in \mathcal{C}mp(\pi_\mathcal{I})} N(\varphi), \max_{\pi \in \mathcal{C}mp(\pi_\mathcal{I})} N(\varphi)]$$

As it is shown in [BHLR11], these measures can be characterized by the means of operations on intervals:

$$\Pi_{\mathcal{I}}(\varphi) = \mathcal{M}\{\pi_{\mathcal{I}}(\omega) : \omega \in \Omega, \omega \models \varphi\}$$
$$N_{\mathcal{I}}(\varphi) = 1 \ominus \Pi_{\mathcal{I}}(\neg\varphi)$$

where $\mathcal{M}\{I_1, \ldots, I_n\} = [\max\{\alpha_1, \ldots, \alpha_n\}, \max\{\beta_1, \ldots, \beta_n\}]$ and $1 \ominus [\alpha, \beta] = [1 - \beta, 1 - \alpha]$.

**Interval-based possibilistic bases** A syntactic representation of interval-based possibilistic logic is obtained by associating necessity-values, in the form of intervals, to formulas. An interval-based possibilistic base, denoted $IK$, is thus defined as:

$$IK = \{\langle \varphi, I \rangle, \varphi \in \mathcal{L} \text{ and } I \in \mathcal{I}\}$$

Likewise possibility distributions, a compatible possibilistic base can be obtained from an interval-based possibilistic base by replacing each interval-based possibilistic formula $\langle \varphi, I \rangle$ by a standard possibilistic formula $\langle \varphi, \delta \rangle$ where $\delta \in I$. The non-finite set of all possible standard possibilistic bases compatible with an interval-based possibilistic base $IK$ is denoted by $\mathcal{C}mp(IK)$. Two particular compatible bases are $IK_{lb} = \{\langle \varphi, \alpha \rangle : \langle \varphi, [\alpha, \beta] \rangle \in IK\}$, $IK_{ub} = \{\langle \varphi, \beta \rangle : \langle \varphi, [\alpha, \beta] \rangle \in IK\}$, which are respectively obtained by selecting either lower or upper bounds of intervals.

Like in the standard possibilistic logic, interval-based possibilistic bases can be partially inconsistent. As it is shown in [BHLR11], the interval-based inconsistency degree can be equivalently computed in the two following ways:

$$\begin{aligned} Inc(IK) &= \{Inc(K) \ : \ K \in Cmp(IK)\} \\ &= [Inc(IK_{lb}), Inc(IK_{ub})] \end{aligned}$$

This central notion allows to characterize the interval-based syntactic inference which can intuitively be defined by considering all compatible bases:

$$IK \models_c \phi \text{ iff } \mathcal{C}ore(IK) \vdash \phi \text{ iff } \forall K \in \mathcal{C}mp(IK),\ \mathcal{C}ore(K) \vdash \phi$$

where $\mathcal{C}ore(IK) = \{\varphi \ : \ \langle \varphi, I \rangle \in IK \text{ and } Inc(IK) \lhd I\}$.

## 3 A semantic approach to interval-based possibilistic merging

In this section, we propose a general semantic approach to merge interval-based possibility distributions. Let $E = \{IK_1, \ldots, IK_n\}$ be a multi-set of $n$ interval-based possibilistic bases. From $E$, we can derive a family of interval-based possibility distributions $\pi_1^{\mathcal{I}} \ldots \pi_n^{\mathcal{I}}$, each $IK_i$ inducing an unique interval-based possibility distribution $\pi_i^{\mathcal{I}}$. This step allows to locally rank order each interpretation $\omega$ of $\Omega$ with respect to each $IK_i$. To compute the result of merging $E$, we now need to aggregate all intervals associated to each interpretation $\omega$ to obtain a global ordering over $\Omega$. We introduce the notion of interval-based aggregation operator, denoted by $\oplus^{\mathcal{I}}$, and then denote by $\pi_{\oplus}^{\mathcal{I}}$ the interval-based possibility distribution obtained by aggregating all distributions obtained from $E$ with $\oplus$.

