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Undecidability of Plane Polygonal Mereotopology

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Abstract

This paper presents a mereotopological model of polygonal regions of the Euclidean plane and an undecidability proof of its first-order theory. Restricted to the primitive operations the model is a Boolean algebra. Its single primitive predicate defines simple polygons as the topologically simplest polygonal regions. It turns out that both the relations usually provided by mereotopologies and more subtle topological relations are elementarily definable in the model. Using these relations, Post's correspondence problem, known as undecidable, can be reduced to the decision problem of the model.

1 Introduction

Formalizing commonsense concepts of space has received much attention both in the philosophical literature and in recent AI research. Mereotopological theories as well as most calculi for spatial reasoning deal with spatial *regions*, i.e. the parts of space occupied by physical bodies, and their topological relations as intuitive concepts of our commonsense space. Whereas mereotopologies are designed to axiomatize the overall structure of regions in terms of topological relations, the research aim of *qualitative spatial reasoning* is the development of calculi to reason about regions and their relations as efficiently as possible (a recent survey is presented by Cohn [6]).

Mereotopologies are formal theories, mostly specified in predicate logic, that describe general properties of spatial regions in terms of topological and mereological relations. A relation is called topological if it is invariant under topological transformations, and mereological if it can be defined entirely by the part-whole relation. For an overview about mereology and mereotopology, especially the ways of defining mereotopology using mereological and topological primitives, we refer to [19] and [9]. The first approach in defining a mereotopology is due to Whitehead [20]; some other approaches are presented by Tarski [18], Randell et al. [16] and Borgo et al. [4].

Most mereotopologies are defined axiomatically by taking as axioms propositions that are considered intuitive for regions and their relations. The overall structure of these regions, together with their mereotopological relations, forms the intended model, which mereotopological theories are designed to axiomatize. Thus, to capture an intended model entirely, mereotopological theories are required to be complete. But the main difficulty in searching for a complete mereotopology is due to the fact that the intended model of regions is not described precisely as a particular model-theoretical structure. Rather than proving if an intuitively defined axiom system is complete it is a better method for such theoretical investigations to define a mereotopology semantically with respect to a particular model $\mathcal{M}$ representing the intended regions and relations. When using first-order logic this means that a mereotopology is defined as the elementary theory $Th(\mathcal{M})$ of $\mathcal{M}$, i.e. the set of all sentences of the elementary predicate language of $\mathcal{M}$ that hold in $\mathcal{M}$ (we use "*elementary*" as a synonym for "first-order"). Then, the problem in finding a complete axiomatic mereotopology is proving whether $Th(\mathcal{M})$ can be axiomatized elementarily. So far, such questions on semantics have gained only little attention in the literature; some models for mereotopology are proposed by Asher and Vieu [1] and by Pratt and Lemon [14].

These considerations are also important for spatial rea-

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soning. Since the underlying semantics of many calculi (e.g. [16], [2], [3]) are too weak to represent the intended regions, these calculi are of limited use, though they can be tractable [17]. Therefore, an appropriate mereotopological model $\mathcal{M}$ can serve equally as a formal semantics for calculi for spatial reasoning, because the statements on regions and relations, which are computed, can be expressed adequately as sentences of $Th(\mathcal{M})$. Thus, it is of considerable interest to know whether $Th(\mathcal{M})$ is decidable. For complete elementary theories like $Th(\mathcal{M})$ decidability is equivalent to axiomatizability (see theorem 3 in section 3), such that the decision problem of $Th(\mathcal{M})$ is relevant both for mereotopology and spatial reasoning.

In this paper we show that $Th(\mathcal{M})$ is already undecidable for the simple mereotopological model $\mathcal{M}$ of polygonal regions of the Euclidean plane. In section 2 we define $\mathcal{M}$ to consist of polygonal regions, some primitive mereological operations and a single primitive topological predicate, such that $\mathcal{M}$ turns out to be a topological Boolean algebra. Section 3 shows first that many topological relations are indeed definable in $\mathcal{M}$, which justifies $\mathcal{M}$ as a mereotopological model, and second, using these definability results, that $Th(\mathcal{M})$ is undecidable by reducing Post's correspondence problem to the decision problem of $Th(\mathcal{M})$. Finally, section 4 offers some conclusions.

2 The Model $\mathcal{M}$ of Plane Polygonal Regions

With the definition of the following mereotopological model $\mathcal{M}$ we propose a simple and plausible semantics for mereotopology of the plane. As usual, the Euclidean plane is represented by the two-dimensional real vector space $\mathbb{R}^2$, whose standard topology and geometry we use to define certain subsets of $\mathbb{R}^2$ as regions. We call $\mathcal{M}$ a *mereotopological model* if $\mathcal{M}$ satisfies the following conditions:

1. The elements of the domain M of $\mathcal{M}$ must be regular closed (or alternatively, regular open) to represent the intended regions, conceptualized as two-dimensional objects, adequately. As a consequence, they are all of the same dimensionality.

2. The primitive relations and functions of $\mathcal{M}$ must be topological or mereological. This ensures that $Th(\mathcal{M})$ contains only propositions about the mereotopological structure of M.

3. As we want $Th(\mathcal{M})$ to include all elementary sentences about the usually considered topological and mereological relations between regions, these relations, defined on M, must be elementarily definable in $\mathcal{M}$, i.e. expressible by formal relations in the predicate language induced by $\mathcal{M}$ (the precise definition is given in section 3).

4. All propositions of $Th(\mathcal{M})$ must hold for the intended regions and their relations.

In the choice of primitives for $\mathcal{M}$ we follow some approaches (e.g. [18], [9], [4], [14]) that distinguish clearly between mereological and topological notions by using primitives of both kinds. So, we take operations for building the sum, product and complement (with respect to $\mathbb{R}^2$) as mereological primitives and a single one-place relation as the topological primitive of $\mathcal{M}$. It is widely agreed that these operations for regions should form a Boolean algebra [19]. Although the null element is usually not allowed to represent a region due to philosophical considerations, we include it for technical convenience. It should be clear that this does not restrict the validity of the results.

Our definition is now as follows. Let x° be the *interior* and $\overline{x}$ the *closure* of $x \subseteq \mathbb{R}^2$. Then $x \subseteq \mathbb{R}^2$ is *regular closed* if $x = \overline{x^\circ}$. The set of all regular closed subsets of $\mathbb{R}^2$ is denoted by $RC(\mathbb{R}^2)$. The following operations are usually defined on $RC(\mathbb{R}^2)$ representing the mereological sum, product and complement, respectively:

$$x + y := x \cup y, \quad x \cdot y := \overline{(x \cap y)^\circ}, \quad -x := \overline{\mathbb{R}^2 \setminus x}.$$

As is well known, the structure $\mathcal{RC}(\mathbb{R}^2) := \langle RC(\mathbb{R}^2), +, \cdot, -, \emptyset, \mathbb{R}^2 \rangle$ is a Boolean algebra [12]. With regard to condition 1, we have to ensure that $\mathcal{M}$, restricted to these operations, turns out as a subalgebra of $\mathcal{RC}(\mathbb{R}^2)$. The elements of $RC(\mathbb{R}^2)$ themselves appear to be too general to represent the intended regions, e.g. $RC(\mathbb{R}^2)$ includes regions composed of infinitely many separate regions. Therefore we define the domain M of $\mathcal{M}$ to be a special proper subset of $RC(\mathbb{R}^2)$.

Let $a_1, \ldots, a_n \in \mathbb{R}^2$, $n > 1$, be pairwise distinct and $[a_i, a_{i+1}]$ be the line segment between a_i and a_{i+1}. We call $[a_1, a_2] \cup \cdots \cup [a_{n-1}, a_n]$, denoted by $[a_1, \ldots, a_n]$, a *polygonal line.* If $a_1 = a_n$ the polygonal line is *closed.* If any point of $[a_1, \ldots, a_n] \backslash \{a_1, \ldots, a_n\}$ lies in exactly one line segment $[a_i, a_{i+1}]$ and any point of $\{a_1, \ldots, a_n\}$ belongs to at most two line segments, then we call the polygonal line *simple.* For simple closed polygonal lines the well-known *Jordan curve theorem* holds [13].

Theorem 1 *Let p be a simple closed polygonal line. Then $\mathbb{R}^2 \setminus p$ is a disjoint union of two open domains,* *a bounded* inner domain D_p^I *and an unbounded* outer domain D_p^O, *each of which has the boundary* p. *Moreover, the following properties are satisfied:*

1. *Any two points* $a, b \in D_p^I$ *are path-connected in* D_p^I, *i.e. there is a simple polygonal line* $q \subset D_p^I$ *with endpoints* $a, b \in q$. *The same holds for* D_p^O.
2. *Any simple polygonal line connecting a point of* D_p^I *with a point of* D_p^O *intersects* p. □

With the help of theorem 1 we first define the topologically simplest elements of M.

Definition 1 *Let* p *be a simple closed polygonal line. Then* $\overline{D_p^I}$ *is the* simple polygon *induced by* p. *With* P *we denote the set of all simple polygons.* □

Simple polygons are obviously in $RC(\mathbb{R}^2)$, such that the operations $+$, $\cdot$ and $-$ are declared for P.

Definition 2 *A* polygon *is a finite sum of simple polygons, i.e.* $x_1 + \cdots + x_n$ *with* $n \geq 1$, $x_i \in P$. *The set* $M := \{\sum_{i=1}^{n} x_i,\ -(\sum_{i=1}^{n} x_i) \mid x_1, \ldots, x_n \in P\} \cup \{\emptyset, \mathbb{R}^2\}$ *denotes the set of* polygonal regions. □

Using some general propositions on polygons [5] we can easily establish theorem 2.

