Modallogiken – Theorie und Anwendungen

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Chapter 2

Unimodal modal logic
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2.6 Reducing Modalities

As we have remarked earlier, Lewis’ system S1 does not contain any theorem of the form □□α. Contrary to this, we have the following K-theorems □⊤, □□⊤, □□□⊤, . . . . This means, for each natural number n there exists a K- (T-, S4-, S5-, . . . ) theorem α with mdegα = n. In this section we will go the other way around and try to reduce the modal degree of a formula. This means we seek to solve the following problem:

Let C be a class of frames and let φ be a formula of \( L(P) \) with mdegφ = n > 2. Does there exist any formula ψ such that C ⊨ φ ↔ ψ and mdegψ < n?

To start with, let us introduce the notion of modality. A modality is any sequence of the unary operators of \( L(P) \), namely ¬ and □. The empty sequence is denoted by −. Thus, for example, ¬, ¬, □, ♦ (= ¬¬□), and ¬¬¬□¬¬¬□¬¬¬□ are modalities. A standard modality is a modality that does not contain ¬¬¬ as a sub-sequence. Hence, in our setting (♦ is a defined symbol) ♦♦ is not standard modality.

**Definition 2.6.1** Let C be a class of Kripke frames, and let X and X’ be modalities.

(a) X and X’ are said to be equivalent in C if for each formula φ of \( L(P) \), C ⊨ Xφ ↔ X’φ.

(b) X is said to be reducible in C to X’ if X and X’ are equivalent in C and if the number of occurrences of □ in X is greater than that in X’, i.e., if for some (or all) formulae φ, mdegXφ < mdegXφ.

Since in each class of frames, C, ¬¬φ and φ are equivalent, we can replace in each formula ψ each subformula of the form ¬¬φ by φ. Thus, each modality is equivalent to some standard modality.

In what is to follow we will investigate whether a given class of Kripke frames, C, allows for reducing modalities in the sense that C has a finite basis of modalities.

**Definition 2.6.2** Let B be a set of modalities. B is said to be a basis of modalities for C if each modality is equivalent in C to a modality in B. A basis of modalities B for C is said to be minimal if it has no proper subset that is a basis of modalities B for C.
Note that modalities of a basis are not required to be standard modalities, but that each basis could be transformed in a basis that contains standard modalities only. In what it to follow we will show that the class of all reflexive and transitive frames has a finite basis. Thereto, we will need some laws that allow for reducing modalities. Let $C(S4)$ be the class of all frames that are reflexive and transitive. As we know from section NN, this class of frames is defined by the schemata

\begin{align*}
(T) & \quad \lozenge \phi \rightarrow \phi \\
(4) & \quad \Box \phi \rightarrow \Box \Box \phi
\end{align*}

It is also worthwhile to mention again the mirror schemata of these

\begin{align*}
(T^*) & \quad \phi \rightarrow \lozenge \phi \\
(4^*) & \quad \lozenge \lozenge \phi \rightarrow \lozenge \phi
\end{align*}

By applying these schemata, we obtain the following reduction laws

\begin{align*}
(R1) & \quad \Box \Box \phi \leftrightarrow \Box \phi \\
(R1^*) & \quad \lozenge \lozenge \phi \leftrightarrow \lozenge \phi \\
(R2) & \quad \Box \lozenge \Box \phi \rightarrow \Box \lozenge \phi \\
(R2^*) & \quad \lozenge \Box \lozenge \Box \phi \rightarrow \lozenge \Box \phi
\end{align*}

**Lemma 2.6.3** The schemata $(R1)$ and $(R2)$ as well as their mirror schemata are valid in $C(S4)$.

**Proof.** Obviously, $(R1)$ and its mirror schema follow from $(T)$ and $(4)$. For $(R2)$ we may argue as follows: Since $\Box \lozenge \phi \rightarrow \phi$ is valid in $C(S4)$, so is $\Box \Box \lozenge \phi \rightarrow \Box \lozenge \phi$. Hence, by $(T^*)$, $\Box \Box \lozenge \phi \rightarrow \Box \phi$ and then $\Box \Box \lozenge \phi \rightarrow \Box \lozenge \phi$ are valid in $C(S4)$. For the other direction of the implication note first that, by $(T^*)$, $\Box \lozenge \phi \rightarrow \Box \lozenge \phi$ is valid in $C(S4)$. Hence, $\Box \Box \lozenge \phi \rightarrow \Box \Box \lozenge \phi$ and then, by $(4)$, $\Box \lozenge \phi \rightarrow \Box \lozenge \phi \rightarrow \Box \Box \lozenge \phi$ are valid in $C(S4)$. Thus, $\Box \Box \phi \leftrightarrow \Box \Box \Box \phi$ is valid in $C(S4)$, since a conjunction of formulae is valid in any class of frames, if both of its conjuncts are so. $(R2^*)$ immediately follows from $(R2)$. $\diamondsuit$

