4. Nonmonotonic Reasoning

4.5 Cumulative and Preferential Logics

Bernhard Nebel

- Cumulativity
- Monotonic Consequence Relations
- Cumulative Models
- A Representation Theorem
- Preferential Consequence Relations
1. **Reflexivity**

   \[ \alpha \vdash \alpha \]

2. **Left Logical Equivalence**

   \[ \models \alpha \leftrightarrow \beta, \ \alpha \vdash \gamma \\
   \beta \vdash \gamma \]

3. **Right Weakening**

   \[ \models \alpha \rightarrow \beta, \ \gamma \vdash \alpha \\
   \gamma \vdash \beta \]

4. **Cut**

   \[ \alpha \land \beta \vdash \gamma, \ \alpha \vdash \beta \\
   \alpha \vdash \gamma \]

5. **Cautious Monotonicity**

   \[ \alpha \vdash \beta, \ \alpha \vdash \gamma \\
   \alpha \land \beta \vdash \gamma \]
**Cumulativity**

**Lemma.** The rules 4 & 5 can be equivalently characterized by

If $\alpha \models \beta$, then the sets of plausible conclusions from $\alpha$ and $\alpha \land \beta$ are identical (**cumulativity**).
Cumulativity

**Lemma.** The rules 4 & 5 can be equivalently characterized by

If $\alpha \not\sim \beta$, then the sets of plausible conclusions from $\alpha$ and $\alpha \land \beta$ are identical (*cumulativity*).

**Proof.**

$\Rightarrow$: Assume that 4 & 5 hold and assume that we have $\alpha \not\sim \beta$. 
Lemma. The rules 4 & 5 can be equivalently characterized by

If $\alpha \models \beta$, then the sets of plausible conclusions from $\alpha$ and $\alpha \land \beta$ are identical (cumulativity).

Proof.

$\Rightarrow$: Assume that 4 & 5 hold and assume that we have $\alpha \models \beta$. Now let $\gamma$ be a plausible consequence of $\alpha$. 
**Cumulativity**

**Lemma.** The rules 4 & 5 can be equivalently characterized by

If $\alpha \not\models \beta$, then the sets of plausible conclusions from $\alpha$ and $\alpha \land \beta$ are identical (cumulativity).

**Proof.**

$\Rightarrow$: Assume that 4 & 5 hold and assume that we have $\alpha \not\models \beta$. Now let $\gamma$ be a plausible consequence of $\alpha$. With rule 5 (CM), we have $\alpha \land \beta \not\models \gamma$. Similarly, from $\alpha \land \beta \not\models \gamma$ it follows with rule 4 (Cut) that $\alpha \not\models \gamma$. 
Lemma. The rules 4 & 5 can be equivalently characterized by

If $\alpha \models \sim \beta$, then the sets of plausible conclusions from $\alpha$ and $\alpha \land \beta$ are identical (cumulativity).

Proof.

$\Rightarrow$: Assume that 4 & 5 hold and assume that we have $\alpha \models \sim \beta$. Now let $\gamma$ be a plausible consequence of $\alpha$. With rule 5 (CM), we have $\alpha \land \beta \models \sim \gamma$. Similarly, from $\alpha \land \beta \models \sim \gamma$ it follows with rule 4 (Cut) that $\alpha \models \sim \gamma$. This means that the sets of plausible conclusions from $\alpha$ and $\alpha \land \beta$ are identical.
Lemma. The rules 4 & 5 can be equivalently characterized by

If $\alpha \not\vdash \beta$, then the sets of plausible conclusions from $\alpha$ and $\alpha \land \beta$ are identical (cumulativity).

Proof.

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$\Leftarrow$: Assume the cumulativity principle and assume that we have $\alpha \not\vdash \beta$. 
**Cumulativity**

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If $\alpha \not\vdash \beta$, then the sets of plausible conclusions from $\alpha$ and $\alpha \land \beta$ are identical (cumulativity).

**Proof.**

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$\Leftarrow$. Assume the cumulativity principle and assume that we have $\alpha \not\vdash \beta$. From that we can derive rule 4 and 5.
**Cumulativity**

**Lemma.** The rules 4 & 5 can be equivalently characterized by

If $\alpha \not\models \beta$, then the sets of plausible conclusions from $\alpha$ and $\alpha \land \beta$ are identical (cumulativity).

**Proof.**

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$\Leftarrow$. Assume the cumulativity principle and assume that we have $\alpha \not\models \beta$. From that we can derive rule 4 and 5.

**Note:** In the presence of rules 1 and 3, it follows that $\alpha \not\models \beta$, provided the sets of plausible conclusions from $\alpha$ and $\alpha \land \beta$ are identical.
Undesirable Properties (1)

- **Monotonicity:**

\[
\models \alpha \rightarrow \beta, \; \beta \sim \gamma \\
\therefore \alpha \sim \gamma
\]
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  \[
  \models \alpha \rightarrow \beta, \beta \sim \gamma \\
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  \]

- **Example:** Let us assume that John goes to the party *normally* implies Mary goes to the party.
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- Monotonicity:

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- Example: Let us assume that John goes to the party *normally implies* Mary goes to the party. Now we will probably not expect that John goes to the party *and* Joan goes to the party *normally implies* that Mary goes to the party.
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- **Monotonicity:**

  \[ \models \alpha \rightarrow \beta, \beta \not\models \gamma \]

  \[ \alpha \not\models \gamma \]

  - **Example:** Let us assume that *John goes to the party normally implies Mary goes to the party*. Now we will probably not expect that *John goes to the party and Joan goes to the party normally implies that Mary goes to the party*.