Theorem 2 $\langle M, +, \cdot, -, \emptyset, \mathbb{R}^2 \rangle$ *is a subalgebra of* $\mathcal{RC}(\mathbb{R}^2)$. □

This completes our construction of the mereological part of $\mathcal{M}$. Since topological relations are known not to be definable in a pure Boolean algebra, we have to include some topological primitive in $\mathcal{M}$. For this, we simply take the one-place relation P defined above, since P characterizes the simplest elements in M. Finally, our model $\mathcal{M}$ of polygonal regions is defined as:

Definition 3 $\mathcal{M} := \langle M, +, \cdot, -, \emptyset, \mathbb{R}^2, P \rangle$. □

3 Metamathematical Analysis of $\mathcal{M}$

In this section we first show by some definability results that $\mathcal{M}$ is indeed a mereotopological model (in particular with regard to condition 3). Using these results we are able to prove the undecidability of $Th(\mathcal{M})$. However, we first introduce the notions and notations of predicate logic [8] we need for analyzing $\mathcal{M}$.

A first-order predicate language L^S is determined by the set S of non-logical symbols. L^S contains all first-order expressions composed of the usual logical symbols and those of S. With L_0^S we denote the set of all *sentences* of L^S, i.e. expressions without any free variables. An *(elementary) theory* T is a consistent subset of L_0^S that is closed with respect to logical consequence. We say T is *(elementary) axiomatizable*, if there is a recursive $A \subseteq T$ from which any sentence of T follows; in this case, A is an *axiom system* of T. Moreover, T is called *complete*, if for any $\varphi \in L_0^S$ either $\varphi \in T$ or $\neg\varphi \in T$ holds. To say that $\varphi \in L_0^S$ is a logical consequence of A or is valid in a structure $\mathcal{S}$ we use the notation $A \models \varphi$ and $\mathcal{S} \models \varphi$ respectively. The next theorem [8] is fundamental for our purpose.

Theorem 3 *Let* T *be a theory. If* T *is axiomatizable, then it is recursively enumerable. If* T *is complete, then the following propositions are equivalent:*

1. T *is axiomatizable.*
2. T *is decidable.*
3. T *is recursively enumerable.* □

The set of non-logical symbols induced by $\mathcal{M}$ is $S_{\mathcal{M}} := \{+, \cdot, -, \mathsf{0}, \mathsf{1}, \mathsf{P}\}$; the symbols are to be interpreted with respect to $\mathcal{M}$ in the obvious manner, e.g. P denotes the relation P on M. Moreover, the *elementary theory of* $\mathcal{M}$ is defined as $Th(\mathcal{M}) := \{\varphi \in L_0^{S_{\mathcal{M}}} \mid \mathcal{M} \models \varphi\}$, which is complete by definition.

We have to make precise what we mean by relations elementarily definable in $\mathcal{M}$. If $\varphi(\mathsf{x_1}, \ldots, \mathsf{x_n}) \in L^{S_{\mathcal{M}}}$ and $\mathsf{x_1}, \ldots, \mathsf{x_n}$ are its free variables then $\mathcal{M} \models \varphi[x_1, \ldots, x_n]$ means that φ is true when interpreted in $\mathcal{M}$ such that the variables $\mathsf{x_i}$ refer to $x_i \in M$ respectively. Then, $R \subseteq M^n$ is *elementarily definable in* $\mathcal{M}$ if there is a $\varphi(\mathsf{x_1}, \ldots, \mathsf{x_n}) \in L^{S_{\mathcal{M}}}$ such that $(x_1, \ldots, x_n) \in M^n$ iff $\mathcal{M} \models \varphi[x_1, \ldots, x_n]$ for any $x_1, \ldots, x_n \in M$. Furthermore, if φ includes some defined relation symbols we write again $\mathcal{M} \models \varphi$ instead of extending $S_{\mathcal{M}}$ and $\mathcal{M}$. Since all defined relation symbols in φ can be replaced equivalently by its defining expressions of $L^{S_{\mathcal{M}}}$ this does not have any effect on definability or decidability questions.

3.1 Elementarily Definable Relations in $\mathcal{M}$

With the following propositions we present definability results[1] first for relations sufficient for defining easily the relations usually provided by mereotopological calculi. The other topological relations that are provably definable in $\mathcal{M}$ are more subtle and used in section 3.2 to encode Post's correspondence problem. In the remainder of this paper we shall abbreviate terms like $\mathsf{x} \cdot \mathsf{y}$ by xy.

[1] The detailed proofs of the propositions given in this section are contained in [7].

Clearly, $\subseteq$ can be defined in $\mathcal{M}$ by the partial order $\leq$ usually associated with a Boolean algebra by $\mathsf{x} \leq \mathsf{y} \equiv_{def} \mathsf{x} + \mathsf{y} = \mathsf{y}$. Also trivially, the set of polygons POL is definable in $\mathcal{M}$, since polygons are exactly the bounded non-null elements in M.

Proposition 1 *Let $POL \subsetneq M$ and $\mathsf{POL(x)} \in L^{S\mathcal{M}}$ be defined as:*

$$POL \;:=\; \{x \in M \mid x = \sum_{i=1}^{n} x_i \text{ with } x_i \in P,\; n \geq 1\},$$

$$\mathsf{POL(x)} \equiv_{def} \;\; \mathsf{x} \neq 0 \wedge \exists \mathsf{y}(\mathsf{P(y)} \wedge \mathsf{x} \leq \mathsf{y}).$$

Then for all $x \in M$:

$$x \in POL \;\; \text{iff} \;\; \mathcal{M} \models \mathsf{POL}[x]. \qquad \square$$

The relations defined in proposition 2, 3 and 4 respectively are crucial in defining topological relations. They distinguish fundamental cases of how the boundaries of two simple polygons x and y are related that share no common interior point. The first one, ECL, is the set of all pairs of simple polygons that are externally connected at a simple non-closed polygonal line. Second, ECP is the set of all pairs (x, y), such that x and y are externally connected at a single point. Figure 1 shows how $\mathsf{ECP}(x, y)$ defines ECP elementarily in $\mathcal{M}$. The set of all pairs of disconnected simple polygons is defined by DC.

Proposition 2 *Let $ECL \subsetneq M^2$ and $\mathsf{ECL(x,y)} \in L^{S\mathcal{M}}$ be defined as:*

$$ECL \;:=\; \{(x, y) \in P^2 \mid x \cap y \text{ is a simple non-closed polygonal line }\},$$

$$\mathsf{ECL(x,y)} \equiv_{def} \;\; \mathsf{P(x)} \wedge \mathsf{P(y)} \wedge \mathsf{xy} = 0 \wedge \mathsf{P(x+y)}.$$

Then for all $x, y \in M$:

$$(x, y) \in ECL \;\; \text{iff} \;\; \mathcal{M} \models \mathsf{ECL}[x, y]. \qquad \square$$

Proposition 3 *Let $ECP \subsetneq M^2$ and $\mathsf{ECP(x,y)} \in L^{S\mathcal{M}}$ be defined as:*

$$ECP \;:=\; \{(x, y) \in P^2 \mid x \cap y = \{a\} \text{ for some } a \in \mathbb{R}^2\},$$

$$\begin{aligned} &\mathsf{ECP(x,y)} \equiv_{def} \\ &\quad \mathsf{P(x)} \wedge \mathsf{P(y)} \wedge \mathsf{xy} = 0 \wedge \neg \mathsf{P(x+y)} \wedge \\ &\quad \exists \mathsf{v} \exists \mathsf{w}(\mathsf{P(v)} \wedge \mathsf{P(w)} \wedge \neg \mathsf{P(v+w)} \wedge \\ &\qquad\quad \mathsf{xv} = 0 \wedge \mathsf{xw} = 0 \wedge \mathsf{yv} = 0 \wedge \mathsf{yw} = 0 \wedge \mathsf{vw} = 0 \wedge \\ &\qquad\quad \mathsf{P(x+v)} \wedge \mathsf{P(x+w)} \wedge \mathsf{P(y+v)} \wedge \mathsf{P(y+w)} \wedge \\ &\qquad\quad \mathsf{P(x+v+y)} \wedge \mathsf{P(x+w+y)} \wedge \mathsf{P(x+v+w)} \wedge \\ &\qquad\quad \mathsf{P(y+v+w)} \wedge \mathsf{P(x+y+v+w)}). \end{aligned}$$

Then for all $x, y \in M$:

$$(x, y) \in ECP \;\; \text{iff} \;\; \mathcal{M} \models \mathsf{ECP}[x, y]. \qquad \square$$

Proposition 4 *Let $DC \subsetneq M^2$ and $\mathsf{DC(x,y)} \in L^{S\mathcal{M}}$ be defined as:*

$$DC \;:=\; \{(x, y) \in P^2 \mid x \cap y = \emptyset\},$$

$$\begin{aligned} \mathsf{DC(x,y)} \equiv_{def} \;\; &\mathsf{P(x)} \wedge \mathsf{P(y)} \wedge \mathsf{xy} = 0 \wedge \\ &\neg(\exists \mathsf{v} \exists \mathsf{w}(\mathsf{v} \leq \mathsf{x} \wedge \mathsf{w} \leq \mathsf{y} \wedge \mathsf{ECP(v,w)})). \end{aligned}$$

Then for all $x, y \in M$:

$$(x, y) \in DC \;\; \text{iff} \;\; \mathcal{M} \models \mathsf{DC}[x, y]. \qquad \square$$

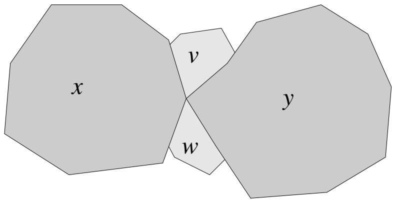

Figure 1: Simple polygons x, y, v, w used to define $\mathsf{ECP(x,y)}$ in proposition 3

It is obvious now that all the usual relations which are provided by mereotopology and spatial reasoning can be defined formally using ECL, ECP and DC. For instance, the connection relation $\mathsf{C(x,y)}$, often taken as the only primitive relation in mereotopological approaches such as the RCC-theory [16], can be defined by

$$\mathsf{C(x,y)} \equiv_{def} \exists \mathsf{v} \exists \mathsf{w}(\mathsf{v} \leq \mathsf{x} \wedge \mathsf{w} \leq \mathsf{y} \wedge \mathsf{ECP(v,w)}).$$

In accordance with the intended interpretation, $\mathcal{M} \models \mathsf{C}[x, y]$ holds for polygonal regions x and y iff x and y have a non-empty intersection. Another example of a RCC-relation that can be easily defined using ECP is $\mathsf{TPP(x,y)}$, the intended meaning of which is that x is a tangential proper part of y:

$$\begin{aligned} \mathsf{TPP(x,y)} \equiv_{def} \; &\mathsf{x} \leq \mathsf{y} \wedge \mathsf{x} \neq \mathsf{y} \wedge \\ &\exists \mathsf{v} \exists \mathsf{w}(\mathsf{v} \leq \mathsf{x} \wedge \mathsf{w} \leq -\mathsf{y} \wedge \mathsf{ECP(v,w)}). \end{aligned}$$

Due to these facts, $\mathcal{M}$ proves to be a mereotopological model, in particular with regard to condition 3 (see section 2). The next propositions deal with more special topological relations or general properties that hold for these relations. They are established with the purpose of preparing for the undecidability proof of $Th(\mathcal{M})$. Proposition 5 means that a simple polygon y, which is contained in the union of simple polygons $x_1, \ldots, x_n$ each two of which have at most one point in common, is already part of some x_i.