**Theorem 2.6.4** Each modality is in $C(S4)$ equivalent to

\begin{align*}
\neg, \Box, \lozenge, \Box \lozenge, \lozenge \Box, \Box \Box, \lozenge \Box \lozenge, \Box \Box \lozenge
\end{align*}

or a negation of these.
Proof. We will show that the set of 14 modalities
\[ B = \{ -, \neg, \Box, \neg\Box, \Diamond, \neg\Diamond, \Box\Diamond, \neg\Box\Diamond, \Box\neg\Diamond, \neg\Box\neg\Diamond, \Box\neg\Box\Diamond, \neg\Box\neg\Box\Diamond, \Box\neg\Diamond\neg\Box, \neg\Box\neg\Diamond\neg\Box \} \]
is a basis of modalities for \( C(S4) \). To prove this it suffices to show that
\[(*) B \text{ is (up to equivalence in } C(S4)) \text{ closed with respect to negation and necessitation,}
\]
i.e., for each \( X \in B \), there exist \( X' \) and \( X'' \) in \( B \) such that \( X' \) is equivalent to \( \Box X \) and \( X'' \) is equivalent to \( \neg X \). Why does \((*)\) suffice? This follows by induction on the length of \( X \). If \( X \) has length 1, then \( X \) is \( \Box \) or \( \neg \) and hence \( X \) contained in \( B \). If \( X \) has length \( n + 1 \) and if each modality of length \( n \) is equivalent to a modality in \( B \), then \( X \) is \( \Box X' \) or \( \neg X' \) for some modality \( X' \). Since \( X' \) is equivalent to a modality in \( B \), so is \( X \) (by applying \((*)\)).
By applying lemma 2.6.3, the proof of \((*)\) becomes an easy exercise. \( \square \)

If a class of frames has a finite basis of modalities we can investigate the logical relations between them and visualize them by a graph. In the case of \( C(S4) \), for example, we obtain the graph, called the \textit{S4-diagram of modalities}, shown in figure 2.6. A graph of the other half of the 14 modalities of \( C(S4) \) can be obtained by

\[\begin{align*}
\square & \quad \Box \Diamond \quad \Diamond \neg \Box \\
\Box \Box & \quad \Box \neg \Box \Diamond & \Diamond \neg \Box \Diamond & \Diamond \\
\Diamond \Box & \quad \Diamond \neg \Diamond \Box & \Diamond \neg \Box \Box & \Diamond \\
\Diamond & \quad \Diamond \Box \Diamond & \Diamond \neg \Diamond \Diamond & \Diamond
\end{align*}\]

Figure 2.6: Affirmative S4-diagram of modalities

contraposing the relations shown in figure 2.6.

Let us now turn to the class \( C(S5) \) of all frames that are reflexive, symmetrical, and transitive. As has been shown in section NN, this class is defined by the schemata
It is obvious that $C(S4)$ is a proper superset of $C(S5)$, i.e., each formula that is true in $C(S4)$ is true in $C(S5)$ as well. In particular, the scheme

(4) $\Box \varphi \rightarrow \Box \Box \varphi$

is true in $C(S5)$. This can also be seen by the following argument. Since $\Diamond \neg \varphi \rightarrow \Box \Diamond \neg \varphi$ is valid in $C(S5)$, so is $\neg \Box \neg \Diamond \neg \varphi \rightarrow \neg \Diamond \neg \varphi$, and hence so is

(S5-1) $\Diamond \Box \varphi \rightarrow \Box \varphi$.

Thus, by (T), we obtain that

(S5-2) $\Diamond \varphi \leftrightarrow \Box \Diamond \varphi$

as well as

(S5-3) $\Box \varphi \leftrightarrow \Diamond \Box \varphi$

are valid in $C(S5)$. Since $\Box \varphi \rightarrow \Diamond \Box \varphi$ and, by (S5-2), $\Diamond \Box \varphi \leftrightarrow \Box \Diamond \Box \varphi$ are valid in $C(S5)$, it follows that $\Box \varphi \rightarrow \Box \Diamond \Box \varphi$ is so, too. Then, by (S5-3),

(4) $\Box \varphi \rightarrow \Box \Box \varphi$

is valid in $C(S5)$.