- **Contraposition:**

  \[ \alpha \not\models \beta \]

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• **Monotonicity:**

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• **Contraposition:**

\[ \alpha \not\models \beta \]
\[ \therefore \neg \beta \not\models \neg \alpha \]

○ **Example:** Let us assume that John goes to the party *normally implies* Mary goes to the party. Now we will probably not expect that *not* Mary goes to the party *normally implies* that *not* John goes to the party.
Undesirable Properties (2)

- Transitivity:

\[
\alpha \sim \beta, \, \beta \sim \gamma \\
\hline
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- Transitivity:

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\frac{\alpha \sim \beta, \beta \sim \gamma}{\alpha \sim \gamma}
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- Example: Let us assume that John goes to the party normally implies Mary goes to the party and that Mary goes to the party normally implies Jack goes to the party.
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- **Transitivity:**

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  \frac{\alpha \sim \beta, \beta \sim \gamma}{\alpha \sim \gamma}
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  - **Example:** Let us assume that John goes to the party *normally implies* Mary goes to the party and that Mary goes to the party *normally implies* Jack goes to the party. Now, should John goes to the party *normally imply* that Jack goes to the party?
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- **Transitivity**:
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  - **Example**: Let us assume that John goes to the party *normally* implies Mary goes to the party and that Mary goes to the party *normally implies* Jack goes to the party. Now, should John goes to the party *normally imply* that Jack goes to the party?

- **Easy Half of Deduction Theorem (EHD)**:
  \[
  \frac{\alpha \nvdash \beta \rightarrow \gamma}{\alpha \land \beta \nvdash \gamma}
  \]
Undesirable Properties (3)

**Theorem.** In the presence of the rules in system C, *monotonicity* and *EHD* are equivalent.
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*Monotonicity ⇒ EHD:*

- $\alpha \models \beta \rightarrow \gamma$ (assumption)
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- $\alpha \land \beta \vdash \beta \rightarrow \gamma$ (monotonicity)
- $\alpha \land \beta \vdash \alpha \land \beta$ (reflexivity)
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- $\alpha \land \beta \vdash \alpha \land \beta$ (reflexivity)
- $\alpha \land \beta \vdash \beta$ (right weakening)
Undesirable Properties (3)

Theorem. In the presence of the rules in system C, monotonicity and EHD are equivalent.

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• $\alpha \land \beta \vdash \gamma$ (MPC)
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**Theorem.** In the presence of the rules in system C, *monotonicity* and *EHD* are equivalent.

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*Monotonicity $\Leftarrow$ EHD:*

- $\alpha \models \beta \rightarrow \gamma$ (assumption)
- $\alpha \land \beta \models \beta \rightarrow \gamma$ (monotonicity)
- $\alpha \land \beta \models \alpha \land \beta$ (reflexivity)
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Undesirable Properties (3)

**Theorem.** In the presence of the rules in system C, *monotonicity* and *EHD* are equivalent.

**Proof.**

*Monotonicity ⇒ EHD:*

- \( \alpha \not\vdash \beta \to \gamma \) (assumption)
- \( \alpha \land \beta \not\vdash \beta \to \gamma \) (monotonicity)
- \( \alpha \land \beta \not\vdash \alpha \land \beta \) (reflexivity)
- \( \alpha \land \beta \not\vdash \beta \) (right weakening)
- \( \alpha \land \beta \not\vdash \gamma \) (MPC)

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- \( \models \alpha \to \beta, \beta \not\vdash \gamma \) (assumption)
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- \( \models \alpha \rightarrow \beta, \beta \vdash \gamma \) (assumption)
- \( \beta \vdash \alpha \rightarrow \gamma \) (right weakening)
- \( \beta \land \alpha \vdash \gamma \) (EHD)
- \( \alpha \vdash \gamma \) (left logical equivalence)
Undesirable Properties (4)

**Theorem.** In the presence of the rules in system C, *monotonicity* and *transitivity* are equivalent.
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- $\alpha \land \beta \not\sim \gamma$ (monotonicity)
**Undesirable Properties (4)**

**Theorem.** In the presence of the rules in system C, *monotonicity* and *transitivity* are equivalent.

**Proof.**

*Monotonicity $\Rightarrow$ transitivity:*

- $\alpha \models \beta, \beta \models \gamma$ (assumption)
- $\alpha \land \beta \models \gamma$ (monotonicity)
- $\alpha \models \gamma$ (cut)
**Theorem.** In the presence of the rules in system C, *monotonicity* and *transitivity* are equivalent.

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*Monotonicity $\Rightarrow$ transitivity:*

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*Monotonicity $\Leftarrow$ transitivity:*

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Theorem. In the presence of the rules in system C, *monotonicity* and *transitivity* are equivalent.