In this framework, the real uncertainty value associated to a formula is unknown, and may be any value of the interval. Therefore, a first approach to define an aggregation operator on intervals is to take into account each possible combination of all existing standard distributions compatible with interval-based distributions to consider. Namely:

**Definition 1.** *Let $\pi_1, \ldots, \pi_n$ be interval-based possibility distributions and let $\oplus$ be a possibilistic aggregation operator, then an interval-based possibilistic aggregation operator $\oplus^{\mathcal{I}}$ based on $\oplus$ can be defined as follows:*

$$\oplus^{\mathcal{I}}(\pi_1^{\mathcal{I}}(\omega), \ldots, \pi_n^{\mathcal{I}}(\omega)) = \bigcup_{\pi_i \in Cmp(\pi_i^{\mathcal{I}})} \{\oplus(\pi_1(\omega), \ldots, \pi_n(\omega))\}.$$

However, the previous definition is not helpful computationally speaking. Indeed, the number of compatible possibility distributions being infinite, computing the result of merging is a difficult problem. We introduce a first characterisation of aggregating intervals, when considering the $minimum$-based, the $maximum$-based and the $product$-based operations, which only relies on considering lower and upper bounds of intervals. Moreover, the following proposition shows that in these cases, the value respectively associated to each interpretation $\omega$ of $\Omega$ by $\pi_{\oplus}^{\mathcal{I}}$ constitutes an interval. Namely:

**Proposition 1.** *Let $\pi_1, \ldots, \pi_n$ be $n$ interval-based possibility distributions, let $\oplus^{\mathcal{I}}$ be an interval-based aggregation operator relying on the $minimum$-based, the $maximum$-based or the $product$-based classical possibilistic operator and let $\pi_{\oplus}^{\mathcal{I}}$ be the interval-based possibility distribution obtained by considering $\oplus^{\mathcal{I}}$, then:*

$$\pi_{\oplus}^{\mathcal{I}}(\omega) = \oplus^{\mathcal{I}}(\pi_1^{\mathcal{I}}(\omega), \ldots, \pi_n^{\mathcal{I}}(\omega)) = [\oplus(\alpha_1, \ldots, \alpha_n), \oplus(\beta_1, \ldots, \beta_n)]$$

*where for each $i = 1 \ldots n$, $\pi_i^{\mathcal{I}}(\omega) = [\alpha_i, \beta_i]$.*

Obviously, many other definitions of $\oplus$ are possible. We then propose three intuitive requirements for an interval-based aggregation operator. Formally:

1. $\oplus^{\mathcal{I}}(I_1, \ldots, I_n)$ is an interval;
2. $\oplus^{\mathcal{I}}([1,1], \ldots, [1,1]) = [1,1]$;
3. If $\forall 1 \leq i \leq n, I_i \triangleleft I'_i$ then $\oplus^{\mathcal{I}}(I_1, \ldots, I_n) \triangleleft \oplus^{\mathcal{I}}(I'_1, \ldots, I'_n)$.

The first requirement means that the result of aggregating some intervals should also be an interval. The second requirement says that if each source agrees that $\omega$ is fully possible, then the result of aggregation should confirm it. The last says that if each source prefers $\omega$ to $\omega'$, then the result of aggregation should prefer so. The following result shows that any interval-based aggregation operator based on a classical aggregation operator straightforwardly ensures the two last requirements. Namely:

**Proposition 2.** *Let $\oplus$ be a n-ary function from $[0,1]^n$ to $[0,1]$. If $\oplus$ is a possibilistic aggregation operator, then $\oplus^{\mathcal{I}}$ (given by Definition 1) satisfies conditions 2 and 3.*

In a general case, Condition 1 is not guaranteed. However, when considering the $minimum$-based, the $maximum$-based or the $product$-based aggregation operations, the three requirements are satisfied. Namely:

**Proposition 3.** *Let $\oplus_{min}$, $\oplus_{max}$ and $\oplus_{prd}$ be respectively the n-ary $minimum$-based, the $maximum$-based and the $product$-based classical aggregation operations. Then $\oplus^{\mathcal{I}}_{min}$, $\oplus^{\mathcal{I}}_{max}$ and $\oplus^{\mathcal{I}}_{prd}$ satisfy conditions 1-3, and are interval-based possibilistic aggregation operators.*

