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Combining Topological and Qualitative Size Constraints for Spatial Reasoning

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Abstract. Information about the relative size of spatial regions is often easily accessible and, when combined with other types of spatial information, it can be practically very useful. In this paper we combine a simple framework for reasoning about qualitative size relations with the Region Connection Calculus RCC-8, a widely studied approach for qualitative spatial reasoning with topological relations. Reasoning about RCC-8 relations is NP-hard, but a large maximal tractable subclass of RCC-8 called $\widehat{\mathcal{H}}_8$ was identified. Interestingly, any constraint in RCC-8 $-$ $\widehat{\mathcal{H}}_8$ can be consistently reduced to a constraint in $\widehat{\mathcal{H}}_8$, when an appropriate size constraint between the spatial regions is supplied. We propose an $O(n^3)$ time path-consistency algorithm based on a novel technique for combining RCC-8 constraints and relative size constraints, where n is the number of spatial regions. We prove its correctness and completeness for deciding consistency when the input contains topological constraints in $\widehat{\mathcal{H}}_8$. We also provide results on finding a consistent scenario in $O(n^3)$ time both for combined topological and relative size constraints, and for topological constraints alone. This is an $O(n^2)$ improvement over the known methods.

1 Introduction

The Region Connection Calculus (RCC) [14] is a well studied topological approach for qualitative spatial reasoning, where regions are non-empty regular subsets of a topological space. Regions need not be one-piece. Binary relations between regions are based on the "connected" relation $C(a, b)$ which is true if the closure of region a and the closure of region b have a non-empty intersection.

RCC-8 is a set of eight jointly exhaustive and pairwise disjoint relations called basic relations definable in the RCC-theory, and of all possible disjunctions of the basic relations, resulting in 2^8 different RCC-8 relations altogether.

An important reasoning problem in this framework is deciding consistency of a set of spatial constraints of the form xRy where x, y are region variables and R is a relation in RCC-8. Another related problem is finding a consistent scenario for a set of RCC-8 constraints, that is a consistent refinement of all the constraints in the set to one of their basic relations. These problems are in general NP-hard, but they can be decided in polynomial time for a large subset of RCC-8 (denoted $\widehat{\mathcal{H}}_8$) which is a maximal tractable subclass of RCC-8 [16]. In particular, Renz and Nebel [16] proved that the consistency of a set of constraints over $\widehat{\mathcal{H}}_8$ can be decided in $O(n^3)$ time, where n is the number of variables involved.

This paper contains two main contributions. In the first part of the paper we address the problem of finding a consistent scenario for a set of $\widehat{\mathcal{H}}_8$ constraints, and we propose an $O(n^3)$ time algorithm for solving this task. This is an $O(n^2)$ improvement over the bound of the previously known methods.

In the second part of the paper, we study the combination of RCC-8 with qualitative information about region sizes, which is often easily accessible and practically useful. As a very simple example, suppose to have three geographical regions A, B and C for which the only topological information available is that B is contained in A. In addition we know that A is smaller than C, and that C is smaller than B. The combined set of topological and relative size information is inconsistent, but we cannot detect this by just independently processing the two kind of information, or by just expressing the size information as topological constraints.[1]

Specifically, we consider the following qualitative relations between region sizes, which have been largely studied in the context of temporal reasoning (e.g., [18, 17, 6]): $<$, $\leq$, $>$, $\geq$, $\neq$, $=$, $<=>$. Interestingly, any constraint in RCC-8 $-$ $\widehat{\mathcal{H}}_8$ can be consistently refined to a constraint in $\widehat{\mathcal{H}}_8$, when an appropriate size constraint of this class is supplied.

We propose an algorithm, Bipath-consistency, based on a novel technique for dealing with combined topological and qualitative size constraints, and we prove that, despite our extended framework is more expressive than $\widehat{\mathcal{H}}_8$ (and therefore it has a larger potential applicability), the problem of deciding consistency can be solved in cubic time (i.e., without additional worst-case cost). Bipath-consistency is a general algorithm, in the sense that it can be applied not only to spatial relations. For example, in the context of temporal reasoning it can be used to combine relations in Allen's Interval Algebra [1] and qualitative constraints on the duration of the temporal intervals. Of course, different classes of relations might need different completeness and complexity proofs.

The proof of the completeness of this algorithm is based on a particular method of constructing a consistent scenario for a set of constraints over $\widehat{\mathcal{H}}_8$, which is analyzed in the first part of the paper. Moreover, this method can also be used in combination with Bipath-consistency to compute, in cubic time, a consistent topological scenario satisfying an input set of qualitative size constraints between the spatial regions.

The rest of the paper is organized as follows. In the first part we briefly introduce RCC-8 (Section 2) and give the results concerning computing a consistent scenario (Section 3). The second part deals with the combination of topological and size constraints (Sections 4 and 5).

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[1] Another more complex example illustrating interdependencies between topological and size constraints is given in Section 5.

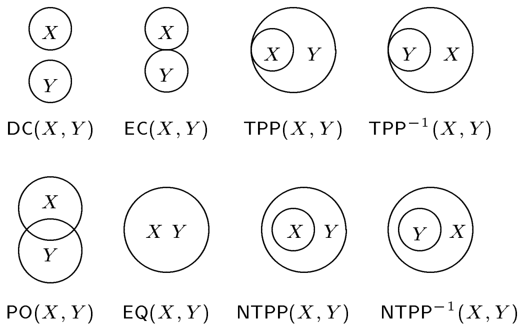

Fig. 1. Two-dimensional examples for the eight basic relations of RCC-8

2 The Region Connection Calculus RCC-8

RCC-8 is a set of binary spatial relations formed by eight jointly exhaustive and pairwise disjoint relations definable in the RCC-theory called *basic relations*,[2] and by all possible disjunctions of the basic relations (resulting in 2^8 different RCC-8 relations altogether). The basic relations are denoted by DC (DisConnected), EC (Externally Connected), PO (Partial Overlap), EQ (EQual), TPP (Tangential Proper Part), NTPP (Non-Tangential Proper Part), and their converse TPP^{-1} and NTPP^{-1} [14]. Figure 1 shows two-dimensional examples of these relations. In the following, an RCC-8 relation will be written as a set of basic relations.