Proposition 5 *For all $n \geq 1$ let $\sigma_n \in L_0^{S\mathcal{M}}$ be*

$$\begin{aligned} \forall \mathsf{y} \forall \mathsf{x_1} \cdots \forall \mathsf{x_n}((&\bigwedge_{\substack{\mathsf{i,j} \in \{1,\ldots,\mathsf{n}\},\\ \mathsf{i} \neq \mathsf{j}}} (\mathsf{DC(x_i,x_j)} \vee \mathsf{ECP(x_i,x_j)}) \\ &\wedge \mathsf{P(y)} \wedge \;\mathsf{y} \leq \mathsf{x_1} + \cdots + \mathsf{x_n}) \\ &\Rightarrow (\mathsf{y} \leq \mathsf{x_1} \vee \ldots \vee \mathsf{y} \leq \mathsf{x_n})). \end{aligned}$$

Then for all $n \geq 1$*:* $\mathcal{M} \models \sigma_n$. □

In definition 4 we introduce a relation that plays an important role not only for encoding Post's correspondence problem but generally in describing relevant relations between spatial objects.

Definition 4 *Let* $\mathsf{COM}(\mathsf{x},\mathsf{y}) \in L^{S\mathcal{M}}$ *be defined as:*

$$\mathsf{COM}(\mathsf{x},\mathsf{y}) \equiv_{def} \mathsf{P}(\mathsf{x}) \wedge \mathsf{x} \leq \mathsf{y} \wedge \forall \mathsf{z}((\mathsf{P}(\mathsf{z}) \wedge \mathsf{x} \leq \mathsf{z} \wedge \mathsf{z} \leq \mathsf{y}) \Rightarrow \mathsf{x} = \mathsf{z}).$$

We call $x \in P$ *a* component *of* $y \in M$ *if* $\mathcal{M} \models \mathsf{COM}[x,y]$. □

Thus, a component x is a maximal simple polygon that is part of y. By definition of M, a polygonal region has at most finitely many different components. But it need not be the case, of course, that a polygonal region has any component at all (consider a simple polygon with a hole inside). The role of components comes up when considering a situation as described above: a union of simple polygons, each two of which have at most one point in common, can be separated entirely into its components, namely the simple polygons it is built of. This is what proposition 6 states, but since this separation property already follows from a simple subset $T'_{\mathcal{M}}$ of $Th(\mathcal{M})$ we formulate proposition 6 more generally.

Definition 5 *Let* $\Phi_{BA} \subset L_0^{\{+,\cdot,-,0,1\}}$ *denote the axiom system of ordinary Boolean algebra (see [12]). Then* $T'_{\mathcal{M}} \subset Th(\mathcal{M})$ *is defined as (for* $\sigma_n \in L_0^{S\mathcal{M}}$ *see proposition 5):*

$$T'_{\mathcal{M}} := \Phi_{BA} \cup \{\sigma_n \mid n \geq 1\} \cup \{\neg \mathsf{P}(0)\}.$$ □

Proposition 6 *For all* $n \geq 1$ *let* $\tau_n \in L_0^{S\mathcal{M}}$ *be*

$$\begin{aligned} &\forall \mathsf{x}_1 \cdots \forall \mathsf{x}_n((\bigwedge_{\substack{i,j\in\{1,\dots,n\},\\ i\neq j}} (\mathsf{DC}(\mathsf{x}_i,\mathsf{x}_j) \vee \mathsf{ECP}(\mathsf{x}_i,\mathsf{x}_j)) \wedge \mathsf{P}(\mathsf{x}_1)) \\ &\qquad \Rightarrow (\bigwedge_{i\in\{1,\dots,n\}} \mathsf{COM}(\mathsf{x}_i, \mathsf{x}_1 + \cdots + \mathsf{x}_n) \wedge \\ &\qquad\quad \forall \mathsf{y}(\mathsf{COM}(\mathsf{y}, \mathsf{x}_1 + \cdots + \mathsf{x}_n) \\ &\qquad\qquad \Rightarrow (\mathsf{y} = \mathsf{x}_1 \vee \cdots \vee \mathsf{y} = \mathsf{x}_n)))). \end{aligned}$$

Then for all $n \geq 1$*:* $T'_{\mathcal{M}} \models \tau_n$*. In particular,* $\mathcal{M} \models \tau_n$. □

The next topological relation *SDC* is the set of all sums of disconnected simple polygons. In proposition 7 $x \in SDC$ is characterized formally as a polygon such that any simple polygon in x is contained in some component of x (μ_1) and, additionally, any distinct components of x are disconnected (μ_2).

Proposition 7 *Let* $SDC \subsetneq M$ *and* $\mathsf{SDC}(\mathsf{x}) \in L^{S\mathcal{M}}$ *be defined as:*

$$\begin{aligned} SDC \;:=\; &\{x \in M \mid x = \sum_{i=1}^{n} x_i \text{ with } x_i \in P,\; n \geq 1, \\ &\qquad (x_i, x_j) \in DC \text{ if } i \neq j\}, \end{aligned}$$

$$\mathsf{SDC}(\mathsf{x}) \equiv_{def} \mathsf{POL}(\mathsf{x}) \wedge \mu_1(\mathsf{x}) \wedge \mu_2(\mathsf{x})$$

with

$$\mu_1(\mathsf{x}) \equiv_{def} \forall \mathsf{y}((\mathsf{P}(\mathsf{y}) \wedge \mathsf{y} \leq \mathsf{x}) \Rightarrow \exists \mathsf{z}(\mathsf{y} \leq \mathsf{z} \wedge \mathsf{COM}(\mathsf{z},\mathsf{x}))),$$

$$\begin{aligned} \mu_2(\mathsf{x}) \equiv_{def} &\forall \mathsf{v} \forall \mathsf{w}((\mathsf{COM}(\mathsf{v},\mathsf{x}) \wedge \mathsf{COM}(\mathsf{w},\mathsf{x})) \\ &\Rightarrow (\mathsf{v} = \mathsf{w} \vee \mathsf{DC}(\mathsf{v},\mathsf{w}))). \end{aligned}$$

Then for all $x \in M$*:*

$$x \in SDC \quad \text{iff} \quad \mathcal{M} \models \mathsf{SDC}[x].$$ □

The last relation *SEQ* defined in proposition 8 describes a sequence of simple polygons, as shown in figure 2. $(x_1, x) \in SEQ$ means that $x = x_1 + \cdots + x_n$ for some simple polygons $x_1, \ldots, x_n$, each two of which either intersect in a single point if their indices are neighbors or are disconnected otherwise. This condition allows x to be considered as a chain, because, by proposition 6, $x_1, \ldots, x_n$ are all the components of x, thus representing the distinct elements of the chain, whose endings are x_1 and x_n. If x_1 is fixed as the first element then x represents a sequence. In proposition 8 *SEQ* is elementarily defined by the expression SEQ requiring first that, similar to SDC, x is a polygon combined entirely of its components each two of which intersect at most in a single point and one of which is x_1, second that any partition of the components into v_1 and v_2, and hence x itself, is connected (ν_3), third that x_1 is externally connected to at most one component (ν_4), and finally that any component is externally connected to at most two distinct components (ν_5).

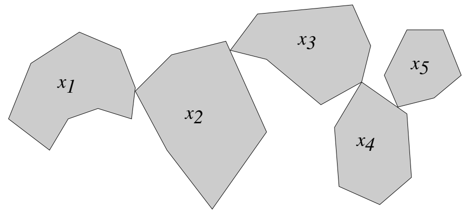

Figure 2: Representing a sequence by polygons: $(x_1, x_1 + \cdots, x_5) \in SEQ$

