Modallogiken – Theorie und Anwendungen

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Chapter 2

Unimodal modal logic
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2.8 Decidability and Complexity

As has been pointed out several times, an important point why modal logics have
got such popular is the fact that many modal logics are decidable. In the light of
section 2.4, many modal logics deal with decidable fragments of first order logic.
In this section, we shall present some techniques for proving decidability results in
modal logic.

For repetition: Let \( C \) be a class of Kripke frames. A formula \( \varphi \) is said to be
satisfiable in a frame \( \mathcal{F} \) if there exists a Kripke model \( \mathcal{M} \) defined on \( \mathcal{F} \) and a
world \( s \) of \( \mathcal{F} \) such that \( \mathcal{M} \models \varphi \). \( \varphi \) is said to be satisfiable in \( C \) if it is satisfiable in a frame of
\( C \). \( \varphi \) is said to be valid in \( C \) if \( \mathcal{F} \models \varphi \). Decidability results, then, provide effective
solutions to the following problems

- \( \text{SAT}(C) \): Is a given formula \( \varphi \) satisfiable in \( C \)?
- \( \text{VAL}(C) \): Is a given formula \( \varphi \) valid in \( C \)?

Since a formula \( \varphi \) is valid in \( C \) if and only if \( \neg \varphi \) is not satisfiable in \( C \), we can
reduce (as is done in propositional logic) each of these problems to the other one.
For if we have found a procedure or an algorithm that decides \( \text{SAT}(C) \) for any
formula \( \varphi \) the same procedure can be used for deciding \( \text{VAL}(C) \) for \( \neg \varphi \), and vice
versa.

In section 2.6 we did already present a procedure for deciding validity in S5, i.e.,
we proved that \( \text{VAL}(C)(S5) \) is decidable. But the method used in the proof of
that result was tailored to the very special situation in S5, namely that in S5 each
formula is modally equivalent to a formula that has modal degree \( \leq 1 \). But for
testing whether a formula of the form \( \Box \varphi \) with \( \text{mdeg}\Box \varphi = 1 \) is satisfiable it is
sufficient to consider models that are defined on frames with exactly two worlds
only (note that there are up to isomorphism only two such frames).\(^4\) In what is to
follow we will generalize this method to deal with such classes of frames that do
not allow for reducing modalities as does S5.

2.8.1 Finite Model Property

Let us assume that we want to prove that a given formula \( \varphi \) is satisfiable in a given
class \( C \) of Kripke frames. Then we have to look at all the various models that are

\(^4\)This does not mean that for testing validity of an arbitrary formula with modal degree \( \leq 1 \) it is
sufficient to consider models with at most two worlds. For example, for testing a formula of the form
\( (\varphi \land \psi) \land \Box (\varphi \land \neg \psi) \land \Box (\neg \varphi \land \psi) \), where \( \varphi \) and \( \psi \) are PL-formulae, we have to consider frames with
three worlds. But if we transform a formula with modal degree \( \leq 1 \) into a formula in conjunctive
normal form ...
defined on some of maybe very huge frames in that class. In section 2.4 we proved that some classes of frames allow for considering ‘small’ models only: If \( C \) is first order definable and if \( \phi \) is satisfiable in \( C \), then \( \phi \) is satisfiable in a model based on a frame in \( C \) that has only countably many worlds. But in case such a model is defined on an infinite set of worlds, it is not decidable by a finite algorithm whether a given formula is satisfiable in that model. Hence the following property of classes of frames is essential for proving decidability results.

**Definition 2.8.1** A class of Kripke frames, \( C \), is said to have the finite model property (FMP) if each formula that is satisfiable in \( C \) is satisfiable in a finite frame of \( C \). A normal modal logic \( \Lambda \) is said to have the finite model property if it is characterized by some class of frames that has the finite model property.

It is clear that we could define the notion of finite model property with respect to classes of Kripke model, too. Then if \( C \) has the finite model property, so does \( \mathcal{K}(C) \).

**Lemma 2.8.2** Let \( \Lambda \) be a class of Kripke frames. Then the following statements are equivalent:

(i) \( \Lambda \) has the finite model property.

(ii) \( \Lambda \) is characterized by some class of finite frames.

The finite model property does not guarantee decidability. But we will now prove that if a class of frames, \( C \), has the finite model property and if it is finitely axiomatizable, then \( \text{VAL}(C) \) is decidable.