Proof.

\[
\begin{align*}
\text{Monotonicity } \Rightarrow \text{ transitivity:} & & \text{Monotonicity } \Leftarrow \text{ transitivity:} \\
\bullet & \alpha \vdash \beta, \beta \vdash \gamma \text{ (assumption)} & \bullet & \vdash \alpha \rightarrow \beta, \beta \vdash \gamma \text{ (assumption)} \\
\bullet & \alpha \land \beta \vdash \gamma \text{ (monotonicity)} & \bullet & \vdash \alpha \rightarrow \beta, \beta \vdash \gamma \text{ (assumption)} \\
\bullet & \alpha \vdash \gamma \text{ (cut)} & \bullet & \vdash \alpha \rightarrow \beta, \beta \vdash \gamma \text{ (assumption)}
\end{align*}
\]
Undesirable Properties (4)

Theorem. In the presence of the rules in system C, monotonicity and transitivity are equivalent.

Proof.

Monotonicity $\Rightarrow$ transitivity:

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- $\alpha \land \beta \not\rightarrow \gamma$ (monotonicity)
- $\alpha \not\rightarrow \gamma$ (cut)

Monotonicity $\Leftrightarrow$ transitivity:

- $\models \alpha \rightarrow \beta, \beta \not\rightarrow \gamma$ (assumption)
- $\alpha \models \beta$ (classical deduction theorem)
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**Theorem.** In the presence of the rules in system C, *monotonicity* and *transitivity* are equivalent.

**Proof.**

\[\text{Monotonicity } \Rightarrow \text{ transitivity:}\]

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- \(\alpha \land \beta \not\sim \gamma\) (monotonicity)
- \(\alpha \not\sim \gamma\) (cut)

\[\text{Monotonicity } \Leftarrow \text{ transitivity:}\]

- \(\models \alpha \rightarrow \beta, \beta \not\sim \gamma\) (assumption)
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Theorem. In the presence of the rules in system C, monotonicity and transitivity are equivalent.

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Monotonicity $\Leftrightarrow$ transitivity:

1. $\models \alpha \rightarrow \beta, \beta \not\sim \gamma$ (assumption)
2. $\alpha \models \beta$ (classical deduction theorem)
3. $\alpha \not\sim \beta$ (super classicality)
4. $\alpha \not\sim \gamma$ (transitivity)
Theorem. In the presence of *right weakening, contraposition* implies *monotonicity*. 
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Proof.

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Theorem. In the presence of *right weakening*, *contraposition* implies *monotonicity*.

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- $\models \alpha \rightarrow \beta, \beta \not\models \gamma$ (assumption)
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Theorem. In the presence of right weakening, contraposition implies monotonicity.

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- $\models \alpha \rightarrow \beta, \beta \not\models \gamma$ (assumption)
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- $\neg \gamma \not\models \neg \alpha$ (right weakening)
Undesirable Properties (5)

**Theorem.** In the presence of right weakening, contraposition implies monotonicity.

**Proof.**

- $\models \alpha \rightarrow \beta, \beta \not\models \gamma$ (assumption)
- $\not\models \gamma \not\models \neg \beta$ (contraposition)
- $\models \neg \beta \rightarrow \neg \alpha$ (classical contraposition)
- $\not\models \neg \gamma \not\models \neg \alpha$ (right weakening)
- $\alpha \not\models \gamma$ (contraposition)
Theorem. In the presence of right weakening, contraposition implies monotonicity.

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• $\models \alpha \rightarrow \beta, \beta \not\models \gamma$ (assumption)

• $\neg \gamma \not\models \neg \beta$ (contraposition)

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• $\neg \gamma \not\models \neg \alpha$ (right weakening)

• $\alpha \not\models \gamma$ (contraposition)

Note: Contraposition does not imply monotonicity, even in the presence of all rules of system C!
• Until now, we have only studied properties of $\sim$. 
Cumulative Closure (1)

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- How do we reason from $\varphi$ to $\psi$?
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- How do we reason from $\varphi$ to $\psi$?
- **Assumption**: We have a set $K$ of conditional statements of the form $\alpha_i \sim \beta_i$. The question is now: If we take rules in $K$ as granted, will it then be plausible to conclude $\psi$ if $\varphi$ is given?
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- **Idea**: We consider all cumulative consequence relations which contain $K$. 
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How do we reason from $\varphi$ to $\psi$?

**Assumption:** We have a set $K$ of conditional statements of the form $\alpha_i \sim \beta_i$. The question is now: If we take rules in $K$ as granted, will it then be plausible to conclude $\psi$ if $\varphi$ is given?

**Idea:** We consider all cumulative consequence relations which contain $K$.

**Further Idea:** We need to consider only the *minimal* cumulative consequence relations containing $K$. 
Lemma. Cumulative consequence relations are closed under intersection.
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**Proof.** Assume two cumulative consequence relations $\sim_1$ and $\sim_2$. If we now have the precondition of a rule satisfied by both relations, then the consequence must of course also be satisfied in both relations (since they are cumulative).
**Lemma.** Cumulative consequence relations are closed under intersection.

**Proof.** Assume two cumulative consequence relations $\mid\sim_1$ and $\mid\sim_2$. If we now have the precondition of a rule satisfied by both relations, then the consequence must of course also be satisfied in both relations (since they are cumulative).