Proposition 8 *Let* $SEQ \subsetneq M^2$ *and* $\mathsf{SEQ}(\mathsf{x}_1, \mathsf{x}) \in L^{S\mathcal{M}}$ *be defined as:*

$$\begin{aligned} SEQ \;:=\; &\{(x_1, x) \in M^2 \mid x = \sum_{i=1}^{n} x_i \text{ with } x_i \in P,\; n \geq 1, \\ &\qquad (x_i, x_j) \in \begin{cases} ECP & \text{if } |i-j| = 1 \\ DC & \text{if } |i-j| > 1 \end{cases} \}, \end{aligned}$$

$$\begin{aligned} \mathsf{SEQ}(\mathsf{x}_1, \mathsf{x}) \equiv_{def} &\mathsf{POL}(\mathsf{x}) \wedge \mathsf{COM}(\mathsf{x}_1, \mathsf{x}) \wedge \\ &\nu_1(\mathsf{x}) \wedge \nu_2(\mathsf{x}) \wedge \nu_3(\mathsf{x}) \wedge \nu_4(\mathsf{x}_1, \mathsf{x}) \wedge \nu_5(\mathsf{x}) \end{aligned}$$

with

$$\nu_1(\mathsf{x}) \equiv_{def} \forall \mathsf{y}((\mathsf{P}(\mathsf{y}) \wedge \mathsf{y} \leq \mathsf{x}) \Rightarrow \exists \mathsf{z}(\mathsf{y} \leq \mathsf{z} \wedge \mathsf{COM}(\mathsf{z},\mathsf{x}))),$$

$$\nu_2(\mathsf{x}) \equiv_{def} \forall \mathsf{z}_1 \forall \mathsf{z}_2(\mathsf{COM}(\mathsf{z}_1,\mathsf{x}) \wedge \mathsf{COM}(\mathsf{z}_2,\mathsf{x})) \Rightarrow (\mathsf{z}_1 = \mathsf{z}_2 \vee \mathsf{ECP}(\mathsf{z}_1,\mathsf{z}_2) \vee \mathsf{DC}(\mathsf{z}_1,\mathsf{z}_2))),$$

$$\begin{aligned}\nu_3(\mathsf{x}) \equiv_{def}\ & \forall \mathsf{v}_1 \forall \mathsf{v}_2((\mathsf{v}_1 \neq 0 \wedge \mathsf{v}_2 \neq 0 \wedge \mathsf{v}_1\mathsf{v}_2 = 0 \wedge \mathsf{v}_1 + \mathsf{v}_2 = \mathsf{x} \wedge \\ & \forall \mathsf{z}(\mathsf{COM}(\mathsf{z},\mathsf{x}) \Rightarrow (\mathsf{z} \leq \mathsf{v}_1 \vee \mathsf{z} \leq \mathsf{v}_2))) \Rightarrow \\ & \exists \mathsf{z}_1 \exists \mathsf{z}_2(\mathsf{COM}(\mathsf{z}_1,\mathsf{x}) \wedge \mathsf{z}_1 \leq \mathsf{v}_1 \wedge \\ & \mathsf{COM}(\mathsf{z}_2,\mathsf{x}) \wedge \mathsf{z}_2 \leq \mathsf{v}_2 \wedge \mathsf{ECP}(\mathsf{z}_1,\mathsf{z}_2))),\end{aligned}$$

$$\begin{aligned}\nu_4(\mathsf{x}_1,\mathsf{x}) \equiv_{def}\ & \forall \mathsf{z}_1 \forall \mathsf{z}_2((\mathsf{COM}(\mathsf{z}_1,\mathsf{x}) \wedge \mathsf{COM}(\mathsf{z}_2,\mathsf{x}) \wedge \\ & \mathsf{ECP}(\mathsf{z}_1,\mathsf{x}_1) \wedge \mathsf{ECP}(\mathsf{z}_2,\mathsf{x}_1)) \\ & \Rightarrow \mathsf{z}_1 = \mathsf{z}_2),\end{aligned}$$

$$\begin{aligned}\nu_5(\mathsf{x}) \equiv_{def}\ & \forall \mathsf{z} \forall \mathsf{z}_1 \forall \mathsf{z}_2 \forall \mathsf{z}_3((\mathsf{COM}(\mathsf{z},\mathsf{x}) \wedge \mathsf{COM}(\mathsf{z}_1,\mathsf{x}) \wedge \mathsf{COM}(\mathsf{z}_2,\mathsf{x}) \\ & \wedge \mathsf{COM}(\mathsf{z}_3,\mathsf{x}) \wedge \mathsf{ECP}(\mathsf{z}_1,\mathsf{z}) \wedge \mathsf{ECP}(\mathsf{z}_2,\mathsf{z}) \\ & \wedge \mathsf{ECP}(\mathsf{z}_3,\mathsf{z})) \\ & \Rightarrow (\mathsf{z}_1 = \mathsf{z}_2 \vee \mathsf{z}_1 = \mathsf{z}_3 \vee \mathsf{z}_2 = \mathsf{z}_3)).\end{aligned}$$

Then for all $x_1, x \in M$*:*

$$(x_1, x) \in SEQ \quad \textit{iff} \quad \mathcal{M} \models \mathsf{SEQ}[x_1, x]. \qquad \square$$

To complete the preparation of the undecidability proof presented in the next section proposition 9 ensures the existence of polygons in M that enable the encoding of Post's correspondence problem. For any n there are simple polygons x_{ij}, $i, j \in \{1, \ldots, n\}$, that are arranged like squares of an $n \times n$ chessboard, as shown in figure 3. Then, the sentence ξ_n in proposition 9 guarantees some topological relations to hold between these x_{ij} as well as between special sums $\Phi_{i,j,k,l}$ of them, which are determined entirely by x_{ij} and x_{kl}.

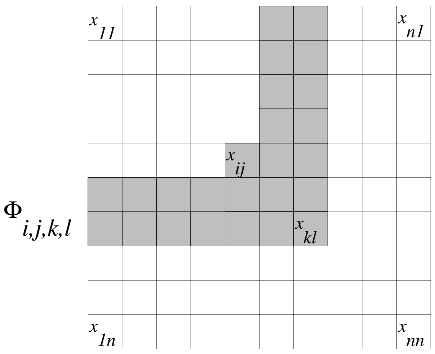

Figure 3: Simple polygons composed by squares of a chessboard

Proposition 9 *For all* $n \geq 1$ *let* $\xi_n \in L_0^{S_\mathcal{M}}$ *be*

$$\begin{aligned}& \exists \mathsf{x}_{11} \cdots \exists \mathsf{x}_{n1} \exists \mathsf{x}_{12} \cdots \exists \mathsf{x}_{n2} \cdots \exists \mathsf{x}_{1n} \cdots \exists \mathsf{x}_{nn} \\ & \Big(\bigwedge_{i \neq k \vee j \neq l} \mathsf{x}_{ij}\mathsf{x}_{kl} = 0 \ \wedge \bigwedge_{i,j} \mathsf{P}(\mathsf{x}_{ij}) \ \wedge \bigwedge_{\substack{|i-k| \geq 2 \\ \vee |j-l| \geq 2}} \mathsf{DC}(\mathsf{x}_{ij}, \mathsf{x}_{kl}) \ \wedge \\ & \bigwedge_{\substack{i<k, \\ j<l}} \mathsf{P}(\Phi_{i,j,k,l}) \wedge \bigwedge_{\substack{i<k \wedge j<l, \\ k+1<k' \wedge l+1<l'}} \mathsf{ECP}(\Phi_{i,j,k,l}, \Phi_{k+1,l+1,k',l'}) \\ & \wedge \bigwedge_{\substack{i<k \wedge j<l, \\ i'<k' \wedge j'<l', \\ k+1<i' \wedge l+1<j'}} \mathsf{DC}(\Phi_{i,j,k,l}, \Phi_{i',j',k',l'}) \Big)\end{aligned}$$

with $\Phi_{i,j,k,l}$:=

$$\begin{aligned}& \mathsf{x}_{i,j} + (\mathsf{x}_{i+1,1} + \cdots + \mathsf{x}_{k,1}) + \cdots + (\mathsf{x}_{i+1,j} + \cdots + \mathsf{x}_{k,j}) \\ & \quad + (\mathsf{x}_{1,j+1} + \cdots + \mathsf{x}_{k,j+1}) + \cdots + (\mathsf{x}_{1,l} + \cdots + \mathsf{x}_{k,l}).\end{aligned}$$

Then for all $n \geq 1$*:* $\mathcal{M} \models \xi_n$. $\square$

3.2 Undecidability of $Th(\mathcal{M})$

Using the propositions of the last section we are now able to show $Th(\mathcal{M})$ to be undecidable. Actually, we prove a slightly stronger result. We define an axiomatic fragment $T_\mathcal{M}$ of $Th(\mathcal{M})$ to consist of all properties of $\mathcal{M}$ that are sufficient for encoding Post's correspondence problem. As we will see in theorem 4, any set T of elementary sentences which is valid in $\mathcal{M}$ and includes $T_\mathcal{M}$ turns out to be undecidable.

Definition 6 *Let* $T_\mathcal{M} \subset Th(\mathcal{M})$ *be defined as*

$$\begin{aligned} T_\mathcal{M} &:= T'_\mathcal{M} \cup \{\xi_n \mid n \geq 1\} \\ &= \Phi_{BA} \cup \{\neg \mathsf{P}(0)\} \cup \{\sigma_n,\ \xi_n \mid n \geq 1\}. \qquad \square \end{aligned}$$

Post's correspondence problem (PCP) (restricted to the alphabet $\{0,1\}$) is defined as follows. A *PCP*-instance is given by a list $(\alpha_1, \beta_1), \ldots, (\alpha_m, \beta_m)$ of pairs of non-empty strings over $\{0,1\}$. This instance has a solution if there is any sequence of integers $i_1, \ldots, i_k$ with $1 \leq i_j \leq m$, such that $\alpha_{i_1} \cdots \alpha_{i_k} = \beta_{i_1} \cdots \beta_{i_k}$. Post's correspondence problem is then to determine if a given *PCP*-instance has a solution. As it is well-known, this problem is undecidable [11].

The key in proving any $T \subset L_0^{S_\mathcal{M}}$ with $T_\mathcal{M} \subseteq T \subseteq Th(\mathcal{M})$ to be undecidable is to reduce Post's correspondence problem to the decision problem of T, that is, to associate with any *PCP*-instance C a sentence $\Psi_C \in L_0^{S_\mathcal{M}}$, such that $T \models \Psi_C$ iff C has a solution. The sentence Ψ_C is given in definition 7 using some operations on strings. For any string $\delta \in \{0,1\}^+$ $|\delta|$ denotes the length of δ, $\delta[i]$ denotes the ith symbol in δ, if $1 \leq i \leq |\delta|$, and $\delta[i,j]$ denotes the substring $\delta[i]\ \delta[i+1] \cdots \delta[j]$, if $1 \leq i \leq j \leq |\delta|$.