**Definition 2.8.3** A class of frames, \( C \), is said to be finitely axiomatizable, if there exists a finite set of \( \mathcal{L}(P) \)-formulae, \( \Sigma \), such that \( \Lambda(C) = \Lambda[\Sigma] \).

For example, the class of \( \mathcal{K} \), \( \mathcal{K}[D] \), \ldots, \( \mathcal{S}4 \), or \( \mathcal{S}5 \)-frames is finitely axiomatizable.

**Lemma 2.8.4** Let \( \mathcal{M} = \langle S, R, V \rangle \) be a finite Kripke model. Then for each \( s \in S \) and each \( \mathcal{L}(P) \)-formula \( \phi \), it is decidable in time \( O(|\phi| \cdot \|\mathcal{M}\|) \) whether \( \mathcal{M} \models \phi \).

Here and in what is to follow \( \|\mathcal{M}\| \) denotes \( |S| + |R| \).
Proof. Let \( \varphi_1, \ldots, \varphi_n \) be an enumeration of all subformulae of \( \varphi \), in the order of degree, i.e., \( \varphi = \varphi_n \). For each \( 1 \leq i \leq n \), label each world \( s' \in S \) by \( ' \varphi_i ' \) or \( ' \neg \varphi_i ' \) depending on whether \( M \models s \varphi_i \) or \( M \models s \neg \varphi_i \). Finally return ‘\( \varphi \) is satisfiable’, if \( s \) is labeled with “\( \varphi \)”. Otherwise, return ‘\( \varphi \) is not satisfiable’.

Obviously, this algorithm terminates after \( n = |\text{Sub}(\varphi)| \) rounds. In each round \( |S| \) are to be checked, If in the \( i \)-th round \( \varphi_i \) has the form \( \neg \psi \) or \( (\psi \rightarrow \psi') \), each world has already been labeled in an earlier round by ‘\( \psi ' \) or by ‘\( \psi ' \)’ (and by ‘\( \neg \psi ' \)’. So it can immediately be seen, whether \( M \models s \varphi_i \) or not. If in the \( i \)-th round \( \varphi_i \) has the form \( \Box \psi \), the algorithm has to check for each pair of worlds \( (s', s'') \) where \( s' \) can see \( s'' \) via \( R \), whether \( s'' \) is labeled by ‘\( \psi ' \)’, or not. Thus, each round needs at most \( O(\max(|S|, |R|)) = O(|M|) \), and the algorithm terminates in time \( O(|\varphi| \cdot |M|) \). 

\[ \square \]

Theorem 2.8.5 Let \( C \) be a class of Kripke models that has the finite model property and is finitely axiomatizable. then \( \text{VAL}(C) \) and \( \text{SAT}(C) \) are decidable.

Proof. We show that both (a) \( \Lambda(C) \) and (b) \( \mathcal{L}(P) \setminus \Lambda(C) \) are recursively enumerable. From this it follows that ‘\( \varphi \in \Lambda(C) \)’ and ‘\( \varphi \notin \Lambda(C) \)’ are semi-decidable, and hence that ‘\( \varphi \in \Lambda(C) \)’ is decidable.

(a) By assumption, \( \Lambda(C) = \Lambda[\Sigma] \) for some finite set of \( \mathcal{L}(P) \)-formulae, \( \Sigma \). But \( \Lambda[\Sigma] \) is recursively enumerable, for \( \Lambda[\Sigma] \) consists of exactly those formulae that are provable from \( \Sigma \). A proof from \( \Sigma \) is a finite sequence of formulae \( \varphi_1, \ldots, \varphi_n \) where each \( \varphi_i \) is (K) or a formula from \( \Sigma \) or the result of applying one of the rules (R-MP), (R-US), or (R-\( \Box \)) on preceeding formulae of that sequence. Obviously, the set of all proofs from \( \Sigma \) is recursively enumerable.

(b) We have to define a recursive enumeration of all formulae \( \varphi \) with \( C \not\models \varphi \). Since \( C \) has the finite model property we may assume that \( C \) is a class of finite frames (otherwise, consider the class of all finite frames of \( C \)). Note that

(a) \( C \) may contain infinitely non-isomorphic frames (for that reason, the procedure presented now, is not a decision procedure).

(b) Each finite frame \( \mathcal{F} = (S, R) \) is isomorphic to a Kripke frame on the set \( \{1, \ldots, n\} \) where \( n = |S| \).

(c) For each set \( \{1, \ldots, n\} \) there exist \( 2n^2 \) distinct Kripke frames defined on it.