Thus, in summary, we have the following reduction theorems:
Applying these theorems on theorem 2.6.4, we obtain

**Theorem 2.6.5** In \( C(S5) \) each modality is equivalent to

\[-, \Box, \Diamond\]

or a negation of one these.

It is clear that each finite sequence of \( \Box \) and \( \Diamond \), \( O_1 \ldots O_n \), is in \( C(S5) \) equivalent to \( O_n \). And the S4-diagram of modalities (as shown in figure 2.6) reduces in S5 to the diagram shown in figure 2.8)

![Diagram](image)

**Figure 2.8:** Affirmative S5-diagram of modalities

In what follows, we aim at proving a stronger result than that of theorem 2.6.5. Consider, for example, the formula \( \Box(\Box p \land \Box q) \). This formula, which has modal degree 2, can be shown to be S4-equivalent to a formula that has modal degree 1, namely

\( \Box p \land \Box q \).

To show this, we can use

\((R4)\) \ \( \Box(\varphi \land \psi) \leftrightarrow (\Box \varphi \land \Box \psi) \)

\((R4^+)\) \ \( \Diamond(\varphi \lor \psi) \leftrightarrow (\Diamond \varphi \lor \Diamond \psi) \)
Contrary to this, the formula \( \Box(p \lor \Box q) \) can be shown to be not S4-equivalent to any formula of modal degree 1. With respect to S5, however, there exists a formula that is equivalent to it, namely \( \Box p \lor \Box q \).

More generally, while both S4 and S5 allow for reducing \textit{iterated} modalities, only S5 allows to reduce \textit{nested} occurrences of modalities.

**Theorem 2.6.6** Each \( L(P) \)-formula \( \varphi \) is in S5 equivalent to an \( L(P) \)-formula \( \psi \) with \( \text{mdeg} \, \psi \leq 1 \).

To prove this theorem we need some further reduction theorems that deal with nested occurrences of modalities, namely

(R5) \( \Box (\varphi \lor \Box \psi) \leftrightarrow (\Box \varphi \lor \Box \psi) \)

(R5\*) \( \lozenge (\varphi \land \lozenge \psi) \leftrightarrow (\lozenge \varphi \land \lozenge \psi) \)

(R6) \( \Box (\varphi \lor \varnothing \psi) \leftrightarrow (\Box \varphi \lor \varnothing \psi) \)

(R6\*) \( \lozenge (\varphi \land \Box \psi) \leftrightarrow (\lozenge \varphi \land \Box \psi) \)

**Lemma 2.6.7** (R5) and (R6) as well as their mirror schemata are valid in \( C(S5) \).

**Proof.** We will prove (R5) only. The proof of the other theorems works analogously. Let \( F \) be a frame in \( C(S5) \), let \( M \) be a model defined on \( F \), and let \( s \) be a world of \( M \). Let us first suppose that \( M \models_s \Box \varphi \lor \Box \psi \). Then if \( M \models_s \Box \varphi \), it trivially holds that \( M \models_s \Box (\varphi \lor \Box \psi) \). If \( M \models_s \Box \psi \), then, by transitivity, \( M \models_s \Box \Box \psi \) and hence \( M \models_s \Box (\varphi \lor \Box \psi) \). For the other direction, suppose that \( M \models_s \Box (\varphi \lor \Box \psi) \). If \( M \models \varphi \), then there exists an \( s' \) with \( sRs' \) and \( M \models_{s'} \varphi \). Then \( M \models_{s'} \Box \psi \), thus \( M \models_s \lozenge \Box \psi \) and hence, by (5), \( M \models_s \Box \psi \) — as to be shown.

\( \Box \)

We are now in a position to prove theorem 2.6.6.

**Proof of 2.6.6.** We will show that

(\#) Each formula \( \varphi \) with \( \text{mdeg} \, \varphi = 2 \) is S5-equivalent to a formula \( \psi \) with \( \text{mdeg} \, \psi = 1 \).
This shows the more general claim of the theorem, since each formula that has a modal degree $n > 2$ contains subformulae of modal degree 2. Replacing each such subformula by an S5-equivalent subformula with modal degree 1 yields a formula that has a modal degree lower than $n$.