Definition 7 *Let* $\Delta_\delta(\mathsf{x},\mathsf{c},\mathsf{a},\mathsf{a}_1,\mathsf{z}) \in L^{S_\mathcal{M}}$ *for any* $\delta \in \{0,1\}^+$ *be defined as*

$$\Delta_\delta(\mathsf{x},\mathsf{c},\mathsf{a},\mathsf{a}_1,\mathsf{z}) \equiv_{def}$$

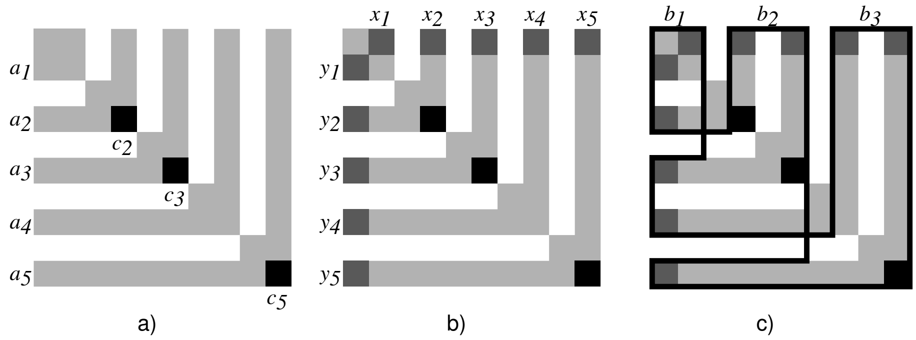

Figure 4: Encoding a solution of a *PCP*-instance by polygons: in a) the a_i are the light-grey colored and the c_l are the black colored simple polygons contained in a_l, in b) the x_i and y_i respectively are the dark-grey colored simple polygons, and in c) the b_j are the simple polygons marked by their borderlines.

$$
\begin{aligned}
&\exists \mathsf{v}_1 \cdots \exists \mathsf{v}_{|\delta|}\\
&(\mathsf{COM}(\mathsf{v}_1,\mathsf{a}) \wedge \cdots \wedge \mathsf{COM}(\mathsf{v}_{|\delta|},\mathsf{a}) \wedge\\
&\mathsf{ECP}(\mathsf{v}_1,\mathsf{v}_2) \wedge \mathsf{ECP}(\mathsf{v}_2,\mathsf{v}_3) \wedge \cdots \wedge \mathsf{ECP}(\mathsf{v}_{|\delta|-1},\mathsf{v}_{|\delta|})\\
&\wedge \bigwedge_{\mathsf{i}\neq \mathsf{j}} \mathsf{v}_\mathsf{i} \neq \mathsf{v}_\mathsf{j} \wedge\\
&\forall \mathsf{s}((\mathsf{s} \leq \mathsf{a} \wedge \forall \mathsf{d}(\mathsf{COM}(\mathsf{d},\mathsf{a}) \Rightarrow (\mathsf{d} \leq \mathsf{s} \vee \mathsf{ds} = 0)) \wedge\\
&\qquad \mathsf{SEQ}(\mathsf{a}_1,\mathsf{s}) \wedge \mathsf{v}_{|\delta|} \leq \mathsf{s}) \Rightarrow \mathsf{v}_1 \leq \mathsf{s}) \wedge\\
&\mathsf{zx} = (\mathsf{v}_1 + \cdots + \mathsf{v}_{|\delta|})\mathsf{x} \wedge\\
&\Omega_{\delta[1]}(\mathsf{v}_1\mathsf{c}) \wedge \cdots \wedge \Omega_{\delta[|\delta|]}(\mathsf{v}_{|\delta|}\mathsf{c}))
\end{aligned}
$$

$$
\textit{with } \Omega_{\delta[i]}(\mathsf{v}_\mathsf{i}\mathsf{c}) = \begin{cases} \mathsf{v}_\mathsf{i}\mathsf{c} \neq 0 & \textit{if} \quad \delta[i] = 1\\ \mathsf{v}_\mathsf{i}\mathsf{c} = 0 & \textit{if} \quad \delta[i] = 0. \end{cases}
$$

Let $C = \{(\alpha_1,\beta_1),\ldots,(\alpha_m,\beta_m)\}$*, for* $\alpha_j,\beta_j \in \{0,1\}^+$*, be an arbitrary* PCP*-instance. Then,* $\Psi_C \in L_0^{\mathcal{SM}}$ *is defined as*

$$
\begin{aligned}
&\Psi_C \equiv_{def}\\
&\quad \exists \mathsf{x}\exists \mathsf{y}\exists \mathsf{a}\exists \mathsf{a}_1\exists \mathsf{b}\exists \mathsf{b}_1\exists \mathsf{c}\\
&\quad (\mathsf{SEQ}(\mathsf{a}_1,\mathsf{a}) \wedge \mathsf{SEQ}(\mathsf{b}_1,\mathsf{b}) \wedge \mathsf{SDC}(\mathsf{x}) \wedge \mathsf{SDC}(\mathsf{y}) \wedge\\
&\quad \mathsf{x} \leq \mathsf{a} \wedge \mathsf{y} \leq \mathsf{a} \wedge \mathsf{x} \leq \mathsf{b} \wedge \mathsf{y} \leq \mathsf{b} \wedge \mathsf{a}_1 \leq \mathsf{b}_1 \wedge\\
&\quad \lambda_1(\mathsf{a},\mathsf{x},\mathsf{y}) \wedge \lambda_2(\mathsf{a},\mathsf{b},\mathsf{x},\mathsf{y}) \wedge \lambda_3(\mathsf{x},\mathsf{y},\mathsf{a},\mathsf{a}_1,\mathsf{b},\mathsf{c}))
\end{aligned}
$$

with

$$
\begin{aligned}
&\lambda_1(\mathsf{a},\mathsf{x},\mathsf{y}) \equiv_{def}\\
&\quad \forall \mathsf{z}(\mathsf{COM}(\mathsf{z},\mathsf{a}) \Rightarrow (\mathsf{COM}(\mathsf{zx},\mathsf{x}) \wedge \mathsf{COM}(\mathsf{zy},\mathsf{y}))),\\
&\lambda_2(\mathsf{a},\mathsf{b},\mathsf{x},\mathsf{y}) \equiv_{def}\\
&\quad \forall \mathsf{z}_1\forall \mathsf{z}_2((\mathsf{COM}(\mathsf{z}_1,\mathsf{b}) \wedge \mathsf{COM}(\mathsf{z}_2,\mathsf{b}) \wedge \mathsf{ECP}(\mathsf{z}_1,\mathsf{z}_2)) \Rightarrow\\
&\qquad\qquad (\exists \mathsf{v}_1\exists \mathsf{v}_2(\mathsf{COM}(\mathsf{v}_1,\mathsf{a}) \wedge \mathsf{COM}(\mathsf{v}_2,\mathsf{a}) \wedge \mathsf{v}_1\mathsf{x} \leq \mathsf{z}_1\\
&\qquad\qquad\qquad \wedge \mathsf{v}_2\mathsf{x} \leq \mathsf{z}_2 \ \wedge \mathsf{ECP}(\mathsf{v}_1,\mathsf{v}_2)) \wedge\\
&\qquad\qquad \exists \mathsf{u}_1\exists \mathsf{u}_2(\mathsf{COM}(\mathsf{u}_1,\mathsf{a}) \wedge \mathsf{COM}(\mathsf{u}_2,\mathsf{a}) \wedge \mathsf{u}_1\mathsf{y} \leq \mathsf{z}_1\\
&\qquad\qquad\qquad \wedge \mathsf{u}_2\mathsf{y} \leq \mathsf{z}_2 \ \wedge \mathsf{ECP}(\mathsf{u}_1,\mathsf{u}_2)))),\\
&\lambda_3(\mathsf{x},\mathsf{y},\mathsf{a},\mathsf{a}_1,\mathsf{b},\mathsf{c}) \equiv_{def}\\
&\quad \forall \mathsf{z}(\mathsf{COM}(\mathsf{z},\mathsf{b}) \Rightarrow\\
&\qquad (\bigvee_{\mathsf{j}=1,\ldots,\mathsf{m}} (\Delta_{\alpha_j}(\mathsf{x},\mathsf{c},\mathsf{a},\mathsf{a}_1,\mathsf{z}) \wedge \Delta_{\beta_j}(\mathsf{y},\mathsf{c},\mathsf{a},\mathsf{a}_1,\mathsf{z})))).
\end{aligned}
$$

□

To illustrate how Ψ_C encodes in $\mathcal{M}$ the solvability of a *PCP*-instance C, consider the case where C is given by $\alpha_1 = 0$, $\alpha_2 = 11$, $\alpha_3 = 01$ and $\beta_1 = 01$, $\beta_2 = 10$, $\beta_3 = 1$. Obviously, $(1,2,3)$ is a solution of C, since $\alpha_1\ \alpha_2\ \alpha_3 = 01101 = \beta_1\ \beta_2\ \beta_3$. The fact that this is a solution of C can be represented by polygons satisfying certain topological relations, as shown in figure 4. The string 01101 induced by the solution is represented by polygon $a = a_1 + \cdots + a_5$ together with polygon $c = c_2 + c_3 + c_5$, shown in a), because a represents a sequence whose ith element, represented by a_i, is "1" if $c_i \subset a_i$, otherwise "0". Since the polygons x_i and y_i respectively are in a one-to-one correspondence to the a_i, as shown in b) by $x_i, y_i \subset a_i$, the string 01101 is also represented both by polygon $x = x_1 + \cdots + x_5$ and by polygon $y = y_1 + \cdots + y_5$. This is what $\mathcal{M} \models \lambda_1[a,x,y]$ means.

Figure c) includes all polygons that are involved in encoding that $(1,2,3)$ is a solution of C. This is due to the fact that each b_j represents the jth index of the solution: b_j includes exactly the parts of x and y which represent the corresponding strings α_{i_j} and β_{i_j} respectively. More formally, this can be expressed by $\mathcal{M} \models \lambda_3[x,y,a,a_1,b,c]$, where b is $b_1 + b_2 + b_3$. For instance, we have $\mathcal{M} \models \Delta_{\alpha_3}[x,c,a,a_1,b_3]$, i.e. there are distinct components v_1 and v_2 of a which represent (by $(v_1,v_2) \in ECP$ and by the formula containing the variable s) a subsequence of a of the same order. Since $v_1 + v_2$ and b_3 must contain the same components x_i, we get $v_1 = a_4$ and $v_2 = a_5$. Thus $v_1 + v_2$ represents the string α_3.

