Principles of Knowledge Representation and Reasoning

4. Nonmonotonic Reasoning

4.5 Introduction to Cumulative Logics

Bernhard Nebel

- Motivation
- Cumulative Consequence Relations
- Some Derived Rules
Motivation

- Conventional NM logics are based on (ad hoc) modifications of the logical machinery (proofs/models)
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• The only general characterization is negative
Motivation

- Conventional NM logics are based on (ad hoc) modifications of the logical machinery (proofs/models).

- The only general characterization is **negative**: If we have $\Theta \models \varphi$, we do not have necessarily $\Theta \cup \{\psi\} \models \varphi$.

- It would be much more enlightening to have a **positive** characterization.
Plausible Consequences

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• Now we will study the relation of plausible consequence (on top of classical propositional logic):
  $\alpha \not\models \beta$ (speak: “$alpha$ snake $beta$” or “if $alpha$ then normally $beta$” or “$alpha$ normally implies $beta$”).

• We will characterize this relation positively, e.g., if we have $\alpha \not\models \beta$ and $\alpha \not\models \gamma$, then we must have $\alpha \not\models \beta \land \gamma$. 

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→ Write down all such postulates!
Plausible Consequences

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• Now we will study the relation of plausible consequence (on top of classical propositional logic):
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• We will characterize this relation positively, e.g., if we have \( \alpha \not\models \beta \) and \( \alpha \not\models \gamma \), then we must have \( \alpha \not\models \beta \land \gamma \).

\[ \rightarrow \] Write down all such postulates!

\[ \sim \] Perhaps, we can even come up with a semantic specification that is equivalent
1. Reflexivity:

\[ \alpha \sim \alpha \]
Desirable Properties of Nonmonotonic Reasoning (1)

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\[ \alpha \vdash \neg \alpha \]

- **Rationale**: If \( \alpha \) holds, this *normally implies* \( \alpha \).
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- **Example**: When *Tom goes to the party* holds, then this *normally implies* that *Tom goes to the party*. 

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1. **Reflexivity**: 

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2. **Left Logical Equivalence**:

\[ \models \alpha \leftrightarrow \beta, \alpha \models \gamma \]

\[ \beta \models \sim \gamma \]
Desirable Properties of Nonmonotonic Reasoning (1)

1. **Reflexivity:**
   
   \[ \alpha \not\sim \alpha \]

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2. **Left Logical Equivalence:**
   
   \[ \models \alpha \leftrightarrow \beta, \ \alpha \not\sim \gamma \]

   \[ \beta \not\sim \gamma \]

   - **Rationale:** It is not the syntactic form, but the logical contents that is responsible for what we conclude normally.
Desirable Properties of Nonmonotonic Reasoning (1)

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\[ \alpha \vdash \alpha \]

○ Rationale: If \( \alpha \) holds, this *normally implies* \( \alpha \).

○ Example: When Tom goes to the party holds, then this *normally implies* that Tom goes to the party.

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○ Example: Let us assume that Tom goes to the party or Peter goes to the party *normally implies* that Mary goes.
Desirable Properties of Nonmonotonic Reasoning (1)

1. Reflexivity:

\[ \alpha \sim \alpha \]

- **Rationale**: If \( \alpha \) holds, this *normally implies* \( \alpha \).
- **Example**: When Tom goes to the party holds, then this *normally implies* that Tom goes to the party.

2. Left Logical Equivalence:

\[ \models \alpha \leftrightarrow \beta, \alpha \sim \gamma \]
\[ \beta \sim \gamma \]

- **Rationale**: It is not the syntactic form, but the logical contents that is responsible for what we conclude normally.
- **Example**: Let us assume that Tom goes to the party or Peter goes to the party *normally implies* that Mary goes. Then we would expect that Peter goes to the party or Tom goes to the party *normally implies* that Mary goes.
3. **Right Weakening:**

\[ \models \alpha \rightarrow \beta, \ \gamma \models \alpha \]

\[ \therefore \gamma \models \beta \]
3. **Right Weakening:**

\[
\models \alpha \rightarrow \beta, \gamma \models \alpha \\
\therefore \gamma \models \beta
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- **Rationale:** If something can be concluded normally, then everything classically implied should also be concluded normally.
3. **Right Weakening:**

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\[ \gamma \models \beta \]

- **Rationale:** If something can be concluded normally, then everything classically implied should also be concluded normally.