An important reasoning problem in this framework is deciding consistency of a set Θ of spatial constraints of the form xRy where x, y are region variables and R is an RCC-8 relation. Θ is consistent if and only if there is a *model* of Θ, i.e., an assignment of spatial regions to the variables of Θ such that all the constraints are satisfied. This problem (denoted by RSAT) is NP-complete [16]. RSAT is a fundamental reasoning problem since several other interesting reasoning problems can be reduced to it by polynomial Turing reduction [7].

A set of constraints in RCC-8 can be processed using an $O(n^3)$ time *path-consistency* algorithm, which makes the set *path-consistent* by eliminating all the impossible labels (basic relations) in every subset of constraints involving three variables [12, 13]. If the empty relation occurs during this process, the set is not consistent, otherwise the resulting set is path-consistent. Since RSAT is NP-complete, in general imposing path-consistency is not sufficient for deciding consistency of a set of RCC-8 constraints. Renz and Nebel [16] identified a subset of RCC-8 (denoted by $\widehat{\mathcal{H}}_8$) for which RSAT can be decided in polynomial time. They also proved that $\widehat{\mathcal{H}}_8$ is maximal with respect to tractability, i.e., if any RCC-8 relation is added to $\widehat{\mathcal{H}}_8$, then the consistency problem becomes NP-complete. Finally they showed that computing path-consistency for a set of constraints over $\widehat{\mathcal{H}}_8$ is sufficient for deciding its consistency.

$\widehat{\mathcal{H}}_8$ contains 148 relations, i.e., almost 60% of the RCC-8 relations. The following 108 RCC-8 relations are not contained in $\widehat{\mathcal{H}}_8$, where (N)TPP denotes TPP

[2] I.e., between any two regions exactly one of the basic relations holds.

or NTPP:

$$\begin{aligned}\mathsf{RCC\text{-}8} \setminus \widehat{\mathcal{H}}_8 = \{R \quad | \quad & (\{(\mathsf{N})\mathsf{TPP}, (\mathsf{N})\mathsf{TPP}^{-1}\} \subset R \text{ and } \{\mathsf{PO}\} \not\subset R) \\ & \text{or } (\{\mathsf{EQ}, \mathsf{NTPP}\} \subset R \text{ and } \{\mathsf{TPP}\} \not\subset R) \\ & \text{or } (\{\mathsf{EQ}, \mathsf{NTPP}^{-1}\} \subset R \text{ and } \{\mathsf{TPP}^{-1}\} \not\subset R)\}\end{aligned}$$

We assume that a set of constraints Θ contains one constraint for each pair of variables involved in Θ, i.e., if no information is given about the relation holding between two variables x and y, then the universal constraint $x\{*\}y$ is implicitly contained in Θ ($*$ denotes the *universal relation*, i.e., the union of all the basic relations). Another assumption that we make is that whenever a constraint xRy is in Θ, also $yR^{\smile}x$ is present, where $R^{\smile}$ is the converse of R.

We say that a set of constraints Θ' is a *refinement* of Θ if and only if the same variables are involved in both the sets, and for every pair of variables x, y, if $xR'y \in \Theta'$ and $xRy \in \Theta$ then $R' \subseteq R$. Θ' is said to be a *consistent refinement* of Θ if and only if Θ' is a refinement of Θ and both Θ and Θ' are consistent. A *consistent scenario* Θ_s of a set of constraints Θ is a consistent refinement of Θ where all the constraints of Θ_s are singletons, i.e., assertions of basic relations.

3 Finding a Consistent Scenario for RCC-8

In order to decide the consistency of a set of RCC-8 constraints Θ, a path-consistent refinement of Θ containing only constraints over $\widehat{\mathcal{H}}_8$ must be found[3] (usually by applying backtracking). However, for other tasks such as finding a model of Θ this is not sufficient. For these tasks it is rather necessary to have basic relations between any pair of variables involved in Θ, i.e., a consistent scenario for Θ is required.

A naive algorithm for finding a consistent scenario for Θ is based on iteratively selecting a label and enforce path-consistency on the restricted set. This requires $O(n^5)$ time if Θ contains only constraints over $\widehat{\mathcal{H}}_8$, and to the best of our knowledge no algorithm with a better worst-case complexity is known. As in the case of qualitative temporal reasoning (e.g., [11], [17]), by exploiting some properties of the particular class of used relations, it is possible to design more efficient algorithms. For instance, a more efficient algorithm is possible for a certain set of relations when path-consistency implies minimal labels or strong consistency [4] for this set. However, this is not the case for $\widehat{\mathcal{H}}_8$ [16].

In the following we will prove that given a consistent set Θ of constraints over $\widehat{\mathcal{H}}_8$, a consistent scenario Θ_s can be obtained in $O(n^3)$ time by reducing all the constraints of Θ to constraints over the basic relations in a very particular way. In order to prove this, we will use several "refinement strategies" of the form "given a particular set $\mathcal{S} \subseteq \widehat{\mathcal{H}}_8$, a relation R', and a path-consistent set Θ of constraints over $\mathcal{S}$, Θ can be consistently refined to Θ' by replacing each constraint xRy in Θ, such that $R' \subset R$, with the constraint $xR'y$." In the following the set $\mathcal{S}$ will be indicated with $\widehat{\mathcal{H}}_{\downarrow_{R'}}$.

[3] This is of course also true for any other tractable subclass of RCC-8 for which path-consistency is sufficient for deciding consistency.