For the proof of (∗) let $ϕ_0 = ϕ$ be a formula with $\text{mdeg}\, ϕ_0 = 2$. We present an algorithm that syntactically transforms $ϕ_0$ into an S5-equivalent formula with modal degree 1.

1. Rewrite $ϕ_0$ is such a way that only $¬$, $□$, $◊$, $∧$, and $∨$ occur in it, i.e., we replace in $ϕ_0$ each subformula of the form $(ψ → ψ′)$ by $(¬ψ ∨ ψ′)$ and each subformula of the form $(ψ ← ψ′)$ by $((¬ψ ∨ ψ′) ∧ (ψ ∨ ¬ψ′))$. Let $ϕ_1$ denote the resulting formula.

2. Replace each subformula of $ϕ_1$ that has the form $¬(ψ ∧ ψ′)$ by $¬ψ ∨ ¬ψ′$ and each subformula of the form $¬(ψ ∨ ψ′)$ by $(¬ψ ∧ ¬ψ′)$ (de Morgan laws). Let $ϕ_2$ denote the formula obtained thereby.

3. Obtain from $ϕ_2$ a formula $ϕ_3$ by replacing in $ϕ_2$ each subformula of the form $¬□ψ$ by $◊¬ψ$ and each subformula of the form $¬◊ψ$ by $□¬ψ$.

4. Repeat steps (1.) and (2.) until the negation symbol occurs in front of propositional variables only. Let $ϕ_4$ denote the new formula.

5. Transform $ϕ_4$ into $ϕ_5$ by reducing iterated modalities to unary ones (apply (R1) and (R2)).

6. If $ϕ_5$ has modal degree 1, then we are set. If $ϕ_5$ still has modal degree 2, then $ϕ_5$ contains subformulae of the form $□ψ'$ or $◊ψ'$, where $\text{mdeg}\, ψ' = 1$ and $ψ'$ is a conjunction or a disjunction of formulae. Proceede for each such subformula as follows

**Case 1:** Let $ψ = □ψ'$ be a subformula of $ϕ_5$.

**Case 1.1:** $ψ'$ is a conjunction of formulae $ψ'_1$ and $ψ'_2$. Then replace $□(ψ'_1 ∧ ψ'_2)$ by $(□ψ'_1 ∧ □ψ'_2)$ and absorb modalities by applying (R1) or (R2). Thereby we reduce the modal degree of $ψ$ by 1.

**Case 1.2:** $ψ'$ is a disjunction of formulae $ψ'_1$ and $ψ'_2$. If $ψ'_1$ or $ψ'_2$ is of the form $□τ$ or $◊τ$, then apply one of the theorems (R5) or (R5*) on $ψ'$ and reduce modalities. The resulting formula $ψ''$ then has modal degree 1. If neither $ψ'_1$ nor $ψ'_2$ is of the form $□τ$ or $◊τ$, both $ψ'_1$ and $ψ'_2$ must be disjunction or conjunctions of formulae. In the first case apply the associative law and in the second case apply the distributive law to the formula $ψ'$.
Case 2: \( \psi = \Diamond \psi' \). The procedure is analogously to that of case 1.

7. By repeating step 6 finitely often, we finally obtain a formula \( \varphi_7 \) from \( \varphi_5 \) that has modal degree of 1.

Since all the transformations applied in this procedure result in equivalent formulae, it is obvious that \( \varphi_7 \) is \( C(\mathbb{S}5) \)-equivalent to \( \varphi_0 \). □

For example, let us consider the formula \( \neg \Diamond (p \land (q \lor \Diamond r)) \). By applying the procedure presented in the proof we obtain

\[
\begin{align*}
(1) & \quad \neg \Diamond (p \land (q \lor \Diamond r)) \\
(2) & \quad \Box \neg (p \land (q \lor \Diamond r)) \\
(3) & \quad \Box (\neg p \lor \neg (q \lor \Diamond r)) \\
(4) & \quad \Box (\neg p \lor (q \land \neg \Diamond r)) \\
(5) & \quad \Box (q \land \neg \Diamond r) \\
(6) & \quad \Box (\neg p \lor (q \land \neg \Diamond r)) \\
(7) & \quad \Box (\neg p \lor \neg q) \\
(8) & \quad \Box (\neg p \lor \neg q) \land \Box (\neg p \lor \Box \neg r)
\end{align*}
\]

Definition 2.6.8 Let \( \Sigma \) be a set of \( \mathcal{L}(P) \)-formulae and let \( \varphi \) be an formula of \( \mathcal{L}(P) \).