One can remark that this approach of aggregating interval-based possibility distributions generalizes aggregation operations defined in the classical case, since in the extreme case, where all intervals provided by sources are only singletons, the possibility distribution $\oplus^{\mathcal{I}}$ recovers the results provided by $\oplus$. More formally:

**Proposition 4.** *In the case where intervals within each $\pi^{\mathcal{I}}_i$ obtained from $E$ only consist in singletons (namely for all $\pi^{\mathcal{I}}_i$ obtained from $E$, for all $\omega$, $\pi^{\mathcal{I}}_i(\omega) = [\alpha, \alpha]$) then:*
*i) each $\pi^{\mathcal{I}}_i$ obtained from $E$ has a unique compatible possibility distribution $\pi_i$ and*
*ii) $\pi^{\mathcal{I}}_{\oplus}(\omega) = \oplus^{\mathcal{I}}(\pi^{\mathcal{I}}_1(\omega), \ldots, \pi^{\mathcal{I}}_n(\omega)) = \oplus(\pi_1(\omega), \ldots, \pi_n(\omega))$, where is $\oplus^{\mathcal{I}}$ an interval-based aggregation operator based on the classical aggregation operator $\oplus$, and each $\pi_i$ is the unique classical distribution compatible with the respective interval-based distribution $\pi^{\mathcal{I}}_i$.*

Let us illustrate these definitions with the following example:

**Example 1** *Let $\pi_1$, $\pi_2$ and $\pi_3$ be three possibility distributions such that:*

| $\omega$ | $\pi_1$ | $\pi_2$ | $\pi_3$ |
|---|---|---|---|
| $\omega_1$ | $[.1, .3]$ | $[.4, .6]$ | $[.7, .9]$ |
| $\omega_2$ | $[.4, .5]$ | $[.4, .5]$ | $[.4, .5]$ |
| $\omega_3$ | $[.4, .5]$ | $[.7, .8]$ | $[1, 1]$ |
| $\omega_4$ | $[.1, .2]$ | $[.1, .5]$ | $[.1, .8]$ |

*Considering merging operators relying respectively on the $minimum$-based, the $product$-based and the $maximum$-based operations, we obtain the following results:*

| $\omega$ | $\oplus_{min}$ | $\oplus_{prd}$ | $\oplus_{max}$ |
|---|---|---|---|
| $\omega_1$ | $[.1, .3]$ | $[.303, .545]$ | $[.7, .9]$ |
| $\omega_2$ | $[.4, .5]$ | $[.4, .5]$ | $[.4, .5]$ |
| $\omega_3$ | $[.4, .5]$ | $[.654, .736]$ | $[1, 1]$ |
| $\omega_4$ | $[.1, .2]$ | $[.1, .430]$ | $[.1, .8]$ |

Note that considering the interval-based comparative relation $\lhd$, $\pi^{\mathcal{I}}_{\oplus}$ only induces a partial pre-order over interpretations $\omega$ of $\Omega$ (different comparative relations are possible but are out of the scope of this paper, see [BHLR11] for more details). This result allows to provide a result for the merging of $E$ without focusing on each combination of all possible classical distributions compatible with considered interval-based distributions. As a corollary, this result also shows that aggregating interval-based possibility distribution can be achieved within only two calls to classical aggregating operations, respectively on lower and upper bounds of intervals.

In the classical possibilistic case, merging operators, relying on standard aggregation operators, are divided into several non-exclusive families. Since these definitions do not hold anymore when considering interval-based possibilistic degrees, we thus provide their counterparts in the interval based possibilistic framework. Formally:

**Definition 2.** *Let* $\pi_1, \ldots, \pi_n$ *be* $n$ *interval-based possibility distributions and let* $\oplus^{\mathcal{I}}$ *be a possibilistic merging operator, then* $\oplus^{\mathcal{I}}$ *is said to be:*