(d) For each finite frame \( \mathcal{F} = (S, R) \) and each finite set \( P' \) of \( P \) there exist at most \( 2^{|S|^m} \) distinct Kripke models, where \( m = |P'| \).
(e) By lemma 2.8.4, for each finite model $\mathcal{M}$ and each formula $\varphi$, it is decidable whether $\mathcal{M} \models \varphi$ for a given formula $\varphi$.

Let now $p_1, p_2, \ldots$ be an enumeration of all propositional variables of $P$. Let $L_0 := \emptyset$ and let $\Phi_0 := \emptyset$. Let $\text{Var}(\varphi)$ denote the set of propositional variables that occur in $\varphi$. Assume now that $n > 0$ and that $L_k$ and $\Phi_k$ are defined for all $k < n$. Construct all models on the set $\{1, \ldots, n\}$. Check for each such model $\mathcal{M}$ whether $\mathcal{M} \models \Sigma$ (note that $\Sigma$ is finite). If so, then add $\mathcal{M}$ to the set $L_n$ and check for each $\mathcal{L}(P)$-formula $\varphi$ with $\deg \varphi \leq n$ and $\text{Var}(\varphi) \subseteq p_1, \ldots, p_n$ whether $\mathcal{M} \not\models \varphi$. If so, add $\varphi$ to the set $\Phi_n$. If $\mathcal{M} \models \varphi$, check whether $\mathcal{M}' \not\models \varphi$ for some model $\mathcal{M}'$ in $L_k$ ($k < n$). If this is the case, then add $\varphi$ to the set $\Phi_n$ as well.

Thus, we obtain a recursively defined set $\Phi$ of all formulae of $\mathcal{L}(P)$ that are not valid in $\mathcal{C}$. For if $\varphi$ is not valid, there exists a finite Kripke model $\mathcal{M}$ defined on some frame in $\mathcal{C}$ with $\mathcal{M} \not\models \varphi$. This model is isomorphic to some model $\mathcal{M}'$ defined on the set $\{1, \ldots, n\}$. Hence $\mathcal{M}' \models \Sigma$. If $\text{Var}(\varphi) \subseteq \{p_1, \ldots, p_m\}$, $\varphi$ appears in $\Phi$ after at most $\max(m, n)$ steps.

**Definition 2.8.6** A class of Kripke frames, $\mathcal{C}$, is said to have the exponentially bounded model property (EXPMP) if there is a polynom $p(x)$ in $\mathbb{Z}[x]$ such that each formula $\varphi$ satisfiable in $\mathcal{C}$ is satisfiable in a frame $\mathcal{F} = \langle S, R \rangle$ with $|S| \leq 2^{p(|\varphi|)}$. $\mathcal{C}$ is said to have the polynomially bounded model property (PMP) if each formula $\varphi$ satisfiable in $\mathcal{C}$ is satisfiable in a frame $\mathcal{F} = \langle S, R \rangle$ with $|S| \leq p(|\varphi|)$ where $p(x)$ is a polynom in $\mathbb{Z}[x]$.

Obviously, FMP follows from EXPMP and the latter one follows from PMP.

**Corollary 2.8.7** Let $\mathcal{C}$ be a class of Kripke frames that has the exponentially bounded model property. Then if ‘$\mathcal{F} \in \mathcal{C}^{\text{fin}}$’ is decidable, then SAT($\mathcal{C}$) and VAL are decidable.

**Proof.** Let $\varphi$ be any formula. Enumerate first all Kripke models that are defined on a frame with at most $2^{p(|\varphi|)}$ worlds. For each of these models, check whether $\mathcal{M}$ is defined on frame in $\mathcal{C}^{\text{fin}}$. If so, check whether $\mathcal{M} \models \varphi$ for some world $s$ of $\mathcal{M}$. If so, return ‘$\varphi$ is satisfiable in $\mathcal{C}$’ and halt. If not so, continue with the next model. Finally, if no model has been found that satisfies $\varphi$ in some world, return ‘$\varphi$ is not satisfiable in $\mathcal{C}$’. \hfill \square
Obviously, ‘\( F \in C^{\text{fin}} \)’ is decidable for many of the frame classes we introduced in the preceding sections. For example, it can easily be checked by an algorithm whether a given finite frame is reflexive, transitive, etc. (A more general result: ‘\( F \in C^{\text{fin}} \)’ is decidable if \( C^{\text{fin}} \) is defined by some first-order formula.) Thus, for each of these classes of Kripke frames, we have reduced the problem of proving decidability to the question whether the frame class under consideration has some bounded model property.

