Lecture 16: Planning with partial observability

- Backward steps with partial observability, preimages + branching
- A representation of sets of plans
- Planning algorithms
Regression/preimages

\[ \text{spreimg}(S) \]

\[ S \]
Branching in backward search

- Observational classes $C_1, \ldots, C_n$

- Branching: split a set $S$ of states to sets $S_1 = S \cap C_1$, $S_2 = S \cap C_2$, \ldots, $S_n \cap C_n$ according to the observational classes $C_i$.

- Branching in backward direction:

  Let $S_1, S_2, \ldots, S_n$ be sets so that for all $i, j$ such that $i \neq j$ and $C \in \{C_1, \ldots, C_n\}$, $S_i \cap C = \emptyset$ or $S_j \cap C = \emptyset$.

  Now they can be combined to $S = S_1 \cup \cdots \cup S_n$. 

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Combination 11

S1

S2

o1 o2 o3 o4 o5 o6 o7
Combination 22

S1

S2

o1  o2  o3  o4  o5  o6  o7

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Combination 21

S1

S2

o1 o2 o3 o4 o5 o6 o7
No observability $\Rightarrow$ No branching
Combination 1

S1

S2

o1
Combination with full observability

S1

S2

o1  o2  o3  o4  o5  o6  o7
Combination of belief states

Let $\Pi = \langle C_1, \ldots, C_n \rangle$ be a partition of the set of all states.

Let $B_1$ and $B_2$ be two sets of states. Define

$$B_1 \oplus B_2 = \{ S_1 \cup \cdots \cup S_n | S_i \in \{ B_1 \cap C_i, B_2 \cap C_i \}, S_i \not\subset B_1 \cap C_i, S_i \not\subset B_2 \cap C_i, \text{ for all } i \in \{1,\ldots,n\} \}.$$
Example: medication with dangerous drugs

- Diseases 1, 2, 3 and 4 can be diagnosed as follows.

<table>
<thead>
<tr>
<th>test F</th>
<th>positive</th>
<th>negative</th>
</tr>
</thead>
<tbody>
<tr>
<td>negative</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>positive</td>
<td>3</td>
<td>4</td>
</tr>
</tbody>
</table>

- Different medication for each disease:
  Correct medication cures.
  Incorrect medication kills.
Example: medication with dangerous drugs

Three state variables

• State of the patient: disease 1, 2, 3 or 4, the patient is cured, the patient is dead (6 values 1, 2, 3, 4, H, D.).

• Test B has tested positive. (Boolean)

• Test F has tested positive. (Boolean)

The states are written as follows (ones reachable from \{1, 2, 3, 4\}).
H, HB, HF, HBF, 1, 1B, 2, 3, 3B, 3F, 3BF, 4, 4F, D, DB, DF, DBF.
Example: medication actions

Action “medicate for disease 1”:

\[ H \rightarrow H \quad HB \rightarrow HB \quad HF \rightarrow HF \quad HBF \rightarrow HBF \]
\[ 1 \rightarrow H \quad 1B \rightarrow HB \]
\[ 2 \rightarrow D \]
\[ 3 \rightarrow D \quad 3B \rightarrow DB \quad 3F \rightarrow DF \quad 3BF \rightarrow DBF \]
\[ 4 \rightarrow D \quad 4F \rightarrow DF \]
\[ D \rightarrow D \quad DB \rightarrow DB \quad DF \rightarrow DF \quad DBF \rightarrow DBF \]
Example: test actions

Action “do test B”:

\[ H \rightarrow H \quad HB \rightarrow H \quad HF \rightarrow H \quad HBF \rightarrow H \]
\[ 1 \rightarrow 1B \quad 1B \rightarrow 1B \]
\[ 2 \rightarrow 2 \]
\[ 3 \rightarrow 3B \quad 3B \rightarrow 3B \quad 3F \rightarrow 3BF \quad 3BF \rightarrow 3BF \]
\[ 4 \rightarrow 4 \quad 4F \rightarrow 4F \]
\[ D \rightarrow D \quad DB \rightarrow D \quad DF \rightarrow D \quad DBF \rightarrow D \]
Example

The goal states $G$ are the following, split to observational classes corresponding to the 4 different observations $BF, F, B, \emptyset$.

\[ \{ HBF \mid HF \mid HB \mid H \} \]
Example: strong preimages of the goal states

The strong preimage of $G$ with respect to action “medicate for 1”.