(a) \( \varphi \) is said to have conjunctive normal form (CNF) w. r. t. \( \Sigma \) if \( \varphi \) has the form

\[
\bigwedge_{i=1}^{n} \bigvee_{j_i} \psi_{ij_i},
\]

(CNF)

where each \( \psi_{ij_i} \) is contained in \( \mathcal{A} \).

(b) \( \varphi \) is said to have ordered CNF w. r. t. \( \Sigma \) if \( \varphi \) has CNF w. r. t. \( \Sigma \) and if each conjunct \( \bigvee_{j_i}^{m_i} \psi_{ij_i} \) of \( \varphi \) has the form

\[
\beta \lor \Box \gamma_1 \lor \cdots \lor \Box \gamma_n \lor \Diamond \delta,
\]

(*)

where \( \beta \) is a formula with mdeg \( \beta = 0 \). (Note that we also allow degenerated cases of (*) where one or more disjuncts do not occur).

(c) \( \varphi \) is said to have modal CNF (MCNF) if \( \varphi \) has CNF w. r. t. to the set

\[
\{ \varphi : \text{mdeg} \varphi = 0 \} \cup \{ \Box \varphi : \text{mdeg} \varphi = 0 \} \cup \{ \Diamond \varphi : \text{mdeg} \varphi = 0 \},
\]

i. e., if in each conjunct each disjunct \( \psi_{ij_i} \) of \( \varphi \) is a PL-formula or has the form \( \Box \gamma \) of \( \Diamond \gamma \) where \( \gamma \) is a PL-formula.

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Obviously, each formula in MCNF has modal degree 1. Vice versa, by simply modifying the syntactical procedure presented in the proof of theorem 2.6.6, one obtains

**Theorem 2.6.9** Each $\mathcal{L}(P)$-formula $\varphi$ is in S5 equivalent to an $\mathcal{L}(P)$-formula $\psi$ with (ordered) MCNF.

*Proof.* By the procedure presented in the proof of theorem 2.6.6 we obtain a formula with modal degree 1 that is a disjunction of conjunctions (or a conjunction of disjunctions) of formulae that are of modal degree 0 or have the form $\Box \gamma$ or $\Diamond \gamma$, where $\gamma$ is a formula with modal degree 0. In the first case we apply the distributive and (if necessary) the associative laws to obtain a formula in MCNF. Obviously, if a formula $\varphi$ has MCNF, it can easily be transformed into an equivalent formula with ordered MCNF by applying the K-theorem $(\Diamond \delta_1 \lor \Diamond \delta_2) \leftrightarrow (\Diamond (\delta_1 \lor \Diamond \delta_2))$. $\triangleleft$

As an example consider the formula $\Box (\Diamond p \rightarrow p) \rightarrow \Box (p \rightarrow \Box p)$. We first reduce this formula to a formula with modal degree 1

1. $\neg \Box (\neg \Diamond p \lor p) \lor \Box (\neg p \lor 
\Diamond p)$
2. $\Diamond (\Diamond p \land 
\neg p) \lor \Box (\neg p \lor \Box p)$
3. $(\Diamond p \land 
\Diamond \neg p) \lor (\Box \neg p \lor \Box p)$

From this one obtains MCNF by the distributivity laws:

4. $(\Diamond p \lor \Box \neg p \lor \Box p) \land (\Diamond \neg p \lor \Box \neg p \lor \Box p)$

By simply reordering (4) we get a formula in ordered MCNF

5. $(\Box \neg p \lor \Box p \lor \Diamond p) \land (\Box \neg p \lor \Box p \lor \Diamond \neg p)$

Using this result, we can now introduce a test for deciding whether a given formula $\varphi$ is valid in $\mathcal{C}(S5)$. Thereto, we can assume that $\varphi$ already has ordered MCNF. Then, by definition, $\varphi$ is a conjunction of formulae that have the form

$\beta \lor \Box \gamma_1 \lor \cdot \cdot \cdot \Box \gamma_n \lor \Diamond \delta$  

(*)

where all formulae $\beta, \gamma_1, \ldots, \gamma_n, \delta$ have modal degree 0. Consider then the disjunctions

$\beta \lor \delta, \gamma_1 \lor \delta, \ldots, \gamma_n \lor \delta,$  

(**)
which are obviously formulae of PL. We say that $\varphi$ passes the disjunction test if at least one of the disjunctions in $\ast\ast$ is valid in PL. A conjunction of formulae of the form $\ast$ is said to pass the disjunction test if each of its conjuncts passes it.