**Theorem.** For each finite set of conditional statements $K$, there exists a unique smallest cumulative consequence relation containing $K$. 
Lemma. Cumulative consequence relations are closed under intersection.

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Theorem. For each finite set of conditional statements $K$, there exists a unique smallest cumulative consequence relation containing $K$.

Proof. Assume the contrary, i.e., there are incomparable minimal sets $K_1, \ldots, K_m$. Then $K = K_1 \cap \ldots \cap K_m$ is a unique smallest cumulative consequence relation containing $K$: contradiction.
Lemma. Cumulative consequence relations are closed under intersection.

Proof. Assume two cumulative consequence relations $\mathord\sim_1$ and $\mathord\sim_2$. If we now have the precondition of a rule satisfied by both relations, then the consequence must of course also be satisfied in both relations (since they are cumulative).

Theorem. For each finite set of conditional statements $\mathbf{K}$, there exists a unique smallest cumulative consequence relation containing $\mathbf{K}$.

Proof. Assume the contrary, i.e., there are incomparable minimal sets $\mathbf{K}_1, \ldots, \mathbf{K}_m$. Then $\mathbf{K} = \mathbf{K}_1 \cap \ldots \cap \mathbf{K}_m$ is a unique smallest cumulative consequence relation containing $\mathbf{K}$: contradiction.

This relation is called cumulative closure of $\mathbf{K}$, in symbols $\mathbf{K}^C$.
Cumulative Models – informal

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**Idea:** Cumulative models consists of states, which are ordered by a preference relation.

*States* characterize beliefs

The preference relation tries to capture the normality.
Cumulative Models – informal

• We will now try to characterize cumulative reasoning model theoretically.

• **Idea:** *Cumulative models* consists of *states*, which are ordered by a *preference relation*.

• *States* characterize beliefs

• The *preference relation* tries to capture the normality.

• We then will say: \( \alpha \models \beta \) is *accepted* in a model if in all most preferred states in which \( \alpha \) is true, also \( \beta \) is true.
• Let \( \prec \) be an arbitrary binary relation on the set \( U \). \( \prec \) is called **asymmetric** iff

\[
\forall s, t \in U.
\]

\[
s \prec t \text{ implies } t \not\prec s.
\]
Preference Relation

- Let $\prec$ be an arbitrary binary relation on the set $U$. $\prec$ is called **asymmetric** iff
  \[ s \prec t \implies t \not\prec s \quad \forall s, t, \in U. \]

- Let $V \subseteq U$ and $\prec$ be a binary relation on $U$.
  - $t \in V$ is **minimal** in $V$ iff $\forall s \in V : s \not\prec t$. 


**Preference Relation**

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- Let \( V \subseteq U \) and \( \prec \) be a binary relation on \( U \).
  
  o \( t \in V \) is **minimal** in \( V \) iff \( \forall s \in V : s \not\prec t \).

  o \( t \in V \) is a **minimum** of \( V \) (**smallest element** in \( V \)) iff \( \forall s \in V \) such that \( s \neq t : t \prec s \).
• Let $\prec$ be an arbitrary binary relation on the set $U$. $\prec$ is called **asymmetric** iff

$$s \prec t \text{ implies } t \not\prec s \ \forall s,t, \in U.$$ 

• Let $V \subseteq U$ and $\prec$ be a binary relation on $U$.
  
  o $t \in V$ is **minimal** in $V$ iff $\forall s \in V : s \not\prec t$.
  
  o $t \in V$ is a **minimum** of $V$ (**smallest element** in $V$) iff $\forall s \in V$ such that $s \neq t : t \prec s$.

• Let $P \subseteq U$ and $\prec$ be a binary relation on $U$. $P$ is **smooth** iff $\forall t \in P :$
  
  Either $t$ is minimal in $P$ or $\exists s \in P : s$ is minimal in $P$ and $s \prec t$. 

**Preference Relation**
Preference Relation

• Let \( \prec \) be an arbitrary binary relation on the set \( U \). \( \prec \) is called **asymmetric** iff

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\text{s \prec t implies } t \not\prec s \quad \forall s, t, \in U.
\]

• Let \( V \subseteq U \) and \( \prec \) be a binary relation on \( U \).
  
  o \( t \in V \) is **minimal** in \( V \) iff \( \forall s \in V : s \not\prec t \).
  
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• Let \( P \subseteq U \) and \( \prec \) be a binary relation on \( U \). \( P \) is **smooth** iff \( \forall t \in P : \) Either \( t \) is minimal in \( P \) or \( \exists s \in P : s \) is minimal in \( P \) and \( s \prec t \).