$$\begin{array}{c|c|c|c}
HBF & HF & 1B \\
\hline
HB & H & 1 \\
\end{array}$$

The strong preimage of $G$ with respect to action “medicate for 2”.

$$\begin{array}{c|c|c|c}
HBF & HF & HB \\
\hline
2 & H & \\
\end{array}$$
The strong preimage of $G$ with respect to action “medicate for 3”.

$$\left\{ \begin{array}{ccc} 3BF & 3F & 3B & 3H \\ HBF & HF & HB & H \end{array} \right\}$$

The strong preimage of $G$ with respect to action “medicate for 4”.

$$\left\{ \begin{array}{ccc} 4F & 4H \\ HBF & HB \end{array} \right\}$$
Example: possible combinations

\[
\left\{ \begin{array}{c|c|c|c}
HBF & HF & HB & H \\
\end{array} \right\}
\left\{ \begin{array}{c|c|c|c}
HBF & HF & 1B \\
HB & H \\
\end{array} \right\}
\]

\[
\left\{ \begin{array}{c|c|c|c}
HBF & HF & HB & 2H \\
\end{array} \right\}
\left\{ \begin{array}{c|c|c|c}
3BF & HBF & 3F \\
HB & H \\
\end{array} \right\}
\]

\[
\left\{ \begin{array}{c|c|c|c}
HBF & HF & HB & 4H \\
\end{array} \right\}
\]
Example: possible combinations 1/16

\[
\{ \begin{array}{c|c|c|c}
HBF & HF & HB & H \\
\end{array} \} \quad \{ \begin{array}{c|c|c|c}
HBF & HF & 1B & 1H \\
\end{array} \}
\]

\[
\{ \begin{array}{c|c|c|c}
HBF & HF & HB & 2H \\
\end{array} \} \quad \{ \begin{array}{c|c|c|c}
3BF & 3F & 3B & 3H \\
\end{array} \}
\]

\[
\{ \begin{array}{c|c|c|c}
HBF & 4F & HB & 4H \\
\end{array} \}
\]
Example: possible combinations 2/16

\[
\{ \begin{array}{c|c|c|c}
HBF & HF & HB & H \\
\end{array} \} \quad \{ \begin{array}{c|c|c|c}
HBF & HF & 1B & 1H \\
\end{array} \} \\
\{ \begin{array}{c|c|c|c}
HBF & HF & HB & 2H \\
\end{array} \} \quad \{ \begin{array}{c|c|c|c}
3BF & 3F & 3B & 3H \\
\end{array} \} \\
\{ \begin{array}{c|c|c|c}
HBF & 4F & HB & 4H \\
\end{array} \} \]
**Example: possible combinations 3/16**

<table>
<thead>
<tr>
<th></th>
<th>HBF</th>
<th>HF</th>
<th>HB</th>
<th>H</th>
<th></th>
<th>HBF</th>
<th>HF</th>
<th>1B</th>
<th>1H</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>HBF</td>
<td>HF</td>
<td>HB</td>
<td>H</td>
<td>2</td>
<td>HBF</td>
<td>HF</td>
<td>3BF</td>
<td>3F</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>3</td>
<td></td>
<td></td>
<td>3BF</td>
<td>3F</td>
</tr>
<tr>
<td></td>
<td>HBF</td>
<td>HF</td>
<td>HB</td>
<td>H</td>
<td>4</td>
<td>HBF</td>
<td>HF</td>
<td>4F</td>
<td>HBF</td>
</tr>
</tbody>
</table>

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Example: possible combinations 4/16