Now for proving the finite model property of \( K \), we could argue as follows: Let us assume that \( \varphi \) is satisfiable in the class of all Kripke frames. Since \( K \) is sound with respect to that class of frames, \( \varphi \) must be \( K \)-consistent. Now the canonical way of proving that \( \varphi \) is satisfiable (which we already know, of course) is to look at the canonical model of \( K \). However, this model is defined on an infinite frame. But we can also define a finite ‘canonical model’ that behaves similar (in many respects) to the real canonical model. Thereto, consider the set of all subformulae of \( \varphi \), \( \text{Sub}(\varphi) \). Define a model \( M_\varphi := \langle S_\varphi, R_\varphi, V_\varphi \rangle \) as follows: Let \( S_\varphi \) be the set of all subsets \( \Gamma \) of \( \text{Sub}(\varphi) \cup \{ \neg \psi : \psi \in \text{Sub}(\varphi) \} \) that are \( \varphi \)-maximal consistent in the following sense: (a) \( \Gamma \) is consistent with respect to \( K \), and (b) for each subformula \( \psi \) of \( \varphi \), \( \psi \in \Gamma \) or \( \neg \psi \in \Gamma \). Then, let \( R_\varphi \) and \( V_\varphi \) be defined as in the canonical model. It can easily be seen that the finite model constructed in this way satisfies \( \varphi \).

### 2.8.2 Filtration

For proving that a given class of Kripke frames has the finite model property (or better: some bounded model property), we have to provide a method of how to transform an infinite model of that class into a finite model that is also contained in that class. The example discussed at the end of the last subsection presents the red line along which such a method can be developed.

Let \( M \) be a Kripke model, and let \( \Gamma \) be a set of formulae. We say that worlds \( s \) and \( s' \) are \( \Gamma \)-equivalent, \( s \sim_{\Gamma} s' \), if for each formula \( \gamma \in \Gamma \),

\[
M \models s \gamma \iff M \models s' \gamma.
\]

Using the notations from section 2.5, \( s \) and \( s' \) are \( \Gamma \)-equivalent if and only if

\[
T_M(s) \cap \Gamma = T_M(s') \cap \Gamma.
\]

Let \( s_\Gamma \) denote the equivalence class of \( s \) with respect to \( \Gamma \)-equivalence.

**Lemma 2.8.8** Let \( M = \langle S, R, V \rangle \) be a Kripke model and let \( \Gamma \) be a finite set of \( \mathcal{L}(P) \)-formulae. Then \( |S/\sim_{\Gamma}| \leq 2^{|
\Gamma|} \).

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Proof. For each \( s \in S \) set \( \Gamma_s := T_M(s) \cap \Gamma \). Then, for each pair of worlds \( s \) and \( s' \), \( s \) and \( s' \) are \( \Gamma \)-equivalent if and only if \( \Gamma_s = \Gamma_{s'} \). Hence the mapping \( t : S/\sim_\Gamma \rightarrow 2^\Gamma \) is injective, and thus \( |S/\sim_\Gamma| \leq 2^{|
abla|} \).

\[ \square \]

**Definition 2.8.9** A set of formulae, \( \Sigma \), is said to be closed under subformulae if for each \( \phi \in \Sigma \), \( \text{Sub}(\phi) \subseteq \Sigma \).

**Definition 2.8.10** Let \( \mathcal{M} \) be a Kripke model, and let \( \Gamma \) be a set of formulae that is closed under subformulae. A model \( \mathcal{M}' = \langle S', R', V' \rangle \) is said to be a filtration of \( \mathcal{M} \) through \( \Gamma \) if each of the following conditions is satisfied:

(i) \( S' = S/\sim_\Gamma \);

(ii) For all \( s, s' \) in \( S \),

\[ s R s' \implies s R' s' ; \]

\[ \text{If } □ \phi \in \Gamma, s R' s' \text{, and } \mathcal{M} \models_s □ \phi, \text{ then } \mathcal{M} \models_{s'} \phi ; \]

(iii) \( V'(p, s) = V(p, s) \), for each \( p \in P \).

By lemma ??, each filtration through a finite set of formulae, \( \Gamma \), is a model that has at most \( 2^{|
abla|} \) distinct worlds.