By using the encoding of RCC-8 in classical propositional logic as specified in [16], it is possible to prove that these refinement strategies preserve consistency. In this encoding, every set of RCC-8 constraints Θ is transformed to a propositional formula $p(\Theta)$ such that Θ is consistent if and only if $p(\Theta)$ is satisfiable. In particular, every variable x involved in Θ is transformed to a set of propositional atoms $\{X_k \mid k = 1, \ldots, m; m \leq cn^2\}$ (c is a constant), and every constraint $xRy \in \Theta$ is transformed to a set of clauses involving literals over X_k and Y_k for some k. Since each constraint xRy is transformed to a propositional Horn clause if $R \in \widehat{\mathcal{H}}_8$, consistency of a set Θ of constraints over $\widehat{\mathcal{H}}_8$ can be decided by applying positive unit resolution (PUR) to $p(\Theta)$.[4] Here we will use a property which was proved in [16], namely, that a new positive unit clause $\{X_w\}$ can be derived from $p(\Theta)$ by using PUR only if (1) there is a variable y in Θ such that the clause $\{Y_w\}$ is present, and (2) Θ contains a so-called *R_Γ-chain from x to y*. An R_Γ-chain from x to y is a sequence of constraints $xRz, zR'z', \ldots, z''R'''y$, where all relations $R, R', \ldots, R'''$ are from a particular set of relations R_Γ.[5] We also need another property which can be proven by using similar methods as those applied in [16]:

Lemma 1. *Let Θ be a set of constraints over $\widehat{\mathcal{H}}_8$ that contains an R_Γ-chain from x to y for some variables x and y. If for some $w \in \{1, \ldots, m\}$ the clause $\{X_w\}$ can be derived from $p(\Theta)$ and $\{Y_w\}$ by using PUR, then $\{X_k\}$ can be derived from $p(\Theta)$ and $\{Y_k\}$ by using PUR for all $k \in \{1, \ldots, m\}$.*

We will only sketch some of the proofs involving the encoding of RCC-8 in classical propositional logic, and refer to the technical report for the full proofs [5]. We will use refinement strategies for the following sets of relations:

Definition 1.

- $\widehat{\mathcal{H}}_{\downarrow \mathsf{DC}} = \widehat{\mathcal{H}}_8$
- $\widehat{\mathcal{H}}_{\downarrow \mathsf{EC}} = \{R \in \widehat{\mathcal{H}}_8 \mid R$ *does not contain both* DC *and* EC$\}$
- $\widehat{\mathcal{H}}_{\downarrow \mathsf{PO}} = \{R \in \widehat{\mathcal{H}}_8 \mid R$ *does not contain any of* DC, EC, *or* EQ, *unless R is a basic relation*$\}$
- $\widehat{\mathcal{H}}_{\downarrow \mathsf{NTPP}} = \{R \in \widehat{\mathcal{H}}_8 \mid R$ *does not contain any of* DC, EC, EQ, *or* PO, *unless R is a basic relation*$\}$

Lemma 2 (DC-refinement). *Let Θ be a path-consistent set of constraints over $\widehat{\mathcal{H}}_{\downarrow \mathsf{DC}}$. Θ can be consistently refined to Θ' by replacing every constraint $xRy \in \Theta$ such that $\{\mathsf{DC}\} \subset R$ with the constraint $x\{\mathsf{DC}\}y$.*

Proof Sketch. Let xRy with $R = \{\mathsf{DC}\} \cup R'$ be one of the constraints of Θ, and suppose that Θ becomes inconsistent if xRy is replaced with $x\{\mathsf{DC}\}y$ resulting in Θ''. Since the propositional encoding of $x\{\mathsf{DC}\}y$ is $p(x\{\mathsf{DC}\}y) = \bigwedge_{k=1}^{m}(\neg X_k \vee \neg Y_k)$ (see [16]), no new positive unit clause can be derived by using these Horn clauses. Thus, the empty clause can be derived from $p(\Theta'')$ by using PUR only

[4] PUR is complete for deciding satisfiability of a set of propositional Horn clauses [8].

[5] R_Γ is the set of relations that contains one of the basic relations $\mathsf{TPP}^{-1}, \mathsf{NTPP}^{-1}$, or EQ, but does not contain any of TPP, NTPP, or PO [16]

when for some $w \in \{1, \ldots, m\}$ both the unit clauses $\{X_w\}$ and $\{Y_w\}$ can be derived from $p(\Theta)$. It follows from [16] that if this were possible, then Θ would not be path-consistent. This contradicts our assumptions. Since no new positive unit clause is derivable from $p(\Theta'')$, any constraint xRy of Θ that contains $\{\mathsf{DC}\}$ can be replaced with $x\{\mathsf{DC}\}y$ simultaneously, for any pair of variables x and y in Θ, without applying a path-consistency algorithm after each refinement. $\square$

The proofs of the following three refinement strategies are more complex than the proof of the previous one since there are more cases to consider. However, all cases can be handled with similar methods as used in the previous proof, namely, by looking at whether the changes to the propositional encodings resulting from the refinement of constraints permit to derive the empty clause by using PUR. In all of these cases it turns out (mostly by applying Lemma 1) that if the empty clause is derivable after the refinements, then it was also derivable before the refinement. Therefore the refinements preserve consistency of the set of constraints.

Lemma 3 (EC-refinement). *Let Θ be a path-consistent set of constraints over $\widehat{\mathcal{H}}_{\downarrow\mathsf{EC}}$. Θ can be consistently refined to Θ' by replacing every constraint $xRy \in \Theta$ such that $\{\mathsf{EC}\} \subset R$ with the constraint $x\{\mathsf{EC}\}y$.*

Lemma 4 (PO-refinement). *Let Θ be a path-consistent set of constraints over $\widehat{\mathcal{H}}_{\downarrow\mathsf{PO}}$. Θ can be consistently refined to Θ' by replacing every constraint $xRy \in \Theta$ such that $\{\mathsf{PO}\} \subset R$ with the constraint $x\{\mathsf{PO}\}y$.*

Lemma 5 (NTPP-refinement). *Let Θ be a path-consistent set of constraints over $\widehat{\mathcal{H}}_{\downarrow\mathsf{NTPP}}$. Θ can be consistently refined to Θ' by replacing every constraint $xRy \in \Theta$ such that $\{\mathsf{NTPP}\} \subset R$ with the constraint $x\{\mathsf{NTPP}\}y$.*

In addition to the four refinement strategies described above, we need a further constraint refinement technique for handling relations containing $\{\mathsf{EQ}\}$.