\[
\begin{align*}
\{ & HBF & | & HF & | & HB & | & H & \} & \{ & HBF & | & HF & | & 1B \_ & | & 1H \\
& HBF & | & HF & | & HB & | & 2H & \} & \{ & 3BF & | & 3F & | & 3B \_ & | & 3H \\
& HBF & | & \_F & | & HB & | & 4H & \\
\end{align*}
\]
\[
\begin{array}{|c|c|c|c|}
\hline
3BF & 3F & 1B & 1 \\
\hline
HBF & HF & HB & H \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|c|}
\hline
3BF & 4F & 1B & 1 \\
\hline
HBF & HF & HB & H \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|c|}
\hline
3BF & 3F & 3B & 1 \\
\hline
HBF & HF & HB & H \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|c|}
\hline
3BF & 4F & 3B & 1 \\
\hline
HBF & HF & HB & H \\
\hline
\end{array}
\]
\[
\begin{align*}
\{ \begin{array}{|c|c|c|c|}
  3BF & 3F & 1B & 2 \\
  HBF & HF & HB & H \\
\end{array} \}
\end{align*}
\[
\begin{align*}
\{ \begin{array}{|c|c|c|c|}
  3BF & 4F & 1B & 2 \\
  HBF & HF & HB & H \\
\end{array} \}
\end{align*}
\[
\begin{align*}
\{ \begin{array}{|c|c|c|c|}
  3BF & 3F & 3B & 2 \\
  HBF & HF & HB & H \\
\end{array} \}
\end{align*}
\[
\begin{align*}
\{ \begin{array}{|c|c|c|c|}
  3BF & 4F & 3B & 2 \\
  HBF & HF & HB & H \\
\end{array} \}
\end{align*}
\]
\[
\begin{align*}
\{ & 3BF & 3F & 1B & 3 \\
& HBF & HF & HB & H \\
\} \\
\{ & 3BF & 4F & 1B & 3 \\
& HBF & HF & HB & H \\
\} \\
\{ & 3BF & 3F & 3B & 3 \\
& HBF & HF & HB & H \\
\} \\
\{ & 3BF & 4F & 3B & 3 \\
& HBF & HF & HB & H \\
\}
\]
\[
\begin{align*}
\{ & 3BF \quad 3F \quad 1B \quad 4 \\
 & HBF \quad HF \quad HB \quad H \\
\} \\
\{ & 3BF \quad 4F \quad 1B \quad 4 \\
 & HBF \quad HF \quad HB \quad H \\
\} \\
\{ & 3BF \quad 4F \quad 3B \quad 4 \\
 & HBF \quad HF \quad HB \quad H \\
\} \\
\end{align*}
\]
Example: continued

We compute the preimage of every one of these sets of states. However, only one is of real interest.

\[
\{ 3BF, HBF \} \quad \{ 4F, HF \} \quad \{ 1B, HB \} \quad \{ 2, H \}
\]

(This set of states includes the one that is reached from the initial states by the two test actions.)
Many of the preimages are rather useless. Consider for example the following set of states.

\[
\begin{align*}
\{ & 3BF \\
& HBF \\
3F & \\
HF & \\
1B & \\
HB & \\
1 & \\
H & \\
\}
\end{align*}
\]

The preimage with “do test B” is the set itself.
Example: continued

The strong preimage of

\[
\begin{array}{|c|c|c|}
\hline
3BF & 4F & 1B \\
HBF & HF & HB \\
\hline
\end{array}
\]

with respect to “do test B” is

\[
\begin{array}{|c|c|c|}
\hline
3BF & 3F & 1B \\
HBF & HF & HB \\
\hline
\end{array}
\]
would now be combined with all other sets of states. However, it suffices to compute just its preimage with “do test F” and get

\[
\begin{align*}
\{ & 3BF \\ & HBF \} \\
\begin{array}{|c|c|c|}
3F & 1B & 1 \\
4F & 3B & 2 \\
 HF & HB & H \\
\end{array}
\]
Example: continued

We kept track how the sets of states were obtained by strong preimages and combinations, and got a plan:

1. Do test F.
2. Do test B.
3. If BF, medicate for 3.
   If only B, medicate for 1.
   If only F, medicate for 4.
   If neither, medicate for 2.
Problem with the algorithm so far

1. $n$ belief states and $m$ observational classes: $m^n$ combinations.

   - 5 belief states and 5 observational classes: $5^5 = 3125$
   - 10 b-states and 7 o-classes: $7^{10} = 282475249 \sim 2.8 \cdot 10^8$
   - 15 b-states and 9 o-classes: $9^{15} = 205891132094649 \sim 2.1 \cdot 10^{14}$

   This looks rather discouraging!!

2. However, there are usually very many belief states that have the same preimages.
Compact representation of sets of sets of states

The problem is the high number of possible combinations, most of which are useless.