1. *conjunctive iff* $\forall I \in \mathcal{I}, I \oplus^{\mathcal{I}} [1,1] = [1,1] \oplus^{\mathcal{I}} I = I$
2. *disjunctive iff* $\forall I \in \mathcal{I}, I \oplus^{\mathcal{I}} [1,1] = [1,1] \oplus^{\mathcal{I}} I = [1,1]$
3. *idempotent iff* $\forall I \in \mathcal{I}, I \oplus^{\mathcal{I}} I = I$
4. *reinforcement iff* $\forall I, I' \in \mathcal{I}$ *s.t.* $I, I' \neq [1,1]$ *and* $I, I' \neq [0,0]$ *then* $(I \oplus^{\mathcal{I}} I')_\alpha \leq min(I_\alpha, I'_\alpha)$ *and* $(I \oplus^{\mathcal{I}} I')_\beta \leq min(I_\beta, I'_\beta)$
5. *averaging iff* $\forall I \in \mathcal{I}, min(I_\alpha, I'_\alpha) \leq (I \oplus^{\mathcal{I}} I')_\alpha \leq max(I_\alpha, I'_\alpha)$ *and* $min(I_\beta, I'_\beta) \leq (I \oplus^{\mathcal{I}} I')_\beta \leq max(I_\beta, I'_\beta)$

*where* $I_\alpha$ *and* $I_\beta$ *are respectively the lower bound and the upper bound of an interval* $I$.

From these definitions, and from previous results introduced in this paper, one can remark that several families of interval-based aggregation operators extend properties associated with the classical families on which they are based. Namely:

**Proposition 5.** *Let* $\pi_1, \ldots, \pi_n$ *be* $n$ *interval-based possibility distributions and let* $\oplus^{\mathcal{I}}$ *be a possibilistic merging operator. If* $\oplus$ *is conjuctive (resp. disjunctive, or idempotent) then* $\oplus^{\mathcal{I}}$ *is conjuctive (resp. disjunctive, or idempotent).*

Let us illustrate this fact with the following example:

**Example 2** *Let us consider again Example 1. On this example, we have:*

- *The interpretation* $\omega_3$ *is associated with merged value of* $[1,1]$ *for the disjunctive operator* $\oplus^{\mathcal{I}}_{max}$*;*
- *The interpretation* $\omega_2$ *has a merged value of* $[.4, .5]$ *for the idempotent operators* $\oplus^{\mathcal{I}}_{min}$*,* $\oplus^{\mathcal{I}}_{prd}$*,*$\oplus^{\mathcal{I}}_{max}$*.*

## 4 A syntactic counterpart

In this section, we provide some syntactic counterparts to the general semantic approach introduced previously.

The definitions given in the previous section allow to define, from the sources, a possibility value for every interpretations. The result of the merging operation is thus define as the interpretations maximal according to their possibility values.

**Definition 3.** *Let* $\pi_1, \ldots, \pi_n$ *be interval-based possibility distributions and let* $\oplus^{\mathcal{I}}_o$ *be a possibilistic merging operator, then:*

$$\Delta^{\mathcal{I}}_o(IK_1, \ldots, IK_n) = \{\pi(\omega) = \oplus^{\mathcal{I}}_o(\pi_1, \ldots, \pi_n))\}$$

There is also a syntactic counterpart to this definition. One can build an interval-based possibilistic base out of the sources the following way:

**Definition 4.** *Let $IK_i = \{\langle \varphi, I_i^j \rangle\}$ be interval-based possibilistic knowledge bases and let $\oplus_o^{\mathcal{I}}$ be a possibilistic merging operator, then:*

$$\blacktriangle_o^{\mathcal{I}}(IK_1, \ldots, IK_n) = \{(D_j, 1 \ominus \oplus_o^{\mathcal{I}}(x_1, \ldots, x_n)) \ : \ j = 1, \ldots, n\}$$

*and $D_j$ are disjunctions of size $j$ between formulas $\phi_i$ taken from different $B_i$ and $x_i = 1 \ominus [\alpha_i, \beta_i]$ if $\phi_i \in D_j$ and $x_i = [1, 1]$ otherwise.*

The consequences from the syntactic merging operation are equivalent to the results of the semantic merging operation.