Lemma 6 (EQ-elimination). *Let Θ be a path-consistent set of constraints over $\widehat{\mathcal{H}}_8$. Θ can be consistently refined to Θ' by eliminating $\{\mathsf{EQ}\}$ from every constraint $xRy \in \Theta$ unless $R = \{\mathsf{EQ}\}$.*

Proof Sketch. Let xRy be one of the constraints of Θ, and suppose that Θ becomes inconsistent if $\{\mathsf{EQ}\}$ is eliminated from xRy resulting in Θ''. Since eliminating $\{\mathsf{EQ}\}$ from a relation R is equivalent to intersecting R with the relation $\overline{\{\mathsf{EQ}\}}$ expressible as $\overline{\{\mathsf{EQ}\}} = \{\mathsf{EC}\} \circ \{\mathsf{DC}, \mathsf{PO}\}$,[6] Θ'' is equivalent to $\Theta \cup \{x\{\mathsf{EC}\}z, z\{\mathsf{DC}, \mathsf{PO}\}y\}$ where z is a fresh variable which is only related with x and y. As neither $\{\mathsf{EC}\}$ nor $\{\mathsf{DC}, \mathsf{PO}\}$ are contained in R_Γ, no positive unit clauses for literals of $p(\Theta)$ can be derived from $p(\Theta'')$ by PUR using the clauses resulting from the propositional encoding of the two new constraints. Thus, the empty clause can only be derived from $p(\Theta'')$ by using the unit clauses in the propositional encodings of the new constraints. Because of Lemma 1 there must be a way in which only the unit clauses of the newly added constraints can be

[6] $\overline{Rel}$ denotes the complement of Rel, $R_1 \circ R_2$ denotes the composition of R_1 with R_2.

used to derive the empty clause, and not the unit clauses derivable from $p(\Theta)$. It follows from [16] that this is possible only if Θ contains an R_Γ-chain from x to y and an R_Γ-chain from y to x. But if this were the case, it follows from [16] that Θ would not be path-consistent, and thus the initial assumptions would be contradicted. Since no new R_Γ-chain is introduced in Θ'', $\{\mathsf{EQ}\}$ can be eliminated from all relations simultaneously without applying the path-consistency algorithm after each elimination. $\square$

We can now combine the five strategies to derive an algorithm for determining a consistent scenario for a consistent set of constraints over $\widehat{\mathcal{H}}_8$.

Theorem 1. *For each path-consistent set Θ of constraints over $\widehat{\mathcal{H}}_8$, a consistent scenario Θ_s can be determined in time $O(n^3)$, where n is the number of variables involved in Θ.*

Proof. The following algorithm, Scenario(Θ), solves the problem:

> (1) apply DC-refinement, (2) apply EC-refinement, (3) apply EQ-elimination, (4) apply PO-refinement, (5) apply NTPP-refinement, (6) return the set of the resulting constraints.
> Impose path-consistency after each of the steps (1)–(5).

Scenario(Θ) terminates in $O(n^3)$ time since each of the steps (1)–(5) takes time $O(n^2)$ and path-consistency, which takes time $O(n^3)$, is computed 5 times. By Definition 1 we have that $\widehat{\mathcal{H}}_{\downarrow\mathsf{NTPP}} \subset \widehat{\mathcal{H}}_{\downarrow\mathsf{PO}} \subset \widehat{\mathcal{H}}_{\downarrow\mathsf{EC}} \subset \widehat{\mathcal{H}}_8$. It follows from Lemmas 2 to 6 that after applying each refinement step, Θ is refined to a set of constraints containing only relations for which the next refinement step is guaranteed to make a consistent refinement. Since the interleaved applications of the path-consistency algorithm can only refine constraints and never add new basic relations to the constraints, the possible set of relations obtained after each step is not extended by applying the path-consistency algorithm.

Thus, since Θ is consistent and contains only constraints over $\widehat{\mathcal{H}}_8$, the output of Scenario(Θ) is consistent. Moreover, since any (non-basic) relation of $\widehat{\mathcal{H}}_8$ contains one of $\mathsf{DC}, \mathsf{EC}, \mathsf{PO}, \mathsf{NTPP}$, or NTPP^{-1} (see the definition of $\widehat{\mathcal{H}}_8$ in Section 2), the interleaved applications of path-consistency and steps (1)–(5) guarantee that the output of Scenario(Θ) is a consistent scenario for Θ. $\square$

By applying Scenario(Θ) to a path-consistent set of constraints over $\widehat{\mathcal{H}}_8$ we obtain a particular consistent scenario Θ_s of Θ. Since exactly this consistent scenario is used in the proof of the main theorem of Section 5, in the following lemma we explicitly describe the relationship between Θ_s and Θ.

Lemma 7. *Let Θ be a path-consistent set of constraints over $\widehat{\mathcal{H}}_8$ involving the variables x and y, and let Θ_s be the output of* Scenario(Θ).

- *The constraint $x\{\mathsf{EQ}\}y$ is contained in Θ_s only if it is also contained in Θ.*
- *The constraint $x\{\mathsf{TPP}\}y$ is contained in Θ_s only if Θ contains either $x\{\mathsf{TPP}\}y$ or $x\{\mathsf{TPP}, \mathsf{EQ}\}y$.*

– *The constraint* $x\{\mathsf{NTPP}\}y$ *is contained in* Θ_s *only if* Θ *contains either* $x\{\mathsf{NTPP}\}y$, $x\{\mathsf{NTPP}, \mathsf{TPP}\}y$ *or* $x\{\mathsf{NTPP}, \mathsf{TPP}, \mathsf{EQ}\}y$.

In all the other cases, $xRy \in \Theta$ *is refined to one of* $x\{\mathsf{DC}\}y$, $x\{\mathsf{EC}\}y$ *or* $x\{\mathsf{PO}\}y$ *in* Θ_s.