Idea: do not produce the combinations explicitly.
Example: possible combinations

\{
  \text{HBF} | \text{HF} | \text{HB} | \text{H}
\} \quad \{ 
  \text{HBF} | \text{HF} | \frac{1}{\text{HB}} | \frac{1}{\text{H}}
\} \\
\{ 
  \text{HBF} | \text{HF} | \text{HB} | \frac{2}{\text{H}}
\} \quad \{ 
  \text{3BF} | \text{HB} | \text{3} | \text{H}
\} \\
\{ 
  \text{HBF} | \frac{4}{\text{HF}} | \text{HB} | \frac{4}{\text{H}}
\}

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Example: maximal combinations concisely

Representation of the combinations of the 5 sets with non-minimal sets eliminated:
Factored belief space

**DEFINITION** Let $\Pi = \langle C_1, \ldots, C_n \rangle$ be a partition of the set of all states. Then a factored belief space (FBS) is $\langle G_1, \ldots, G_n \rangle$ where $s \subset s'$ for no $\{s, s'\} \subseteq G_i$ and $G_i \subseteq 2^{C_i}$ for all $i \in \{1, \ldots, n\}$.

The representation of one belief state $B$ is

$$\mathcal{F}(B) = \langle \{C_1 \cap B\}, \ldots, \{C_n \cap B\} \rangle$$
Example: the goal belief state

There is one goal belief state

\[
\{ HBF \mid HF \mid HB \mid H \}
\]

that will now be represented as the 4-tuple

\[
\langle \{\{ HBF \}\} \mid \{\{ HF \}\} \mid \{\{ HB \}\} \mid \{\{ H \}\} \rangle
\]
Combining (sets of) belief states

**DEFINITION** Let $G = \langle G_1, \ldots, G_n \rangle$ and $H = \langle H_1, \ldots, H_2 \rangle$. Define $G \oplus H$ as $\langle G_1 \cup H_1, \ldots, G_n \cup H_n \rangle$, where

$$G \cup H = \{ R \in G \cup H | R \subset K \text{ for no } K \in G \cup H \}. $$

A FBS $G = \langle G_1, \ldots, G_n \rangle$ represents the set of belief states

$$\text{flat}(G) = \{ s_1 \cup \cdots \cup s_n | s_i \in G_i \text{ for all } i \in \{1, \ldots, n\} \},$$

and its cardinality is $|G_1| \cdot |G_2| \cdot \ldots \cdot |G_n|$. 

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Example: the goal belief state

The strong preimage of $G$ with respect to action “medicate for 1”.

$$\langle \{\{ HBF \}\} \left| \{\{ HF \}\} \right| \{\{ 1B \}\} \left| \{\{ 1 \}\} \right\rangle$$

combined with the strong preimage of $G$ with “medicate for 3” is

$$\langle \{\{ 3BF \} \left| \{\{ 3F \} \right| \{\{ 1B \}\} \left| \{\{ 1 \}\} \right\rangle$$
DEFINITION A FBS $G$ is included in FBS $H$ if for all $B \in \text{flat}(G)$ there is $B' \in \text{flat}(H)$ such that $B \subseteq B'$. We write this $G \sqsubseteq H$.

Clearly, $B \subseteq B'$ for some $B' \in \text{flat}(G)$ if and only if $\mathcal{F}(B) \sqsubseteq G$. 

Testing inclusion and membership

THEOREM Testing $G \subseteq H$ for FBSs $G$ and $H$ is polynomial time.

Proof: $\langle G_1, \ldots, G_n \rangle \subseteq \langle H_1, \ldots, H_n \rangle$ if and only if for all $i \in \{1, \ldots, n\}$ and all $s \in G_i$ there is $t \in H_i$ such that $s \subseteq t$. 
Finding new belief states

PROCEDURE findnew(o, A, F, H);

IF $F = \emptyset$ AND $\text{spreimg}_o(A) \notin \text{flat}(H)$ THEN RETURN $A$;
IF $F = \emptyset$ THEN RETURN $\emptyset$;

$F$ is $\langle\{f_1, \ldots, f_m\}, F_2, \ldots, F_k\rangle$ for $k \geq 1$;
FOR $i := 1$ TO $m$ DO

    $B := \text{findnew}(o, B \cup f_i, \langle F_2, \ldots, F_k\rangle, H)$;

    IF $B \neq \emptyset$ THEN RETURN $B$;

END;