Finally, $\mathcal{M} \models \lambda_2[a,b,x,y]$ ensures that the concatenation of the corresponding strings, i.e. $\alpha_1\ \alpha_2\ \alpha_3$ and $\beta_1\ \beta_2\ \beta_3$, results in the same string, namely 01101. Hence, $b = b_1 + b_2 + b_3$ encodes the whole solution $(1,2,3)$.

The polygons a, a_1, b, b_1, x, y, c satisfy all properties Ψ_C requires to hold, thus $\mathcal{M} \models \Psi_C$. In the following lemma 1 we show in detail how to construct a particular solution of a given PCP-instance by such polygons.

Lemma 1 *Let* $C = \{(\alpha_1,\beta_1),\dots,(\alpha_m,\beta_m)\}$ *be a* PCP*-instance. Then* C *has a solution if* $\mathcal{M} \models \Psi_C$.

Proof: Suppose $\mathcal{M} \models \Psi_C$. By propositions 7 and 8 it follows that some polygonal regions $x, y, a, a_1, b, b_1, c \in M$ exist satisfying $(a_1, a), (b_1, b) \in SEQ$ and $x, y \in SDC$. Thus, $a_1 \in P$, $a = a_1 + \dots + a_{n_a}$, $n_a \geq 1$ for some $a_2, \dots, a_{n_a} \in P$, and $b_1 \in P$, $b = b_1 + \dots + b_k$, $k \geq 1$ for some $b_2, \dots, b_k \in P$. a_i, a_j with $i, j \in \{1, \dots, n_a\}$ and b_i, b_j with $i, j \in \{1, \dots, k\}$ respectively satisfy the condition

$$(a_i, a_j),\ (b_i, b_j) \in \left\{ \begin{array}{lll} ECP & \text{if} & |i-j| = 1 \\ DC & \text{if} & |i-j| > 1. \end{array} \right. \quad (1)$$

In particular, $a_i a_j = \emptyset$ and $b_i b_j = \emptyset$ if $i \neq j$. From proposition 6 we obtain that the a_i are the only components of a, that is $(z, a) \in COM$ holds exactly for every $z \in \{a_1, \dots, a_{n_a}\}$. Equally, $b_1, \dots, b_k$ are exactly the components of b.

Furthermore, $x = x_1 + \dots + x_{n_x}$, $n_x \geq 1$ for some $x_1, \dots, x_{n_x} \in P$ and $y = y_1 + \dots + y_{n_y}$, $n_y \geq 1$ for some $y_1, \dots, y_{n_y} \in P$. x_i, x_j with $i, j \in \{1, \dots, n_x\}$ and y_i, y_j with $i, j \in \{1, \dots, n_y\}$ respectively satisfy the condition

$$(x_i, x_j),\ (y_i, y_j) \in DC \text{ if } i \neq j \quad (2)$$

with the consequence that $x_i x_j = \emptyset$ and $y_i y_j = \emptyset$ if $i \neq j$. Again by proposition 6, x_i and y_i are the only components of x and y, respectively. In the following steps we construct a solution of C.

1) We show the functions F_x and F_y defined as

$$\begin{array}{l} F_x : \{a_1, \dots a_{n_a}\} \longrightarrow \{x_1, \dots, x_{n_x}\},\ F_x(a_i) := a_i x, \\ F_y : \{a_1, \dots a_{n_a}\} \longrightarrow \{y_1, \dots, y_{n_y}\},\ \ F_y(a_i) := a_i y \end{array}$$

to be one-to-one mappings. For the remainder of the proof we then define $n := n_x = n_a = n_y$.

This follows from $\mathcal{M} \models \lambda_1[a, x, y]$. For any a_j there exists exactly one x_i such that $a_j x = x_i$. Thus, F_x is indeed an overall defined function which maps different a_j to different x_i. From $x \subseteq a$ and proposition 5 it further follows for any x_i that $x_i \subseteq a_j$ holds for some a_j. This implies $x_i = a_j x = F_x(a_j)$. So F_x, and analogously F_y, turn out to be one-to-one functions.

2) We define the string $\pi \in \{0,1\}^+$ with length $|\pi| = n$ by

$$\pi[i] := \left\{ \begin{array}{lll} 1 & \text{if} & a_i c \neq \emptyset \\ 0 & \text{if} & a_i c = \emptyset. \end{array} \right.$$

π is the string that is induced by the special solution of C we construct in the following.

3) Let $G_x, H_x, G_y, H_y : \{1, \dots, k\} \longrightarrow \{1, \dots, n\}$ be defined as:

$$\begin{array}{l} G_x(i) := \min\{j \mid F_x(a_j) \subseteq b_i\}, \\ H_x(i) := \max\{j \mid F_x(a_j) \subseteq b_i\}, \\ G_y(i) := \min\{j \mid F_y(a_j) \subseteq b_i\}, \\ H_y(i) := \max\{j \mid F_y(a_j) \subseteq b_i\}. \end{array}$$

We show that for all $i \in \{1, \dots, k\}$, $j \in \{1, \dots, n\}$:

a) $G_x(i) \leq j \leq H_x(i)$ iff $F_x(a_j) \subseteq b_i$.

b) $G_y(i) \leq j \leq H_y(i)$ iff $F_y(a_j) \subseteq b_i$.

c) $\pi[G_x(i), H_x(i)] = \alpha_{l(i)}$, $\pi[G_y(i), H_y(i)] = \beta_{l(i)}$ for some $l(i) \in \{1, \dots, m\}$.

This can be proven from $\mathcal{M} \models \lambda_3[x, y, a, a_1, b, c]$. For any b_i there exists some $j \in \{1, \dots, m\}$ satisfying

$$\begin{array}{l} \mathcal{M} \models \Delta_{\alpha_j}[x, c, a, a_1, b_i], \\ \mathcal{M} \models \Delta_{\beta_j}[y, c, a, a_1, b_i]. \end{array} \quad (3)$$

In the following we consider only (3). The $v_1, \dots, v_{|\alpha_j|} \in M$ are pairwise distinct elements from $\{a_1, \dots, a_n\}$. Suppose $v_{|\alpha_j|} = a_r$. Then we define $s := a_1 + \dots + a_r$. From (3) we can conclude that $v_1 \subseteq s$.

In case of $|\alpha_j| > 1$ we get by (3) $(v_{|\alpha_j|-1}, v_{|\alpha_j|}) \in ECP$, thus by (1) either $v_{|\alpha_j|-1} = a_{r+1}$ or $v_{|\alpha_j|-1} = a_{r-1}$. Suppose $v_{|\alpha_j|-1} = a_{r+1}$. From $(v_{|\alpha_j|-2}, v_{|\alpha_j|-1}) \in ECP$ and $v_{|\alpha_j|-2} \neq v_{|\alpha_j|}$ it then follows $v_{|\alpha_j|-2} = a_{r+2}$. Concluding in the same way leads to $v_1 = a_{r+|\alpha_j|-1}$, which contradicts $v_1 \subseteq s$. Therefore we obtain

$$v_{|\alpha_j|} = a_r,\ v_{|\alpha_j|-1} = a_{r-1}, \dots,\ v_1 = a_{r-|\alpha_j|+1}.$$

Let $q := r - |\alpha_j| + 1$ such that $1 \leq q \leq r \leq n$. Then we get by (3)

$$b_i x = (a_q + \dots + a_r)x = F_x(a_q) + \dots + F_x(a_r). \quad (4)$$

Thus, $F_x(a_e) \subseteq b_i$ for any $q \in \{q, \dots, r\}$. Conversely, suppose $F_x(a_e) \subseteq b_i$ for some $e \in \{1, \dots, n\}$. Since $F_x(a_e) \subseteq x$ it follows from (4) that $F_x(a_e) = F_x(a_p)$ for some p with $q \leq p \leq r$. Because F_x is one-to-one we get $e = p$. This means in fact that $G_x(i) = q$, $H_x(i) = r$ and proposition a) holds.

Moreover, it follows from (3) that

$$a_{q+e-1} c = v_e c \left\{ \begin{array}{lll} \neq \emptyset & \text{if} & \alpha_j[e] = 1 \\ = \emptyset & \text{if} & \alpha_j[e] = 0 \end{array} \right.$$

and thus, by definition of π, $\pi[q+e-1] = \alpha_j[e]$ for any $e \in \{1, \dots, |\alpha_j|\}$. This means

$$\pi[q, r] = \pi[q] \cdots \pi[r] = \alpha_j[1] \cdots \alpha_j[|\alpha_j|] = \alpha_j.$$

Similarly we can show by $\mathcal{M} \models \Delta_{\beta_j}[y, c, a, a_1, b_i]$ that proposition b) and $\pi[G_y(i), H_y(i)] = \beta_j$ must hold. Hence, proposition c) holds with $l(i) := j$.

4) We show that

d) $G_x(1) = 1,\ H_x(k) = n,$
$G_x(i) = H_x(i-1) + 1$ for $i \in \{2, \ldots, k\}$

e) $G_y(1) = 1,\ H_y(k) = n,$
$G_y(i) = H_y(i-1) + 1$ for $i \in \{2, \ldots, k\}$.

To show d) we first note that

$$\{1, \ldots, n\} \;=\; \dot{\bigcup_{i=1,\ldots,k}} \{G_x(i), \ldots, H_x(i)\}. \qquad (5)$$

This holds since by a) each two sets of the right-hand side do not intersect. Conversely, we have for any $i \in \{1, \ldots, n\}$ $F_x(a_i) \subseteq x \subseteq b$ and thus, by proposition 5, $F_x(a_i) \subseteq b_j$ for some $j \in \{1, \ldots, k\}$. With a) this means $i \in \{G_x(j), \ldots, H_x(j)\}$.

Since $a_1 \subseteq b_1$ we get by a) $G_x(1) \le 1 \le H_x(1)$ and thus $G_x(1) = 1$. Assuming $k > 1$ we show $G_x(i) = H_x(i-1) + 1$ for $i \in \{2, \ldots, k\}$ by induction using $\mathcal{M} \models \lambda_2[a, b, x, y]$.