Proof Sketch. If the path-consistency algorithm were not applied after each of the steps (1) - (5) in SCENARIO(Θ), the proof would immediately follow from the applications of the refinements in the algorithm. Considering the interleaved path-consistency computations, it might be possible that after refining a constraint of Θ to one of $\{\mathsf{DC}\}, \{\mathsf{EC}\}$, or $\{\mathsf{PO}\}$ by one of the steps (1), (2), or (4), another constraint $xRy \in \Theta$, such that $R \cap \{\mathsf{DC}, \mathsf{EC}, \mathsf{PO}\} \neq \emptyset$ and $R \cap \{\mathsf{TPP}, \mathsf{TPP}^{-1}, \mathsf{NTPP}, \mathsf{NTPP}^{-1}, \mathsf{EQ}\} \neq \emptyset$, is refined by the path-consistency algorithm to $xR'y$ such that $R' \cap \{\mathsf{DC}, \mathsf{EC}, \mathsf{PO}\} = \emptyset$. In this case, xRy would be refined by SCENARIO(Θ) to one of $x\{\mathsf{NTPP}\}y$, $x\{\mathsf{TPP}\}y$, $x\{\mathsf{NTPP}^{-1}\}y$, $x\{\mathsf{TPP}^{-1}\}y$, or $x\{\mathsf{EQ}\}y$ in Θ_s which contradicts the lemma. However, by analyzing the composition table of the RCC-8 relations (see, e.g.,[16]) it follows that this is never possible for the sets of relations used by the different refinement strategies. □

4 Combining Topological and Qualitative Size Relations

In this section we introduce $\mathcal{QS}$, a class of qualitative relations between region sizes, and we combine this class with RCC-8. We also give some technical results that will be used in the next section, where we present an algorithm for processing constraints in the combined framework.

In the following we will assume that all the spatial regions are measurable sets in R^n [2]. Note that this assumption does not compromise the computational properties of $\widehat{\mathcal{H}}_8$, because from [15] it follows that the regions of every consistent set of RCC-8 constraints can always be interpreted as measurable sets (e.g., as sets of spheres in R^3). We will also assume that the size of an n-dimensional region corresponds to its n-dimensional measure [2]. For example, the size of a sphere in R^3 corresponds to its volume.

Given a set V of spatial region variables, a set of $\mathcal{QS}$ constraints over V is a set of constraints of the form $size(x)\ S\ size(y)$, where $S \in \mathcal{QS}$, $size(x)$ is the size of the region x, $size(y)$ is the size of the region y, and $x, y \in V$.

Definition 2. *$\mathcal{QS}$ is the class formed by the following eight qualitative relations between the size of spatial regions:* $<$, $>$, $\leq$, $\neq$, $=$, $\geq$, $<=>$ *and* $\emptyset$, *where* $<=>$ *is the universal relation, and* $<$, $>$, *and* $=$ *are the basic relations.*

Proposition 1. *The relations of $\mathcal{QS}$ form a Point Algebra.*

It is obvious that the topological RCC-8 relations and the relative size relations are not independent from each other. Table 1 gives the size relations that are entailed by the basic RCC-8 relations, and the topological relations that are entailed by the basic size relations. $Sizerel(R)$ indicates the strongest size relation entailed by the topological relation R, and $Toprel(S)$ indicates the strongest topological relation entailed by the size relation S.

The dependencies from a non-basic relation R can be obtained by disjunctively combining the relations entailed by each basic relation in R. For example, {TPP,EQ} entails "$\leq$".

r		$Sizerel(r)$	r		$Sizerel(r)$	s		$Toprel(s)$
TPP	$\models$	$<$	DC	$\models$	$<=>$	$=$	$\models$	DC, EC, PO, EQ
NTPP	$\models$	$<$	EC	$\models$	$<=>$	$>$	$\models$	DC, EC, PO, TPP^{-1}, NTPP^{-1}
TPP^{-1}	$\models$	$>$	PO	$\models$	$<=>$	$<$	$\models$	DC, EC, PO, TPP, NTPP
NTPP^{-1}	$\models$	$>$	EQ	$\models$	$=$			

Table 1. Interdependencies of basic RCC-8 relations (r) and basic $\mathcal{QS}$ relations (s)

Since any topological relation – and any sub-relation thereof – entailed by the basic size relations $<, >, =$ is contained in $\widehat{\mathcal{H}}_8$, the following proposition is true.

Proposition 2. *The relation $R \in \mathsf{RCC\text{-}8} \setminus \widehat{\mathcal{H}}_8$ of any constraint xRy can be consistently refined to a relation $R' \in \widehat{\mathcal{H}}_8$, if an appropriate size constraint between x and y is given. In particular, if definite size information is given, then R can always be consistently refined to a relation $R' \in \widehat{\mathcal{H}}_8$.*

For example, the RCC-8 $\setminus\widehat{\mathcal{H}}_8$ constraint $x\{\mathsf{TPP}, \mathsf{TPP}^{-1}, \mathsf{NTPP}, \mathsf{DC}, \mathsf{EC}\}y$ can be consistently refined to the $\widehat{\mathcal{H}}_8$ constraint $x\{\mathsf{TPP}, \mathsf{NTPP}, \mathsf{DC}, \mathsf{EC}\}y$ if the size constraint $size(x) \leq size(y)$ is given. Before giving an algorithm for processing RCC-8 constraints combined with qualitative size constraints, we need to give some further technical definitions and results that will be used in the next section to prove the formal properties of the algorithm.

Definition 3 (Model for Σ). *Given a set Σ of constraints in $\mathcal{QS}$, we say that an assignment σ of spatial regions to the variables of Σ is a* model *of Σ if and only if σ satisfies all the constraints in Σ.*

Definition 4 (Consistency for $\Theta \cup \Sigma$). *Given a set Θ of constraints in* RCC-8 *and a set Σ of constraints in $\mathcal{QS}$, $\Theta \cup \Sigma$ is* consistent *if there exists a model of Θ which is also a model of Σ.*

We say that a consistent scenario for a set Θ of constraints is *size-consistent* relative to a set Σ of constraints if and only if there exists a model for the scenario that is also a model of Σ.

The next lemma states that non-forced equalities can be omitted from a path-consistent set of size constraints in $\mathcal{QS}$ without losing consistency.

Lemma 8. *Let Σ be a path-consistent set of size constraints over $\mathcal{QS}$ and Σ' the set of size constraints such that, for each constraint size(i) S size(j) in Σ,*

1. *if $S \in \{<, >\}$ then size(i) S size(j) $\in \Sigma'$,*
2. *if S = "$\leq$" then $size(i) < size(j) \in \Sigma'$,*
3. *if S = "$\geq$" then $size(i) > size(j) \in \Sigma'$,*
4. *if S = "$=$" then $size(i) = size(j) \in \Sigma'$,*
5. *if S = "$<=>$" then $size(i) \neq size(j) \in \Sigma'$.*

Σ' is consistent and any model of Σ' is also a model of Σ.