RETURN $\emptyset$
\[ \langle \{ 3BF, HBF \} \rangle \mid \{ 3F, HF \} \mid \{ 1B, HB \} \mid \{ 1, H \}, 2, H, 3, H, 4, H \rangle \]
\[ \langle \{ 3BF, HBF \} \rangle \]
\begin{align*}
\langle \{ 3BF \} \{ HBF \} \rangle & \quad \{ 3F \} \{ HF \} \{ 4F \} \{ HF \} \\
& \quad \{ 1B \} \{ HB \} \{ 3B \} \{ HB \} \\
& \quad \{ 1 \} \{ H \} \{ 2 \} \{ H \} \{ 3 \} \{ H \} \{ 4 \} \{ H \} \end{align*}
\( \langle \{ \{ 3BF, HBF \} \} \rangle \) \( \mid \{ \{ 3F, HF \}, 4F, HF \} \rangle \) \( \mid \{ \{ 1B, HB \}, 3B, HB \} \rangle \) \( \mid \{ \{ 1, H \}, 2, H \} \) \( \{ 3, H \} \) \( \{ 4, H \} \rangle \)
\[
\langle \{ \{ 3BF, HBF \} \} \rangle \quad \{ \{ 3F, HF \}, \{ 4F, HF \} \} \quad \{ \{ 1B, HB \}, \{ 3B, HB \} \} \quad \{ \{ 1H \}, \{ 2H \}, \{ 3H \}, \{ 4H \} \} 
\]
\[
\langle \{ 3BF, HBF \} \rangle \quad \{ 3F, HF \} \quad \{ 1B, HB \} \quad \{ 1, H \} \\
\{ 4F, HF \} \quad \{ 4B, HB \} \quad \{ 2, H \} \\
\{ H \} \quad \{ 3, H \} \quad \{ 3, H \} \quad \{ 4, H \}
\]
\[
\langle \{ \{ 3BF \}, HBF \} \rangle = \{ \{ 3F, HF \}, \{ 4F, HF \} \} = \{ \{ 1B, HB \}, \{ 3B, HB \} \} = \{ \{ 1H \}, \{ 2H, 3H, 4H \} \}
\]
\[ \left\{ \begin{array}{c} 3BF \\ HBF \end{array} \right\} \mid \left\{ \begin{array}{c} 3F \\ HF \\ 4F \\ HF \end{array} \right\} \mid \left\{ \begin{array}{c} 1B \\ HB \\ 3B \\ HB \end{array} \right\} \mid \left\{ \begin{array}{c} 1 \\ H \\ 2 \\ H \\ 3 \\ H \\ 4 \\ H \end{array} \right\} \]
Complexity of finding new belief states

THEOREM Testing whether $G = \langle G_1, \ldots, G_n \rangle$ contains a belief state $B$ such that $\text{spreimg}_o(B)$ is not in $G$ is NP-complete. This holds also for deterministic operators $o$.

PROOF: Membership in NP is trivial: nondeterministically choose $s_i \in G_i$ for every $i \in \{1, \ldots, n\}$, compute the preimage $r$ of $s_1 \cup \cdots \cup s_n$, verify that $r \cap C_i$ for some $C_i$ is not in $G_i$. 
Complexity of finding new belief states

NP-hardness by reduction from SAT. We illustrate the proof by an example. Let \( T = \{ A \lor B \lor C, \neg A \lor B, \neg C \} \).

Construct FBS so that \( T \) is satisfiable iff strong preimage of \( o(x) = x_0 \) is not in the FBS: clause is mapped to the set of literals not in it; satisfying valuation = a new belief state.

\[
\langle \{\{\hat{A}, \hat{B}, \hat{C}\}, \{A, \hat{B}, C, \hat{C}\}, \{A, \hat{A}, B, \hat{B}, C\}\},
\{\{A_0\}, \{\hat{A}_0\}\},
\{\{B_0\}, \{\hat{B}_0\}\},
\{\{C_0\}, \{\hat{C}_0\}\} \rangle
\]
A planning algorithm: \( \text{plan}(I,O,G); \)

\[
H := \mathcal{F}(G); \text{ progress } := \text{ true;}
\]

\[
\text{WHILE progress and } I \notin \text{ flat}(H) \text{ DO}
\]

\[
\text{progress } := \text{ false;}
\]

\[
\text{FOR EACH } o \in O \text{ DO}
\]

\[
B := \text{findnew}(o,\emptyset,H,H);
\]

\[
\text{IF } B \neq \emptyset \text{ THEN}
\]

\[
\text{BEGIN}
\]

\[
H := H \oplus \mathcal{F}(\text{spreimg}_o(B));
\]

\[
\text{progress } := \text{ true;}
\]

\[
\text{END; END; END;}
\]

\[
\text{IF } I \in \text{ flat}(H) \text{ THEN plan found;}
\]

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