• **Note:** \( \prec \) is not partial order, but an arbitrary relation!
Cumulative Models – formal

- Let $\mathcal{U}$ be the set of all possible worlds (propositional interpretations).
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- The set of states satisfying $\alpha$ is denoted by $\hat{\alpha}$. 
Cumulative Models – formal

- Let $\mathcal{U}$ be the set of all possible worlds (propositional interpretations).
- A **cumulative model** $\mathcal{W}$ is a tuple $\langle S, l, \prec \rangle$, where
  - $S$ is a set of **states**,
  - $l$ is a mapping $l : S \rightarrow 2^\mathcal{U}$,
  - $\prec$ is an arbitrary **binary relation**, such that the **smoothness condition** is satisfied (see below).
- A state $s \in S$ **satisfies** a formula $\alpha$ ($s \models \alpha$) iff for all propositional interpretations $m \in l(s)$: $m \models \alpha$.
- The set of states satisfying $\alpha$ is denoted by $\hat{\alpha}$.
- **Smoothness condition:** A cumulative model satisfies this condition iff $\forall \alpha : \hat{\alpha}$ is smooth.
Consequence relation induced by a cumulative model

A cumulative model $W$ induces a consequence relation $\sim_W$ as follows:

$$\alpha \sim_W \beta \iff \text{for each minimal } s \text{ in } \hat{\alpha}: s \models \beta$$

**Example:** Model $W = \langle \{s_0, s_1, s_2, s_3\}, l, \prec \rangle$ with $s_0 \prec s_1 \prec s_2 \prec s_3$ (transitive!)
A cumulative model $W$ induces a consequence relation $\vdash_W$ as follows:

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**Example:** Model $W = \langle \{s_0, s_1, s_2, s_3 \}, l, \prec \rangle$ with $s_0 \prec s_1 \prec s_2 \prec s_3$ (transitive!)

$$l(s_0) = \{ \{\neg p, \neg b, \neg f \}, \{\neg p, \neg b, f \}, \{\neg p, b, \neg f \}, \{\neg p, b, f \} \}$$
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- $l(s_0) = \{\{\neg p, \neg b, \neg f\}, \{\neg p, \neg b, f\},$ \{\neg p, b, \neg f\}, \{\neg p, b, f\}\}$
- $l(s_1) = \{\{\neg p, b, f\}\}$
- $l(s_2) = \{\{p, b, \neg f\}\}$
- $l(s_3) = \{\{\neg p, \neg b, f\}, \{\neg p, \neg b, \neg f\}\}$

- Does $W$ satisfy the smoothness condition?
Consequence relation induced by a cumulative model

A cumulative model $W$ induces a consequence relation $\sim_W$ as follows:

$$\alpha \sim_W \beta \quad \text{iff} \quad \text{for each minimal } s \text{ in } \hat{\alpha}: s \models \beta$$

**Example:** Model $W = \langle \{s_0, s_1, s_2, s_3\}, l, \prec \rangle$ with $s_0 \prec s_1 \prec s_2 \prec s_3$ (transitive!)

$$
l(s_0) = \{\{\neg p, \neg b, \neg f\}, \{\neg p, \neg b, f\},$
$$\quad \{\neg p, b, \neg f\}, \{\neg p, b, f\}\}$$

$$l(s_1) = \{\{\neg p, b, f\}\}$$

$$l(s_2) = \{\{p, b, \neg f\}\}$$

$$l(s_3) = \{\{\neg p, \neg b, f\}, \{\neg p, \neg b, \neg f\}\}$$

- Does $W$ satisfy the smoothness condition?
- Which pairs are in $\sim_W$?
Soundness (1)

**Theorem.** If $W$ is a cumulative model, then $\sim_W$ is a cumulative consequence relation.
Soundness (1)

Theorem. If $W$ is a cumulative model, then $\sim_W$ is a cumulative consequence relation.

Proof.

- Reflexivity:
Soundness (1)

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**Proof.**

- *Reflexivity*: satisfied $\sqrt{\text{.}}$. 
Soundness (1)

Theorem. If $W$ is a cumulative model, then $\sim_W$ is a cumulative consequence relation.

Proof.

- Reflexivity: satisfied √.

- Left logical equivalence:
Soundness (1)

**Theorem.** If $W$ is a cumulative model, then $\sim_W$ is a cumulative consequence relation.

**Proof.**

- **Reflexivity:** satisfied $\sqrt{\cdot}$.

- **Left logical equivalence:** satisfied $\sqrt{\cdot}$.
Soundness (1)

**Theorem.** If $W$ is a cumulative model, then $\sim_W$ is a cumulative consequence relation.

**Proof.**

- **Reflexivity:** satisfied $\checkmark$.

- **Left logical equivalence:** satisfied $\checkmark$.

- **Right weakening:**
**Soundness (1)**

**Theorem.** If $W$ is a cumulative model, then $\sim_W$ is a cumulative consequence relation.

**Proof.**

- **Reflexivity:** satisfied $\checkmark$.

- **Left logical equivalence:** satisfied $\checkmark$.

- **Right weakening:** satisfied $\checkmark$. 
Theorem. If $W$ is a cumulative model, then $\sim_W$ is a cumulative consequence relation.

Proof.

- **Reflexivity**: satisfied $\sqrt{\cdot}$.
  
  - **Left logical equivalence**: satisfied $\sqrt{\cdot}$.
  
  - **Right weakening**: satisfied $\sqrt{\cdot}$.
  
  - **Cut:**
Soundness (1)

**Theorem.** If $W$ is a cumulative model, then $\sim_W$ is a cumulative consequence relation.

**Proof.**

- **Reflexivity:** satisfied ✓
- **Left logical equivalence:** satisfied ✓
- **Right weakening:** satisfied ✓
- **Cut:** $\alpha \land \beta \vdash \gamma$, $\alpha \vdash \beta \Rightarrow \alpha \vdash \gamma$. 
Soundness (1)

Theorem. If $W$ is a cumulative model, then $\sim_W$ is a cumulative consequence relation.