Proof Sketch. It follows from van Beek's method of computing a consistent scenario for a set of relations in the temporal Point Algebra [17]. $\square$

Let Θ be a set of constraints in RCC-8, Σ a set of of constraints in $\mathcal{QS}$, t_{ij} the relation between i and j in Θ, and s_{ij} the relation between $size(i)$ and $size(j)$ in Σ. We say that: t_{ij} entails the negation of s_{ij} ($t_{ij} \models \neg s_{ij}$) if and only if $Sizerel(t_{ij}) \cap s_{ij} = \emptyset$; s_{ij} entails the negation of t_{ij} ($s_{ij} \models \neg t_{ij}$) if and only if $Toprel(s_{ij}) \cap t_{ij} = \emptyset$.

Proposition 3. *A consistent set Θ of constraints in* RCC-8 *entails the negation of a $\mathcal{QS}$ relation s_{ij} between $size(i)$ and $size(j)$ if and only if $Sizerel(\hat{t}_{ij}) \cap s_{ij} = \emptyset$, where $\hat{t}_{ij}$ is the strongest entailed relation between i and j in Θ.*

Proof. It follows from the fact that, for any i and j, $\Theta \models \neg s_{ij}$ if and only if $\hat{t}_{ij} \models \neg s_{ij}$, and from the definition of $Sizerel$. $\square$

Proposition 4. *A consistent set Σ of constraints in $\mathcal{QS}$ entails the negation of a* RCC-8 *relation t_{ij} between i and j if and only if $Toprel(\hat{s}_{ij}) \cap t_{ij} = \emptyset$, where $\hat{s}_{ij}$ is the strongest entailed relation between i and j in Σ.*

Proof. It follows from the fact that, for any i and j, $\Sigma \models \neg t_{ij}$ if and only if $\hat{s}_{ij} \models \neg t_{ij}$, and from the definition of $Toprel$. $\square$

Lemma 9. *Let Θ be a consistent set of constraints in* RCC-8*, Σ a consistent set of $\mathcal{QS}$ constraints over the variables of Θ, t_{ij} the relation between i and j in Θ, and s_{ij} the relation between $size(i)$ and $size(j)$ in Σ.*

- *$t_{ij} \models \neg s_{ij}$ if and only if $s_{ij} \models \neg t_{ij}$;*
- *$\Theta \models \neg \hat{s}_{ij}$ if and only if $\Sigma \models \neg \hat{t}_{ij}$.*

Proof Sketch. It follows from Table 1. $\square$

In the next lemma t_{ij} indicates the *basic* relation between i and j in a consistent scenario Θ_s for a set Θ of topological relations.

Lemma 10. *Let Θ_s be a consistent scenario for a (consistent) set Θ of topological constraints in $\widehat{\mathcal{H}}_8$. It is possible to construct a model of Θ that is also a model for the set of size constraints obtained in the following way. For each variable i and j,*

(1) if $Sizerel(t_{ij})$ is one of $<$, $>$, $=$, then $size(i) < size(j)$, $size(i) > size(j)$, and $size(i) = size(j)$, respectively, is in Σ;
(2) if $Sizerel(t_{ij})$ is the universal relation ("$<=>$"), then one of $size(i) < size(j)$ or $size(i) > size(j)$ can be arbitrarily chosen to be added to Σ (provided that Σ remains consistent).

Proof. Let s_{ij} be the relation between $size(i)$ and $size(j)$ in Σ. We show that it is possible to construct a model θ for Θ_s in which the values (spatial regions) assigned to the variables satisfy Σ. Suppose that this were not true. We would have that (a) there would exist h and k such that $\Theta_s \models \neg s_{hk}$, or (b) there would exist h' and k' such that $\Sigma \models \neg t_{h'k'}$ (i.e., there is no model of Θ_s satisfying s_{hk}, or there is no model of Σ consistent with $t_{h'k'}$.) Since by construction of Θ_s, for any pair of variables in Θ_s the strongest entailed relation between i and j is t_{ij}, (a) can hold only if (a') $t_{hk} \models \neg s_{hk}$ holds. For analogous reasons we have that (b) can hold only if (b') $s_{h'k'} \models \neg t_{h'k'}$ holds. But both (a') and (b') cannot hold. In fact, since for any i, j Sizerel$(t_{ij}) \in \{<, >, =, <=>\}$ (because t_{ij} is basic), by (1) and (2) it cannot be the case that $t_{hk} \models \neg s_{hk}$, and hence by Lemma 9 also $s_{h'k'} \models \neg t_{h'k'}$ cannot hold. □

5 Reasoning about Size and Topology Relations

A natural method for deciding the consistency of a set of RCC-8 constraints and a set of $\mathcal{QS}$ constraints, would be to first extend each set of constraints with the constraints entailed by the other set, and then independently check the consistency of the extended sets by using a path-consistency algorithm. However, as the example below shows, this method is not complete for $\widehat{\mathcal{H}}_8$ constraints.

Another possibility, would be to compute the strongest entailed relations (minimal relations) between each pair of variables before propagating constraints from one set to the other. However, this method has the disadvantage that it is computationally expensive.[7]

Finally, a third method could be based on iteratively using path-consistency as a preprocessing technique and then propagating the information from one set to the other.[8] The following example shows that the information would need to be propagated more than once, and furthermore it is not clear whether in general this method would be complete for detecting inconsistency.[9]

Example. Consider the set Θ formed by the following $\widehat{\mathcal{H}}_8$ constraints

$$x_0\{\mathsf{TPP}, \mathsf{EQ}\}x_2,\ x_1\{\mathsf{TPP}, \mathsf{EQ}, \mathsf{PO}\}x_0,\ x_1\{\mathsf{TPP}, \mathsf{EQ}\}x_2,\ x_4\{\mathsf{TPP}, \mathsf{EQ}\}x_3,$$

and the set Σ formed by the of following $\mathcal{QS}$ constraints

$$size(x_0) < size(x_2),\ size(x_3) \leq size(x_1),\ size(x_2) \leq size(x_4).$$

[7] The best known algorithm for computing the minimal network of a set of $\widehat{\mathcal{H}}_8$ constraints requires $O(n^5)$ time.