In general, filtrations are not uniquely determined. Let \( \mathcal{M}' \) be a filtration of \( \mathcal{M} \) through a set of formulae, \( \Gamma \). \( \mathcal{M}' \) is said to be the finest \( \Gamma \)-filtration if for all worlds \( s \) and \( t \),

\[ s R' t \iff \text{there exist } s', t' \in t \text{ with } s R t' . \tag{2.5} \]

\( \mathcal{M}' \) is the coarsest \( \Gamma \)-filtration if for all worlds \( s \) and \( t \),

\[ s R' t \iff \text{for all } □ \phi \in \Gamma \text{, if } \mathcal{M} \models_t \phi \text{ then } \mathcal{M} \models_s □ \phi . \tag{2.6} \]

It is worth noting that the defining clause of 2.6 follows from that of 2.5. To see this, let \( □ \gamma \) be any formula in \( \Gamma \) with \( \mathcal{M} \models_s □ \gamma \). Choose \( s' \in s \) and \( t' \in t \) such that \( s' R t' \). Then \( \mathcal{M} \models_{s'} □ \gamma \), hence \( \mathcal{M} \models_{s'} □ \gamma \), consequently \( \mathcal{M} \models_t \gamma \), and thus \( \mathcal{M} \models_t \gamma \).

**Theorem 2.8.11** Let \( \Gamma \) be a finite set of \( \mathcal{L}(P) \)-formulae that is closed under subformulae. Let \( \mathcal{M} \) be a Kripke model and let \( \mathcal{M}' \) be a filtration of \( \mathcal{M} \) through \( \Gamma \). Then \( T_M(s) \cap \Gamma = T_M(s') \cap \Gamma \), i.e., for each \( \gamma \in \Gamma \) and each world \( s \) of \( \mathcal{M} \),

\[ \mathcal{M} \models_s \gamma \iff \mathcal{M}' \models_{s'} \gamma . \]
**Proof.** Straight forward by induction on the degree of $\gamma$. Note first that each sub-formula of $\gamma$ is contained in $\Gamma$, too.

**Theorem 2.8.12** The normal modal logic $K$ has the exponentially bounded model property. In particular, each formula satisfiable in some Kripke model is satisfiable in a Kripke model with at most $2^{\lvert \phi \rvert}$ worlds.

**Proof.** By assumption, there exist a model $M$ and a world $s$ such that $M \models_s \phi$. Set $\Gamma := \text{Sub}(\phi)$. Then $\Gamma$ is a finite set that is closed under subformulae. Furthermore, $\lvert \Gamma \rvert \leq \lvert \phi \rvert$. Let $M'$ be the coarsest or the finest filtration of $M$ through $\Gamma$. Then $M'$ has at most $2^{\lvert \Gamma \rvert} \leq 2^{\lvert \phi \rvert}$ worlds and $M' \models_{s'} \phi$. 

As an immediate conclusion of corollary, we thus obtain

**Corollary 2.8.13** SAT($K$) and VAL($K$) are decidable.

At this point it is worthwhile to reconsider the proof of Theorem 2.8.12. There the finest filtration and the coarsest filtration were sufficiently small models, since both filtrations are defined on a frame for $K$. In general this is not the case. For example, in general the coarsest filtration of a model does not preserves symmetry. To see this, let us consider the set $\Gamma := \{p, \Box p\}$, which is closed under subformulae, and a Kripke model $M = \langle S, R, V \rangle$ with $S = \{0, 1\}$, $R = \{(1, 1)\}$, and $V(p, s) = 1$ iff $s = 1$. Then the coarsest filtration of $M$ through $\Gamma$ is given by $M' = \langle S', R', V' \rangle$ where $S' = \{0_\Gamma, 1_\Gamma\}$ and $R' = \{(0_\Gamma, 1_\Gamma), (1_\Gamma, 1_\Gamma)\}$, which is not symmetrical. However, the

- $p, \Box p \quad 1$
- $\neg p, \Box p \quad 0$
- $\neg p, \Box p \quad 0_\Gamma$
- $p, \Box p \quad 1_\Gamma$

Figure 2.9: Coarsest filtration of a symmetrical model

finest filtration preserves symmetry (as can easily be checked by its definition),
**Lemma 2.8.14** Let $\mathcal{M}$ be a Kripke model and let $\mathcal{M}'$ be a filtration of $\mathcal{M}$ through some set of $\mathcal{L}(P)$-formulae.

- If $\mathcal{M}$ is serial, then so is $\mathcal{M}'$.
- If $\mathcal{M}$ is reflexive, then so is $\mathcal{M}'$.

**Corollary 2.8.15** $\text{SAT}(KD)$ and $\text{SAT}(KD)$ as well as $\text{VAL}(KD)$ and $\text{VAL}(KT)$ are decidable.