Proof.

- **Reflexivity:** satisfied $\sqrt{}$.

- **Left logical equivalence:** satisfied $\sqrt{}$.

- **Right weakening:** satisfied $\sqrt{}$.

- **Cut:** $\alpha \land \beta \models \gamma$, $\alpha \models \beta \Rightarrow \alpha \models \gamma$. Assume, all minimal elements of $\hat{\alpha}$ satisfy $\beta$. 

Soundness (1)

**Theorem.** If $W$ is a cumulative model, then $\sim_W$ is a cumulative consequence relation.

**Proof.**

- *Reflexivity:* satisfied $\checkmark$.

- *Left logical equivalence:* satisfied $\checkmark$.

- *Right weakening:* satisfied $\checkmark$.

- *Cut:* $\alpha \land \beta \sim \gamma$, $\alpha \sim \beta \Rightarrow \alpha \sim \gamma$. Assume, all minimal elements of $\hat{\alpha}$ satisfy $\beta$, and all minimal elements of $\hat{\alpha} \land \hat{\beta}$ satisfy $\gamma$. 
**Soundness (1)**

**Theorem.** If $W$ is a cumulative model, then $|\sim W$ is a cumulative consequence relation.

**Proof.**

- **Reflexivity:** satisfied $\sqrt{\cdot}$.

- **Left logical equivalence:** satisfied $\sqrt{\cdot}$.

- **Right weakening:** satisfied $\sqrt{\cdot}$.

- **Cut:** $\alpha \land \beta \vdash (\alpha \vdash \beta \Rightarrow \alpha \vdash \gamma)$. Assume, all minimal elements of $\hat{\alpha}$ satisfy $\beta$, and all minimal elements of $\hat{\alpha} \land \hat{\beta}$ satisfy $\gamma$. Each minimal element of $\hat{\alpha}$ satisfies $\alpha \land \beta$. 
Theorem. If $W$ is a cumulative model, then $\sim_W$ is a cumulative consequence relation.

Proof.

• **Reflexivity:** satisfied $\sqrt{\ }$.

• **Left logical equivalence:** satisfied $\sqrt{\ }$.

• **Right weakening:** satisfied $\sqrt{\ }$.

• **Cut:** $\alpha \land \beta \mid \sim \gamma, \alpha \mid \sim \beta \Rightarrow \alpha \mid \sim \gamma$. Assume, all minimal elements of $\hat{\alpha}$ satisfy $\beta$, and all minimal elements of $\hat{\alpha} \land \hat{\beta}$ satisfy $\gamma$. Each minimal element of $\hat{\alpha}$ satisfies $\alpha \land \beta$. Since $\hat{\alpha} \land \hat{\beta} \subseteq \hat{\alpha}$
Soundness (1)

Theorem. If $W$ is a cumulative model, then $\sim_W$ is a cumulative consequence relation.

Proof.

• **Reflexivity:** satisfied $\checkmark$.

• **Left logical equivalence:** satisfied $\checkmark$.

• **Right weakening:** satisfied $\checkmark$.

• **Cut:** $\alpha \land \beta \sim \gamma, \alpha \sim \beta \Rightarrow \alpha \sim \gamma$. Assume, all minimal elements of $\hat{\alpha}$ satisfy $\beta$, and all minimal elements of $\hat{\alpha} \land \hat{\beta}$ satisfy $\gamma$. Each minimal element of $\hat{\alpha}$ satisfies $\alpha \land \beta$. Since $\hat{\alpha} \land \hat{\beta} \subseteq \hat{\alpha}$, all minimal elements of $\hat{\alpha}$ are also minimal elements of $\hat{\alpha} \land \hat{\beta}$.
Theorem. If $W$ is a cumulative model, then $\sim_W$ is a cumulative consequence relation.

Proof.

- **Reflexivity:** satisfied $\sqrt{\phantom{a}}$.

- **Left logical equivalence:** satisfied $\sqrt{\phantom{a}}$.

- **Right weakening:** satisfied $\sqrt{\phantom{a}}$.

- **Cut:** $\alpha \land \beta \sim \gamma$, $\alpha \sim \beta \Rightarrow \alpha \sim \gamma$. Assume, all minimal elements of $\hat{\alpha}$ satisfy $\beta$, and all minimal elements of $\hat{\alpha} \land \hat{\beta}$ satisfy $\gamma$. Each minimal element of $\hat{\alpha}$ satisfies $\alpha \land \beta$. Since $\hat{\alpha} \land \hat{\beta} \subseteq \hat{\alpha}$, all minimal elements of $\hat{\alpha}$ are also minimal elements of $\hat{\alpha} \land \hat{\beta}$. This means $\alpha \sim_W \gamma$. 


Soundness (2)

- **Cautious Monotonicity**: To show: $\alpha \vdash \beta$, $\alpha \vdash \gamma \Rightarrow \alpha \land \beta \vdash \gamma$. 
Soundness (2)

- **Cautious Monotonicity**: To show: $\alpha \not\models \beta, \alpha \not\models \gamma \Rightarrow \alpha \land \beta \not\models \gamma$.
  