[8] Note that imposing a path-consistency algorithm is sufficient for consistency checking of $\widehat{\mathcal{H}}_8$ and $\mathcal{QS}$ constraints, but is incomplete for computing the minimal relations [17, 16].

[9] A similar method is used by Ladkin and Kautz to combine qualitative and metric constraints in the context of temporal reasoning [9].

We have that Θ and Σ are independently consistent, but their union is not consistent. Moreover, the following propagation scheme does not detect the inconsistency: (a) enforce path-consistency to Σ and Θ independently; (b) extend Σ with the size constraints entailed by the constraints in Θ; (c) extend Θ with the topological constraints entailed by the constraints in Σ; (d) enforce path-consistency to Θ and Σ again. In order to detect that $\Theta \cup \Sigma$ is inconsistent, we need an additional propagation of constraints from the topological set to the size set.

Instead of directly analyzing the complexity and completeness of the propagation scheme illustrated in the previous example, we propose a new method for dealing with combined topological and qualitative size constraints. In particular, we propose an $O(n^3)$ time and $O(n^2)$ space algorithm, BIPATH-CONSISTENCY, for imposing path-consistency to a set of constraints in RCC-8 $\cup$ $\mathcal{QS}$. We prove that BIPATH-CONSISTENCY solves the problem of deciding consistency for any input set Θ of topological constraints in $\widehat{\mathcal{H}}_8$ combined with any set of size constraints in $\mathcal{QS}$ involving the variables of Θ. Thus, despite this framework is more expressive than $\widehat{\mathcal{H}}_8$ (and therefore has a larger potential applicability), the problem of deciding consistency can be solved without additional worst-case cost.

BIPATH-CONSISTENCY is a modification of Vilain and Kautz's path-consistency algorithm [18] as described by Bessière [3], which in turn is a slight modification of Allen's algorithm [1]. The main novelty of our algorithm is that BIPATH-CONSISTENCY operates on a graph of *pairs* of constraints. The vertices of the graph are constraint variables, which in our context correspond to spatial regions. Each edge of the graph is labeled by a pair of relations formed by a topological relation in RCC-8 and a size relation in $\mathcal{QS}$. The function BIREVISION(i, k, j) has the same role as the function REVISE used in path consistency algorithms for constraint networks (e.g., [12]). The main difference is that BIREVISION(i, k, j) considers pairs of (possibly interdependent) constraints, instead of single constraints.

Note that BIPATH-CONSISTENCY is a general algorithm, in the sense that it can be applied not only to spatial reasoning. For example, it can be applied to pairs of temporal relations, where each pair is formed by a relation in the Allen's Interval Algebra [1] and a qualitative constraint on the duration of the intervals. Of course, different classes of relations might need different completeness and complexity proofs.

A formal description of BIPATH-CONSISTENCY is given in Figure 2, where R_{ij} is a pair formed by a relation t_{ij} in RCC-8 and a relation s_{ij} in $\mathcal{QS}$; $R_{ij} = \emptyset$ when $t_{ij} = \emptyset$ or $s_{ij} = \emptyset$; U_t indicates the universal relation in RCC-8 and U_s the universal relation in $\mathcal{QS}$.

Theorem 2. *Given a set Θ of constraints in $\widehat{\mathcal{H}}_8$ and a set Σ of constraints in $\mathcal{QS}$ involving variables in Θ,* BIPATH-CONSISTENCY *applied to Σ and Θ decides the consistency of $\Sigma \cup \Theta$.*

Proof. It is clear that, if the algorithms returns `fail`, then $\Sigma \cup \Theta$ is inconsistent. Otherwise (the algorithm does not return `fail`) both the output set of size

[10] As in the function REVISE given in [3], this step is used to avoid processing the triple i, j, k when it is known that R_{ij} would not be revised.

Algorithm: BIPATH-CONSISTENCY
Input: A set Θ of RCC-8 constraints, and a set Σ of $\mathcal{QS}$ constraints over the variables $x_1, x_2, \ldots, x_n$ of Θ.
Output: `fail`, if $\Sigma \cup \Theta$ is not consistent; path-consistent sets equivalent to Σ and Θ, otherwise.

1. Q $\leftarrow \{(i,j) \mid i < j\}$; ($i$ indicates the i-th variable of Θ. Analogously for j)
2. *while* Q $\neq \emptyset$ *do*
3. select and delete an arc (i,j) from Q;
4. *for* $k \neq i, k \neq j$ ($k \in \{1...n\}$) *do*
5. *if* BIREVISION(i,j,k) *then*
6. *if* $R_{ik} = \emptyset$ *then* return `fail`
7. *else* add (i,k) to Q;
8. *if* BIREVISION(k,i,j) *then*
9. *if* $R_{kj} = \emptyset$ *then* return `fail`
10. *else* add (k,j) to Q.

Function: BIREVISION(i,k,j)
Input: three region variables i, k and j
Output: true, if R_{ij} is revised; false otherwise.
Side effects: R_{ij} and R_{ji} revised using the operations $\cap$ and $\circ$ over the constraints involving i, k, and j.

1. *if* one of the following cases hold, *then* return false:[10]
   (a) $Toprel(s_{ik}) \cap t_{ik} = U_t$ *and* $Sizerel(t_{ik}) \cap s_{ik} = U_s$,
   (b) $Toprel(s_{kj}) \cap t_{kj} = U_t$ *and* $Sizerel(t_{kj}) \cap s_{kj} = U_s$
2. oldt := t_{ij}; olds := s_{ij};
3. $t_{ij} := (t_{ij} \cap Toprel(s_{ij})) \cap ((t_{ik} \cap Toprel(s_{ik})) \circ (t_{kj} \cap Toprel(s_{kj})))$;
4. $s_{ij} := (s_{ij} \cap Sizerel(t_{ij})) \cap ((s_{ik} \cap Sizerel(t_{ik})) \circ (s_{kj} \cap Sizerel(t_{kj})))$;
5. $t_{ij} := (t_{ij} \cap Toprel(s_{ij}))$;
6. *if* (oldt $= t_{ij}$) *and* (olds $= s_{ij}$) *then* return *false*;
7. $t_{ji} := Converse(t_{ij})$; $s_{ji} := Converse(s_{ij})$;
8. return *true*.