**Proposition 6.** *Let $IK_i = \{\langle \varphi, I_i^j \rangle\}$ be interval-based possibilistic knowledge bases, let $\phi$ be a formula and let $\oplus_o^{\mathcal{I}}$ be a possibilistic merging operator, then:*

$$\Delta_o^{\mathcal{I}}(IK_1, \ldots, IK_n) \models \phi \text{ iff } \blacktriangle_o^{\mathcal{I}}(IK_1, \ldots, IK_n) \vdash \phi$$

Now, let us instantiate three particular cases of Definition 4, namely with $\oplus_{min}^{\mathcal{I}}$ (interval-based idempotent conjunctive merging), $\oplus_{max}^{\mathcal{I}}$ (interval-based idempotent disjunctive merging), $\oplus_{prd}^{\mathcal{I}}$ (interval-based product-based conjunctive merging).

We restrict ourselves to the case of two knowledge $IK_1$ and $IK_2$ (since all operations are associative and commutative).

For the interval-based idempotent conjunctive operation, we have:

$$\begin{aligned}
\Delta_{min}^{\mathcal{I}}(IK_1, IK_2) = \; & \{\langle \varphi_i, [1,1] \ominus min([1,1] \ominus I_i, [1,1]) \rangle : (\varphi_i, I_i) \in IK_1\} \\
& \cup \{\langle \psi_j, [1,1] \ominus min([1,1] \ominus I_j, [1,1]) \rangle : (\psi_j, I_j) \in IK_2\} \\
& \cup \{\langle \varphi_i \vee \psi_j, [1,1] \ominus min([1,1] - I_i, [1,1] - I_j) \rangle : \\
& \quad (\varphi_i, I_i) \in IK_1 \text{ and } (\psi_j, I_j) \in IK_2\} \\
& = \{\langle \varphi_i, I_1 \rangle : \langle \varphi_i, I_1 \rangle \in IK_1\} \\
& \cup \{\langle \psi_j, I_j \rangle : \langle \psi_j, I_j \rangle \in IK_2\} \\
& \cup \{\langle \varphi_i \vee \psi_j, max(I_i, I_j) \rangle : \\
& \quad \langle \varphi_i, I_1 \rangle \in IK_1 \text{ and } \langle \psi_j, I_j \rangle \in IK_2\}
\end{aligned}$$

we can check that is equivalent to

$$\Delta_{min}^{\mathcal{I}}(IK_1, IK_2) = IK_1 \cup IK_2$$

For the interval-based idempotent disjunctive operation, we have:

$$\begin{aligned}
\Delta_{max}^{\mathcal{I}}(IK_1, IK_2) = \; & \{\langle \varphi_i, [1,1] \ominus max([1,1] \ominus I_i, [1,1]) \rangle : (\varphi_i, I_i) \in IK_1\} \\
& \cup \{\langle \psi_j, [1,1] \ominus max([1,1] \ominus I_j, [1,1]) \rangle : (\psi_j, I_j) \in IK_2\} \\
& \cup \{\langle \varphi_i \vee \psi_j, [1,1] \ominus max([1,1] - I_i, [1,1] - I_j) \rangle : \\
& \quad (\varphi_i, I_i) \in IK_1 \text{ and } (\psi_j, I_j) \in IK_2\} \\
& = \{\langle \varphi_i, [0,0] \rangle : \langle \varphi_i, I_1 \rangle \in IK_1\} \\
& \cup \{\langle \psi_j, [0,0] \rangle : \langle \psi_j, I_j \rangle \in IK_2\} \\
& \cup \{\langle \varphi_i \vee \psi_j, min(I_i, I_j) \rangle : \\
& \quad \langle \varphi_i, I_1 \rangle \in IK_1 \text{ and } \langle \psi_j, I_j \rangle \in IK_2\}
\end{aligned}$$

we can check that is equivalent to

$$\Delta_{max}^{\mathcal{I}}(IK_1, IK_2) = \{\langle \varphi_i \vee \psi_j, min(I_i, I_j) \rangle : \langle \varphi_i, I_1 \rangle \in IK_1 \text{ and } \langle \psi_j, I_j \rangle \in IK_2\}$$