  Assume $\alpha \not\models_W \beta$ and $\alpha \not\models_W \gamma$.  

Soundness (2)

- **Cautious Monotonicity**: To show: \( \alpha \not\models \beta, \alpha \not\models \gamma \Rightarrow \alpha \land \beta \not\models \gamma \).

  Assume \( \alpha \not\models W \beta \) and \( \alpha \not\models W \gamma \). We have to show: \( \alpha \land \beta \not\models W \gamma \), i.e., for all minimal \( s \in \overline{\alpha \land \beta} \), \( s \models W \gamma \).
Soundness (2)

- **Cautious Monotonicity:** To show: $\alpha \models \beta$, $\alpha \models \gamma \Rightarrow \alpha \wedge \beta \models \gamma$.

  Assume $\alpha \models_W \beta$ and $\alpha \models_W \gamma$. We have to show: $\alpha \wedge \beta \models_W \gamma$, i.e., for all minimal $s \in \hat{\alpha} \wedge \beta$, $s \models \gamma$.

  We know that all minimal $s \in \hat{\alpha} \wedge \beta$ are in $\hat{\alpha}$. 
• **Cautious Monotonicity:** To show: \( \alpha \not\models \beta, \alpha \not\models \gamma \Rightarrow \alpha \land \beta \not\models \gamma \).

Assume \( \alpha \not\models_w \beta \) and \( \alpha \not\models_w \gamma \). We have to show: \( \alpha \land \beta \not\models_w \gamma \), i.e., for all minimal \( s \in \overline{\alpha \land \beta} \), \( s \models \gamma \).

We know that all minimal \( s \in \overline{\alpha \land \beta} \) are in \( \hat{\alpha} \). We will show that they are all minimal in \( \hat{\alpha} \).
Soundness (2)

- **Cautious Monotonicity**: To show: $\alpha \not\models \beta$, $\alpha \not\models \gamma \Rightarrow \alpha \land \beta \not\models \gamma$.

  Assume $\alpha \not\models_W \beta$ and $\alpha \not\models_W \gamma$. We have to show: $\alpha \land \beta \not\models_W \gamma$, i.e., for all minimal $s \in \widehat{\alpha \land \beta}$, $s \models \gamma$.

  We know that all minimal $s \in \widehat{\alpha \land \beta}$ are in $\hat{\alpha}$. We will show that they are all minimal in $\hat{\alpha}$.

  **Assumption**: There exists an $s$ minimal in $\widehat{\alpha \land \beta}$ that is not minimal in $\hat{\alpha}$.
Soundness (2)

• **Cautious Monotonicity:** To show: $\alpha \vdash \beta$, $\alpha \vdash \gamma \Rightarrow \alpha \land \beta \vdash \gamma$.

Assume $\alpha \vdash_W \beta$ and $\alpha \vdash_W \gamma$. We have to show: $\alpha \land \beta \vdash_W \gamma$, i.e., for all minimal $s \in \widehat{\alpha \land \beta}$, $s \equiv \gamma$.

We know that all minimal $s \in \widehat{\alpha \land \beta}$ are in $\widehat{\alpha}$. We will show that they are all minimal in $\widehat{\alpha}$.

**Assumption:** There exists an $s$ minimal in $\widehat{\alpha \land \beta}$ that is not minimal in $\widehat{\alpha}$. Because of the **smoothness condition** there exists $s' \in \widehat{\alpha}$, such that $s' \prec s$. 
Soundness (2)

- **Cautious Monotonicity**: To show: \( \alpha \not\leadsto \beta, \alpha \not\leadsto \gamma \Rightarrow \alpha \land \beta \not\leadsto \gamma. \)

  Assume \( \alpha \not\leadsto_W \beta \) and \( \alpha \not\leadsto_W \gamma \). We have to show: \( \alpha \land \beta \not\leadsto_W \gamma \), i.e., for all minimal \( s \in \widehat{\alpha \land \beta} \), \( s \models \gamma \).

  We know that all minimal \( s \in \widehat{\alpha \land \beta} \) are in \( \hat{\alpha} \). We will show that they are all minimal in \( \hat{\alpha} \).

  **Assumption**: There exists an \( s \) minimal in \( \widehat{\alpha \land \beta} \) that is not minimal in \( \hat{\alpha} \). Because of the *smoothness condition* there exists \( s' \in \hat{\alpha} \), such that \( s' \prec s \). We know, however, that \( s' \models \beta \), which means \( s' \in \widehat{\alpha \land \beta} \).
Soundness (2)

- **Cautious Monotonicity**: To show: $\alpha \not\leq \beta, \alpha \not\leq \gamma \Rightarrow \alpha \land \beta \not\leq \gamma$.

Assume $\alpha \not\leq W \beta$ and $\alpha \not\leq W \gamma$. We have to show: $\alpha \land \beta \not\leq W \gamma$, i.e., for all minimal $s \in \alpha \land \beta$, $s \equiv \gamma$.

We know that all minimal $s \in \alpha \land \beta$ are in $\hat{\alpha}$. We will show that they are all minimal in $\hat{\alpha}$.