Fig. 2. BIPATH-CONSISTENCY

constraints Σ_p and the output set Θ_p of topological constraints are independently path-consistent. Hence, by Proposition 1 and the fact that a path-consistent set of constraints either in $\widehat{\mathcal{H}}_8$ or in a Point Algebra is consistent [16, 10], Σ and Θ are independently consistent.

Let Θ_p be the path-consistent set of topological constraints given as output of BIPATH-CONSISTENCY applied to Σ and Θ, and Σ_p the path-consistent set of the size constraints. We show that $\Sigma_p \cup \Theta_p$ is consistent (and therefore that $\Sigma \cup \Theta$ is consistent). In order to do that, we show that it is possible to construct a consistent scenario Θ_s for Θ_p in which the region variables can be consistently interpreted as regions satisfying the constraints of Σ (i.e., it is possible to construct a model of $\Sigma \cup \Theta$).

Let Θ_s be a consistent scenario for Θ_p in which, for any pair of variables i and j, the (basic) relation r_{ij} between i and j is

– EQ if $i\{\mathsf{EQ}\}j \in \Theta_p$,
– one of DC, EC, PO, if $R \cap \{\mathsf{DC}, \mathsf{EC}, \mathsf{PO}\} \neq \emptyset$, where $iRj \in \Theta_p$,
– one of TPP, NTPP, TPP^{-1}, NTPP^{-1}, otherwise.

Note that one of EQ, TPP, NTPP, TPP^{-1}, NTPP^{-1}, is chosen only if one of them *must* be chosen, and that Lemma 7 guarantees the existence of Θ_s.

From Θ_s we can derive an assignment to the variables of Θ_s satisfying the constraints of Σ_p (and the topological constraints of Θ_s) in the following way. Let Σ'_p be the set of size constraints derived from Σ_p by applying the five transformation rules of Lemma 8, and let σ'_p be a consistent scenario for Σ'_p. By Lemma 8 σ'_p is also a consistent scenario for Σ_p (and hence for Σ).

For each pair of variables i and j, consider the size relation $Sizerel(r_{ij})$ between i and j. By construction of Θ_s and steps 3–7 of Birevision (the subroutine used by Bipath-consistency to revise topological and size constraints), it is clear that if $Sizerel(r_{ij})$ is one of "$<$", "$>$", "$=$", then the relation between i and j in Σ'_p (and in σ'_p) is the same as $Sizerel(r_{ij})$. So, any assignment satisfying r_{ij} satisfies also the size relation between i and j in Σ'_p (and in σ'_p).

Consider now the case in which $Sizerel(r_{ij})$ is the indefinite relation ("$<=>$"). (Note that since r_{ij} is a basic relation it cannot be the case that $Sizerel(r_{ij}) \in \{\leq, \geq, \neq\}$ – see Table 1.) By construction of Θ_s we have that r_{ij} must be one of $\{\mathsf{DC}\}, \{\mathsf{EC}\}, \{\mathsf{PO}\}$, and by construction of σ'_p that either $size(i) < size(j)$ or $size(j) < size(i)$. Since Σ_p is consistent, by construction of σ'_p and by Lemma 10 we can consistently assign regions to i and j satisfying r_{ij} *and* the size relation between i and j in σ'_p (and hence in Σ'_p). Consequently, since from Θ_s we can derive a consistent assignment satisfying the relations in Σ'_p, by Lemma 8 we can also derive a consistent assignment satisfying the relations in Σ_p (i.e., a model for $\Theta_p \cup \Sigma_p$). □

Theorem 3. *Given a set Θ of constraints over* RCC-8 *and a set Σ of constraints in $\mathcal{QS}$ involving variables in Θ, the time and space complexity of* Bipath-consistency *applied to Σ and Θ are $O(n^3)$ and $O(n^2)$ respectively, where n is the number of variables involved in Θ and Σ.*

Proof. Since any relation in $\mathcal{QS}$ can be refined at most three times, any relation in RCC-8 can be refined at most eight times, and there are $O(n^2)$ relations, the total number of edges that can enter into Q is $O(n^2)$. For each arc in Q, Bipath-consistency runs Birevision $2n$ times. Birevision has a constant time complexity. The quadratic space complexity is trivial. □

Theorem 4. *Given a set Θ of constraints in $\widehat{\mathcal{H}}_8$ and a set Σ of constraints in $\mathcal{QS}$ involving variables in Θ, the consistency of $\Sigma \cup \Theta$ can be determined in $O(n^3)$ time and $O(n^2)$ space, where n is the number of variables involved in Θ and Σ.*

Proof. It follows from Theorems 2 and 3. □

Theorem 5. *Given a set Θ of constraints in $\widehat{\mathcal{H}}_8$ and a set Σ of $\mathcal{QS}$ constraints involving variables in Θ, a size-consistent consistent scenario Θ_s for $\Theta \cup \Sigma$ can be computed in $O(n^3)$ time and $O(n^2)$ space, where n is the number of variables involved in Θ and Σ.*

Proof. From Theorem 1, the proof of Theorem 2 and Theorem 3, it follows that Θ_s can be computed by first applying BIPATH-CONSISTENCY to Θ and Σ, and then running the algorithm described in the proof of Theorem 1 on the set of the topological constraints in the output of BIPATH-CONSISTENCY. □

6 Conclusions

In this paper we have addressed the problem of integrating a basic class of spatial relations, expressing information about the relative size of spatial regions, with RCC-8, a well known class of topological relations. We developed an $O(n^3)$ time algorithm for processing a set of combined topological and relative size constraints, and we proved the correctness and completeness of the algorithm for deciding consistency when the topological constraints are in the $\widehat{\mathcal{H}}_8$ class.

We have also presented an $O(n^3)$ time method for computing a consistent scenario both for combined topological and relative size constraints, and for topological constraints alone.

Future work includes extending the class of size relations to (relative) quantitative size constraints, such as "the size of a certain region is at least two times, and at most six times, the size of another region".

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