In the example presented above, for example, formula (5) has two conjuncts and we have to test both of them. In the disjunction test for the first conjunct, $\square\neg p \lor \square p \lor \lozenge p$, we have to test whether one of the disjunctions

\[(6) \quad \neg p \lor p, \ p \lor p\]

is PL-valid, which of course is the case. For the second conjunct $\square\neg p \lor \square p \lor \lozenge \neg p$ the disjunction test gives

\[(7) \quad \neg p \lor \neg p, \ p \lor \neg p\]

and the second formulae of these is obviously PL-valid. Thus, both conjuncts pass the disjunction test, and hence so does (6).

**Lemma 2.6.10** Let $\varphi$ be a $\mathcal{L}(P)$-formula of the form $\ast$.

(a) If $\varphi$ passes the disjunction test, then $\varphi$ is valid in $C(S5)$.

(b) If $\varphi$ is valid in $C(S5)$, then $\varphi$ passes the disjunction test.

**Proof (sketch).** (a) Let $\varphi$ be a formula of the form $\ast$ that passes the disjunction test. Then at least one of the disjunctions in $\ast\ast$ is PL-valid. For example, if $\gamma_1 \lor \delta$ is valid in PL, then it is valid in $C(S5)$, too. Hence, $\square(\gamma_1 \lor \delta)$ is valid in $C(S5)$, hence, by reflexivity, so is $\square(\gamma_1 \lor \lozenge \delta)$, and thus by (R6) so is $\square \gamma_1 \lor \lozenge \delta$. From this we obtain that $\varphi$ is valid in $C(S5)$.

(b) Let $\varphi$ be a formula of the form $\ast$ that does not pass the disjunction test. This means that none of the disjuncts $\beta \lor \delta, \gamma_1 \lor \delta, \ldots, \gamma_n \lor \delta$ is valid in PL. We have to show that $\varphi$ is not valid in $C(S5)$. Since each of these disjunctions is not valid in PL, there exist PL-valuations $V_0, \ldots, V_n$ such that $V_0(\beta \lor \delta) = 0$ and $V_i(\gamma_i \lor \delta) = 0$, for each $1 \leq i \leq n$. Consider then the Kripke model $\mathcal{M} = \langle S, R, V \rangle$ defined by

\[
S := \{0, \ldots, n\} \\
R := S \times S \\
V(p, s) := \begin{cases} 1 & \text{if } V_s(p) = 1 \\ 0 & \text{else.} \end{cases}
\]

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It is clear that this model is an S5-model. Moreover, for each $0 \leq i \leq n$ and each formula $\psi$ with $\text{mdeg} \, \psi = 0$,

$$M \models_i \psi \iff V_i \models \psi,$$

i.e., $M \not\models_0 \beta \lor \delta$ and, for each $1 \leq i \leq n$, $M \not\models_i \gamma \delta$. Thus, $M \not\models_0 \varphi$. ◁

Since a conjunction of formulae is valid in $\mathcal{C}(S5)$ if and only if each of the conjuncts is so (the same holds true for any class of frames, of course), we can weaken the restrictions of the lemma, and obtain

**Theorem 2.6.11** An $\mathcal{L}(P)$-formula is valid in $\mathcal{C}(S5)$ if and only if it is $\mathcal{C}(S5)$-equivalent to a formula in ordered MCNF that passes the disjunction test. ◁

Since transforming a formula into ordered MCNF can be done by an algorithm (note that by applying the distributive laws, the length of the formula may increase exponentially) and since testing PL-validity is decidable, we obtain

**Theorem 2.6.12** The validity problem w. r. t. $\mathcal{C}(S5)$ is decidable. ◁

In fact, what we have done is the following: We have reduced the problem of testing S5-validity to the problem of testing PL-validity.

Finally, in the light of theorem 2.6.11, we can read part (a) of lemma 2.6.10 as: ‘Passing the disjunction test is sound with respect to S5-validity’, and analogously part (b) as: ‘Passing the disjunction test is complete with respect to S5-validity’. These notions will be investigated in detail in the following section.