**Assumption**: There exists an $s$ minimal in $\alpha \land \beta$ that is not minimal in $\hat{\alpha}$. Because of the *smoothness condition* there exists $s' \in \hat{\alpha}$, such that $s' \prec s$. We know, however, that $s' \equiv \beta$, which means $s' \in \alpha \land \beta$. This implies that $s$ is not minimal in $\alpha \land \beta$. 
Soundness (2)

- **Cautious Monotonicity**: To show: $\alpha \models \beta$, $\alpha \models \gamma \Rightarrow \alpha \land \beta \models \gamma$.

  Assume $\alpha \models_W \beta$ and $\alpha \models_W \gamma$. We have to show: $\alpha \land \beta \models_W \gamma$, i.e., for all minimal $s \in \hat{\alpha} \land \beta$, $s \models \gamma$.

  We know that all minimal $s \in \hat{\alpha} \land \beta$ are in $\hat{\alpha}$. We will show that they are all minimal in $\hat{\alpha}$.

  **Assumption**: There exists an $s$ minimal in $\hat{\alpha} \land \beta$ that is not minimal in $\hat{\alpha}$. Because of the smoothness condition there exists $s' \in \hat{\alpha}$, such that $s' \prec s$. We know, however, that $s' \models \beta$, which means $s' \in \hat{\alpha} \land \beta$. This implies that $s$ is not minimal in $\hat{\alpha} \land \beta$. **Contradiction!**
Soundness (2)

- **Cautious Monotonicity:** To show: \( \alpha \vdash \beta, \alpha \vdash \gamma \Rightarrow \alpha \land \beta \vdash \gamma. \)

Assume \( \alpha \vdash_W \beta \) and \( \alpha \vdash_W \gamma \). We have to show: \( \alpha \land \beta \vdash_W \gamma \), i.e., for all minimal \( s \in \widehat{\alpha \land \beta} \), \( s \models \gamma \).

We know that all minimal \( s \in \widehat{\alpha \land \beta} \) are in \( \widehat{\alpha} \). We will show that they are all minimal in \( \widehat{\alpha} \).

**Assumption:** There exists an \( s \) minimal in \( \widehat{\alpha \land \beta} \) that is not minimal in \( \widehat{\alpha} \). Because of the **smoothness condition** there exists \( s' \in \widehat{\alpha} \), such that \( s' \prec s \). We know, however, that \( s' \models \beta \), which means \( s' \in \widehat{\alpha \land \beta} \). This implies that \( s \) is not minimal in \( \widehat{\alpha \land \beta} \). **Contradiction!** This means that \( s \) must be minimal in \( \widehat{\alpha} \),
• **Cautious Monotonicity**: To show: $\alpha \not\sim_\beta$, $\alpha \not\sim_\gamma \Rightarrow \alpha \land \beta \not\sim_\gamma$.

Assume $\alpha \not\sim_W \beta$ and $\alpha \not\sim_W \gamma$. We have to show: $\alpha \land \beta \not\sim_W \gamma$, i.e., for all minimal $s \in \hat{\alpha} \land \hat{\beta}$, $s \equiv \gamma$.

We know that all minimal $s \in \hat{\alpha} \land \hat{\beta}$ are in $\hat{\alpha}$. We will show that they are all minimal in $\hat{\alpha}$.

**Assumption**: There exists an $s$ minimal in $\hat{\alpha} \land \hat{\beta}$ that is not minimal in $\hat{\alpha}$. Because of the *smoothness condition* there exists $s' \in \hat{\alpha}$, such that $s' \prec s$. We know, however, that $s' \equiv \beta$, which means $s' \in \hat{\alpha} \land \hat{\beta}$. This implies that $s$ is not minimal in $\hat{\alpha} \land \hat{\beta}$. **Contradiction!** This means that $s$ must be minimal in $\hat{\alpha}$, i.e. $s \equiv \gamma$. 

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**Soundness (2)**
Soundness (2)

- **Cautious Monotonicity:** To show: $\alpha \vdash \beta$, $\alpha \vdash \gamma \Rightarrow \alpha \land \beta \vdash \gamma$.

  Assume $\alpha \vdash_W \beta$ and $\alpha \vdash_W \gamma$. We have to show: $\alpha \land \beta \vdash_W \gamma$, i.e., for all minimal $s \in \widehat{\alpha \land \beta}$, $s \equiv \gamma$.

  We know that all minimal $s \in \widehat{\alpha \land \beta}$ are in $\hat{\alpha}$. We will show that they are all minimal in $\hat{\alpha}$.

  **Assumption:** There exists an $s$ minimal in $\widehat{\alpha \land \beta}$ that is not minimal in $\hat{\alpha}$. Because of the *smoothness condition* there exists $s' \in \hat{\alpha}$, such that $s' \prec s$. We know, however, that $s' \equiv \beta$, which means $s' \in \widehat{\alpha \land \beta}$. This implies that $s$ is not minimal in $\widehat{\alpha \land \beta}$. **Contradiction!** This means that $s$ must be minimal in $\hat{\alpha}$, i.e. $s \equiv \gamma$. Because this is true for all minimal elements in $\widehat{\alpha \land \beta}$, it follows that $\alpha \land \beta \vdash_W \gamma$. 