Suppose $i = 2$. From $\mathcal{M} \models \lambda_2[a, b, x, y]$ and (1) it follows that $a_s x \subseteq b_1$ and $a_t x \subseteq b_2$ for some components a_s, a_t of a satisfying $(a_s, a_t) \in \mathit{ECP}$, i.e. $|s - t| = 1$. Then, by a),

$$1 = G_x(1) \le s \le H_x(1) \text{ and } G_x(2) \le t \le H_x(2)$$

from which $s = H_x(1)$ and $t = s + 1 = G_x(2)$ follows by (5), that is $G_x(2) = H_x(1) + 1$.

Suppose $i \in \{2, \ldots, k-1\}$ and $G_x(j) = H_x(j-1) + 1$ to hold for all j with $2 \le j \le i$. Again, we have $a_s x \subseteq b_i$ and $a_t x \subseteq b_{i+1}$ for some components a_s, a_t of a satisfying $(a_s, a_t) \in \mathit{ECP}$, i.e. $|s - t| = 1$, and

$$G_x(i) \le s \le H_x(i) \text{ and } G_x(i+1) \le t \le H_x(i+1).$$

Thus, by (5), either $s = H_x(i)$, $t = s+1 = G_x(i+1)$ or $s = G_x(i)$, $t = s-1 = H_x(i+1)$. Since from the latter we can conclude $H_x(i+1) = G_x(i) - 1 = H_x(i-1)$, which contradicts (5), $G_x(i+1) = H_x(i) + 1$ must necessarily hold.

Finally, we get $H_x(k) = n$ from (5) and

$$\bigcup_{i=1,\ldots,k-1} \{G_x(i), \ldots, H_x(i)\} = \{1, \ldots, H_x(k-1)\}.$$

The proof of e) is analogous.

5) Using propositions c), d) and e) we can now easily show that $(l(1), \ldots, l(k))$ is a solution of C:

$$\begin{aligned} &\alpha_{l(1)}\ \alpha_{l(2)} \cdots \alpha_{l(k)} \;= \\ &\pi[G_x(1), H_x(1)]\ \pi[G_x(2), H_x(2)] \cdots \pi[G_x(k), H_x(k)] \\ &= \;\pi\; = \\ &\pi[G_y(1), H_y(1)]\ \pi[G_y(2), H_y(2)] \cdots \pi[G_y(k), H_y(k)] \\ &= \;\beta_{l(1)}\ \beta_{l(2)} \cdots \beta_{l(k)}. \end{aligned}$$

□

Conversely, lemma 2 shows that Ψ_C is already a consequence of $T_{\mathcal{M}}$ provided that C has a solution.

Lemma 2 *Let* $C = \{(\alpha_1, \beta_1), \ldots, (\alpha_m, \beta_m)\}$ *be a solvable* PCP*-instance. Then,* $T_{\mathcal{M}} \models \Psi_C$ *holds.*

Proof: Let $(i_1, \ldots, i_k)$ be a solution of C. Let $\mathcal{A} = \langle A, +^A, \cdot^A, -^A, 0^A, 1^A, P^A \rangle$ be an arbitrary model of $T_{\mathcal{M}}$. Then we have to show $\mathcal{A} \models \Psi_C$. For this, let $\le^A$, POL^A, ECP^A, DC^A, COM^A, SDC^A and SEQ^A be the relations defined on A by $\le$, POL, ECP, DC, COM, SDC and SEQ, respectively, such that $x \le^A y$ iff $\mathcal{A} \models \le[x, y]$ for any $x, y \in A$, etc.

We define $\pi := \alpha_{i_1} \cdots \alpha_{i_k} = \beta_{i_1} \cdots \beta_{i_k} \in \{0,1\}^+$ and $n := |\pi|$. From $\xi_{2n} \in T_{\mathcal{M}}$ it follows that there exist elements $d_{i,j} \in A$, $i, j \in \{1, \ldots, 2n\}$, satisfying the following properties:

$$\begin{aligned} &d_{i,j} \cdot^A d_{k,l} = 0^A \text{ if } (i,j) \ne (k,l), \\ &d_{i,j} \in P^A, \\ &(d_{i,j}, d_{k,l}) \in \mathit{DC}^A \text{ if } |i-k| \ge 2 \text{ or } |j-l| \ge 2, \\ &\Phi_{i,j,k,l} \in P^A, \\ &(\Phi_{i,j,k,l}, \Phi_{k+1,l+1,k',l'}) \in \mathit{ECP}^A, \\ &(\Phi_{i,j,k,l}, \Phi_{i',j',k',l'}) \in \mathit{DC}^A, \text{ if } k+1 < i',\ l+1 < j' \end{aligned}$$

where $\Phi_{i,j,k,l}$ is defined, if $i < k$ and $j < l$, as

$$\begin{aligned} \Phi_{i,j,k,l} \;:=\; d_{i,j} &+^A \textstyle\sum^A_{r=1,\ldots,j} (d_{i+1,r} +^A \cdots +^A d_{k,r}) \\ &+^A \textstyle\sum^A_{r=j+1,\ldots,l} (d_{1,r} +^A \cdots +^A d_{k,r}). \end{aligned}$$

Next we define elements $a, a_1, c, x, y, b, b_1 \in A$ that validate Ψ_C in $\mathcal{A}$:

1. $a := a_1 +^A \cdots +^A a_n$ with $a_i := \Phi_{2i-1,2i-1,2i,2i}$
2. $c := c_1 +^A \cdots +^A c_n$ with
$$c_i := \begin{cases} d_{2i,2i} & \text{if} \quad \pi[i] = 1 \\ 0^A & \text{if} \quad \pi[i] = 0 \end{cases}$$
3. $x := x_1 +^A \cdots +^A x_n$ with $x_i := d_{2i,1}$
4. $y := y_1 +^A \cdots +^A y_n$ with $y_i := d_{1,2i}$

5. $b := b_1 +^A \cdots +^A b_k$ with $b_j := \Phi_{e,f,g,h}$ where

$$\begin{aligned} e &= 2|\alpha_{i_1} \cdots \alpha_{i_{j-1}}| + 1, \\ f &= 2|\beta_{i_1} \cdots \beta_{i_{j-1}}| + 1, \\ g &= 2|\alpha_{i_1} \cdots \alpha_{i_j}|, \\ h &= 2|\beta_{i_1} \cdots \beta_{i_j}|. \end{aligned}$$

1) We show that $(a_1, a) \in SEQ^A$. By definition of the a_i we get

$$(a_i, a_j) \in \begin{cases} ECP^A & \text{if} \quad |i-j| = 1 \\ DC^A & \text{if} \quad |i-j| > 1 \end{cases} \tag{1}$$

and thus by $\mathcal{A} \models \tau_n$ (see proposition 6) that $a_1, \ldots, a_n$ are the only components of a, i.e. $(z, a) \in COM^A$ is satisfied exactly by each $z \in \{a_1, \ldots, a_n\}$. Then, $\mathcal{A} \models \nu_2[a]$ immediately follows. Clearly, $a \in POL^A$, since by $0^A \notin P^A$ $a \neq 0^A$ and $a \leq^A \Phi_{1,1,2n,2n} \in P^A$. By (1) and $\mathcal{A} \models \sigma_n$ (see proposition 5) we have $\mathcal{A} \models \nu_1[a]$. Again by (1) both $\mathcal{A} \models \nu_4[a_1, a]$ and $\mathcal{A} \models \nu_5[a]$ hold.

To prove $\mathcal{A} \models \nu_3[a]$ suppose $v_1, v_2 \in A \backslash \{0^A\}$ satisfying $v_1 \cdot^A v_2 = 0^A$, $v_1 +^A v_2 = a$ and for any a_i either $a_i \leq^A v_1$ or $a_i \leq^A v_2$. Assuming $a_1 \leq^A v_1$ we define

$$q := \max\{i \in \mathbb{N} \mid a_1 \leq^A v_1, \ldots, a_i \leq^A v_1\}.$$

In case of $n = 1$ $\mathcal{A} \models \nu_3[a]$ holds trivially. Otherwise, from the conditions given above it follows $1 \leq q < n$ and thus, $a_{q+1} \leq^A v_2$. Hence by $(a_q, a_{q+1}) \in ECP^A$ we have indeed $\mathcal{A} \models \nu_3[a]$.

2) By similar arguments we can show $(b_1, b) \in SEQ^A$ and $x, y \in SDC^A$. Additionally, we have

$$(b_i, b_j) \in \begin{cases} ECP^A & \text{if} \quad |i-j| = 1 \\ DC^A & \text{if} \quad |i-j| > 1, \end{cases} \tag{2}$$

and

$$(x_i, x_j), (y_i, y_j) \in DC^A \text{ if } i \neq j. \tag{3}$$

The $b_1, \ldots, b_k$ are the only components of b, $x_1, \ldots, x_n$ the only components of x and $y_1, \ldots, y_n$ the only components of y.

3) Clearly, $x \leq^A a$, $y \leq^A a$, since $x_i, y_i \leq^A a_i$, and $a_1 \leq^A b_1$. To see that $x \leq^A b$, and analogously $y \leq^A b$, let $j \in \{1, \ldots, n\}$ and $p := \min\{q \mid |\alpha_{i_1} \cdots \alpha_{i_q}| \geq j\}$. Then, by definition of $b_p = \Phi_{e,f,g,h}$, we obtain $d_{e+1,1} +^A \cdots +^A d_{g,1} \leq^A b_p$ and $e + 1 \leq 2j \leq g$, so that $x_j = d_{2j,1} \leq^A b_p$.