For the interval-based product-based conjunctive merging operation, we have:

$$\begin{aligned}
\Delta^{\mathcal{I}}_{prd}(IK_1, IK_2) &= \{\langle \varphi_i, [1,1] \ominus (([1,1] \ominus I_i) \times [1,1])\rangle : (\varphi_i, I_i) \in IK_1\} \\
&\cup \{\langle \psi_j, [1,1] \ominus ([1,1] \times ([1,1] \ominus I_j))\rangle : (\psi_j, I_j) \in IK_2\} \\
&\cup \{\langle \varphi_i \vee \psi_j, [1,1] \ominus (([1,1] \ominus I_i) \times ([1,1] \ominus I_j))\rangle : \\
&\quad (\varphi_i, I_i) \in IK_1 \text{ and } (\psi_j, I_j) \in IK_2\} \\
&= IK_1 \cup IK_2 \\
&\cup \{\langle \varphi_i \vee \psi_j, [\alpha_i + \alpha_j - \alpha_i \times \alpha_j, \beta_i + \beta_j - \beta_i \times \beta_j]\rangle : \\
&\quad \langle \varphi_i, I_1\rangle \in IK_1 \text{ and } \langle \psi_j, I_j\rangle \in IK_2\}
\end{aligned}$$

**Example 3** *Let $E = \{IK_1, IK_2\}$ be a belief profile with $IK_1 = \{\langle a, [.5, .7]\rangle, \langle \neg a \vee b, [.4, .8]\rangle\}$ and $IK_2 = \{\langle \neg b \vee a, [.2, .3]\rangle, \langle \neg b, [.6, .7]\rangle\}$. The interval-based possibility distribution is given in the following table.*

| $\omega_i$ | $\pi_{\mathcal{I}}(IK_1)$ | $\pi_{\mathcal{I}}(IK_2)$ | $\oplus^{\mathcal{I}}_{min}$ | $\oplus^{\mathcal{I}}_{max}$ | $\oplus^{\mathcal{I}}_{prd}$ |
|---|---|---|---|---|---|
| $a \wedge b$ | $[1,1]$ | $[.3, .4]$ | $[.3, .4]$ | $[1,1]$ | $[.547, .632]$ |
| $a \wedge \neg b$ | $[.2, .6]$ | $[1,1]$ | $[.2, .6]$ | $[1,1]$ | $[.447, .774]$ |
| $\neg a \wedge b$ | $[.3, .5]$ | $[.3, .4]$ | $[.3, .4]$ | $[.3, .5]$ | $[.3, .447]$ |
| $\neg a \wedge \neg b$ | $[.3, .5]$ | $[1,1]$ | $[.3, .5]$ | $[1,1]$ | $[.547, .707]$ |

*In the following, we give the resulting base for our three main operators, the result of the disjunction of $a \vee \neg b$ and $\neg b \vee a$ and $\neg a \vee b$ and $\neg b$ are not given as they produce $\top$.*

- $\blacktriangle^{\mathcal{I}}_{min}(IK_1, IK_2) = IK_1 \cup IK_2 \cup \{\langle a \vee \neg b \vee a, [.5, .7]\rangle, \langle a \vee \neg b, [.6, .7]\rangle\}$
- $\blacktriangle^{\mathcal{I}}_{max}(IK_1, IK_2) = \{\langle a \vee \neg b \vee a, [..2, .3]\rangle, \langle a \vee \neg b, [.5, .7]\rangle\}$
- $\blacktriangle^{\mathcal{I}}_{prd}(IK_1, IK_2) = IK_1 \cup IK_2 \cup \{\langle a \vee \neg b \vee a, [.368, .542]\rangle, \langle a \vee \neg b, [.553, .7]\rangle\}$

*One can easily verify that the syntactic and semantic operators have the same consequences.*

## 5 Conclusion

This paper addressed a first approach for merging interval-based possibilistic belief bases. More precisely, we have extended the possibilistic merging operators introduced in the classical case to handle the concept of interval-based possibilistic degrees. This way, our study shown that convenient and intuitive properties associated to this framework still hold when dealing with more tricky issues, in particular the problem of belief merging.

A future work is to consider the belief revision problem in the context of interval-based possibilistic logic. This problem consists in integrating a higher priority information in a belief base, such that this information must be deduced from the base after the process. Despite this problem is a particular case of merging, namely a belief base is merged with a higher priority piece of information, it still raises some difficult issues.

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