4) We have $\mathcal{A} \models \lambda_1[a, x, y]$ since $x_j \cdot^A a_i = 0^A$ and $y_j \cdot^A a_i = 0^A$ respectively if $j \neq i$, hence

$$a_i \cdot^A x = a_i \cdot^A x_i = x_i \text{ and } a_i \cdot^A y = a_i \cdot^A y_i = y_i. \tag{4}$$

5) Next we show $\mathcal{A} \models \lambda_2[a, b, x, y]$. This holds trivially if $k = 1$, since $(z_1, z_1) \in ECP^A$ implies $z_1 \cdot^A z_1 \neq 0^A$. Otherwise, we consider components b_j, b_{j+1} of b such that $(b_j, b_{j+1}) \in ECP^A$. Let $i := |\alpha_{i_1} \cdots \alpha_{i_j}|$. Then, $1 \leq i < n$ and, by (1), $(a_i, a_{i+1}) \in ECP^A$. Using (4) we obtain immediately from the definition of b_j and b_{j+1} that $a_i \cdot^A x = x_i \leq^A b_j$ and $a_{i+1} \cdot^A x = x_{i+1} \leq^A b_{j+1}$. Equally, $a_l \cdot^A y \leq^A b_j$ and $a_{l+1} \cdot^A y \leq^A b_{j+1}$ hold with $l := |\beta_{i_1} \cdots \beta_{i_j}|$, which verifies $\mathcal{A} \models \lambda_2[a, b, x, y]$.

6) Finally, we prove $\mathcal{A} \models \lambda_3[x, y, a, a_1, b, c]$. Let b_j be a component of b. We only show that $\mathcal{A} \models \Delta_{\alpha_{i_j}}[x, c, a, a_1, b_j]$ since the proof of $\mathcal{A} \models \Delta_{\beta_{i_j}}[y, c, a, a_1, b_j]$ is quite similar. For this, we define $q := |\alpha_{i_1} \cdots \alpha_{i_{j-1}}| + 1$ and $r := |\alpha_{i_1} \cdots \alpha_{i_j}|$, such that $a_q, \ldots, a_r$ are pairwise distinct components of a satisfying $(a_q, a_{q+1}), \ldots, (a_{r-1}, a_r) \in ECP^A$.

Suppose that $(a_1, s) \in SEQ^A$, $s \leq^A a$, $a_r \leq^A s$ for some $s \in A$ such that $a_i \leq^A s$ or $a_i \cdot^A s = 0^A$ for each a_i. Moreover, let $a_{l_1}, \ldots, a_{l_t}$ denote all components of a satisfying $a_i \leq^A s$. Then we get by $s = s \cdot^A a$

$$\begin{aligned} s &= s \cdot^A (a_1 +^A \cdots +^A a_n) \\ &= s \cdot^A (a_{l_1} +^A \cdots +^A a_{l_t}) \\ &= a_{l_1} +^A \cdots +^A a_{l_t}. \end{aligned}$$

By (1) and $\mathcal{A} \models \tau_t$ $a_{l_1}, \ldots, a_{l_t}$ are exactly the components of s. $(a_1, s) \in SEQ^A$ implies especially $a_1 \in \{a_{l_1}, \ldots, a_{l_t}\}$ and $\mathcal{A} \models \nu_3[s]$ (see proposition 8).

Next we show $\{a_1, \ldots, a_t\} = \{a_{l_1}, \ldots, a_{l_t}\}$, that is $s = a_1 +^A \cdots +^A a_t$. From this we get by $a_r \leq^A s$ immediately $q \leq r \leq t$, thus $a_q \leq^A s$. Let $p := \max\{i \mid a_1, \ldots, a_i \in \{a_{l_1}, \ldots, a_{l_t}\}\ \}$. If $p = t$ we are finished. Otherwise, if $1 \leq p < t$, we define

$$v_1 := a_1 +^A \cdots +^A a_p, \ v_2 := \sum_{i \in \{l_1, \ldots, l_t\} \backslash \{1, \ldots, p\}} a_i$$

such that $v_1, v_2 \neq 0^A$, $v_1 \cdot^A v_2 = 0^A$ and $v_1 +^A v_2 = a_{l_1} +^A \cdots +^A a_{l_t}$. Additionally, for any component a_{l_i} of s either $a_{l_i} \leq^A v_1$ or $a_{l_i} \leq^A v_2$ holds. From $\mathcal{A} \models \nu_3[s]$ it follows that $(z_1, z_2) \in ECP^A$ for some $z_1 \in \{a_1, \ldots, a_p\}$ and $z_2 \in \{a_{l_1}, \ldots, a_{l_t}\} \backslash \{a_1, \ldots, a_p\}$. Hence by (1), $z_1 = a_p$ and $z_2 = a_{p+1}$. But this is a contradiction to the definition of p, therefore $p = t$.

Because of the way b_j is defined $x_q, \ldots, x_r$ are the only x_i with $x_i \cdot^A b_j \neq 0^A$, hence by (4)

$$b_j \cdot^A x = x_q +^A \cdots +^A x_r = (a_q +^A \cdots +^A a_r) \cdot^A x.$$

The last property that remains to be shown is

$$a_{q+h-1} \cdot^A c \begin{cases} \neq 0^A & \text{if} \quad \alpha_{i_j}[h] = 1 \\ = 0^A & \text{if} \quad \alpha_{i_j}[h] = 0 \end{cases}$$

for any $h \in \{1, \ldots, |\alpha_{i_j}|\}$. This follows directly from the definition of $c_{q+h-1} = a_{q+h-1} \cdot^A c$ and the fact that

$$\pi[q+h-1] = \pi[|\alpha_{i_1} \cdots \alpha_{i_{j-1}}| + h] = \alpha_{i_j}[h].$$

$\square$

Using lemma 1 and 2 we can easily establish our undecidability result.

Theorem 4 *Any $T \subsetneq L_0^{S_\mathcal{M}}$ with $T_\mathcal{M} \subseteq T \subseteq Th(\mathcal{M})$ is undecidable. In particular, $Th(\mathcal{M})$ is undecidable.*

Proof: Let $T \subsetneq L_0^{S_\mathcal{M}}$ with $T_\mathcal{M} \subseteq T \subseteq Th(\mathcal{M})$ and C denote an arbitrary *PCP*-instance. We associate effectively to C the $S_\mathcal{M}$-sentence Ψ_C given in definition 7. Suppose $T \models \Psi_C$. Since $T \subseteq Th(\mathcal{M})$ is equivalent to $\mathcal{M} \models T$ we immediately get $\mathcal{M} \models \Psi_C$. Then, by lemma 1, C has a solution. Conversely, if C has a solution lemma 2 states that $T_\mathcal{M} \models \Psi_C$, from which $T \models \Psi_C$ follows by $T_\mathcal{M} \subseteq T$. Thus, we have

$$T \models \Psi_C \text{ iff } C \text{ has a solution,}$$

which implies the undecidability of T. $\square$

By theorem 3 we immediately get

Corollar *$Th(\mathcal{M})$ is neither elementarily axiomatizable nor recursively enumerable.* $\square$

4 Conclusions

We have presented a model $\mathcal{M}$ of polygonal regions of the Euclidean plane. $\mathcal{M}$ turned out to be mereotopological, i.e. the topological relations usually provided by mereotopologies proved to be elementarily definable in $\mathcal{M}$. Thus, $Th(\mathcal{M})$ includes all elementary propositions on polygonal regions and their topological relations. Furthermore, we established the undecidability of $Th(\mathcal{M})$ by reduction from Post's correspondence problem. For this, we identified a small axiomatic fragment $T_\mathcal{M}$ of $Th(\mathcal{M})$, such that any T with $T_\mathcal{M} \subseteq T \subseteq Th(\mathcal{M})$ is undecidable. As a consequence, $Th(\mathcal{M})$ cannot be axiomatized in the first-order logic.

Let us finally discuss to what extent this result can be applied to differently defined mereotopological models. Obviously, the proof of theorem 4 works as well for three-dimensional regions which are defined similarly to the planar regions of $\mathcal{M}$: replace the squares shown in figure 4 with three-dimensional cubes arranged on the same plane. Probably, the closest mereotopological model to $\mathcal{M}$ is proposed by Pratt, Lemon and Schoop [14, 15]. Its complete axiomatization is grounded on an infinitary rule of inference allowing proofs of infinite length and is therefore without any consequence for its decision problem (in contrast to the elementary case). Although defined in a different way, its domain also consists of polygonal regions. It differs from $\mathcal{M}$ in the choice of taking "x is connected" as the primitive topological relation. Since in both models polygons are always finite sums of simple polygons we have reason to conjecture that our proof method applies as well.

To encode Post's correspondence problem in the way figure 4 shows it is crucial that finite sums of regions are definable. So, what about mereotopological models including sums of infinitely many regions ? In this case, finite sums can no longer be characterized by their boundedness. A method of defining finite sums can be obtained from Grzegorczyk's undecidability proof of closure algebra, i.e. Boolean algebra extended by some axioms for the predicate "x is closed" [10]. The main idea is to interpret the arithmetic of natural numbers, well-known as undecidable, by elementarily definable sets of regions and relations between regions. In particular, the natural number n is represented by the class of all regions of the underlying space that consist of exactly n components (then, equality between numbers corresponds to equivalence between regions with respect to the number of their components). To represent the set of all natural numbers in closure algebra Grzegorczyk defines a predicate which is satisfied only by regions composed of finitely many components. Consider the simple case of two disjoint regions x and y each of which consists entirely of possibly infinitely many components. Then, we can establish a one-to-one correspondence between their components iff there exists a region such that each of its components contains exactly one component of x and one component of y. x and y represent a natural number only if they consist of finitely many components. In fact, this is the case iff the set of components of x, minus one of its components, cannot be in a one-to-one correspondence to the set of components of y. Thus, finite sums of regions prove to be elementarily definable.

This way of defining finite sums of regions seems to apply to most mereotopological models. We therefore conjecture that Grzegorczyk's method or, alternatively, the method we use in theorem 4 for proving undecidability works as well for these models. Thus, it could be argued that the problem of first-order axiomatizations of spatial regions in terms of topological relations yields in general a negative solution. Moreover, Grzegorczyk's concept of encoding the arithmetic of natural numbers can also be used to show the undecidability of $Th(\mathcal{M})$ when modified in several ways. In comparison to this, our method has the advantage of being based upon few definability results for intuitive topological relations and few properties of polygonal regions.

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