- **Example:** Let us assume that *if* Mary goes to the party *then normally* Cleve goes to the party *and* John goes to the party.
3. **Right Weakening:**

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\models \alpha \rightarrow \beta, \; \gamma \vdash \alpha \\
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\]

- **Rationale:** If something can be concluded normally, then everything classically implied should also be concluded normally.

- **Example:** Let us assume that *if* Mary goes to the party *then normally* Cleve goes to the party *and* John goes to the party. Then we would expect that Mary goes to the party *normally implies* that Cleve goes to the party.
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\frac{\models \alpha \rightarrow \beta, \gamma \models \alpha}{\gamma \models \beta}
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- **Rationale:** If something can be concluded normally, then everything classically implied should also be concluded normally.

- **Example:** Let us assume that *if* Mary goes to the party *then normally* Cleve goes to the party *and* John goes to the party. Then we would expect that Mary goes to the party *normally implies* that Cleve goes to the party.

- **Consequences of rules 1–3:**
  - From 1 & 3 we can conclude **super classicality**:

\[
\frac{\alpha \models \beta}{\alpha \models \beta}
\]
3. **Right Weakening:**

\[ \models \alpha \rightarrow \beta, \; \gamma \models \alpha \]

\[ \gamma \models \beta \]

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- **Example:** Let us assume that *if* Mary goes to the party *then normally* Cleve goes to the party *and* John goes to the party. Then we would expect that Mary goes to the party *normally implies* that Cleve goes to the party.

- **Consequences of rules 1–3:**
  - From 1 & 3 we can conclude **super classicality**:

\[ \alpha \models \beta \]

\[ \alpha \models \beta \]

- All known nonmonotonic reasoning systems satisfy 1–3!
Desirable Properties of Nonmonotonic Reasoning (3)

4. **Cut:**

\[
\alpha \land \beta \not\models \gamma, \quad \alpha \not\models \beta
\]

\[
\alpha \not\models \gamma
\]
Desirable Properties of Nonmonotonic Reasoning (3)

4. Cut:

\[
\alpha \land \beta \not\sim \gamma, \ \alpha \not\sim \beta
\]

\[
\alpha \not\sim \gamma
\]

- **Rationale**: If part of the premise is plausibly implied by another part of the premise, then the latter is enough for the plausible conclusion.
4. **Cut:**

\[
\alpha \land \beta \mid \sim \gamma, \quad \alpha \mid \sim \beta \\
\hline
\alpha \mid \sim \gamma
\]

- **Rationale:** If part of the premise is plausibly implied by another part of the premise, then the latter is enough for the plausible conclusion.

- **Example:** Assume that John goes to the party and Mary goes to the party *normally implies* Cleve goes to the party.
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\[ \alpha \vdash \gamma \]

- **Rationale**: If part of the premise is plausibly implied by another part of
the premise, then the latter is enough for the plausible conclusion.

- **Example**: Assume that John goes to the party and Mary goes to the party normally implies Cleve goes to the party. Assume further that John goes to the party normally implies Mary goes to the party.
4. **Cut:**

\[
\alpha \land \beta \models \gamma, \quad \alpha \models \beta \\
\alpha \models \gamma
\]

- **Rationale:** If part of the premise is plausibly implied by another part of the premise, then the latter is enough for the plausible conclusion.

- **Example:** Assume that John goes to the party and Mary goes to the party normally implies Cleve goes to the party. Assume further that John goes to the party normally implies Mary goes to the party. Then we would expect that John goes to the party alone already normally implies that Cleve goes to the party.

- **Note:** Monotonic logics as well as sceptical reasoning in default logic satisfy this principle.
Desirable Properties of Nonmonotonic Reasoning (4)

5. **Cautious Monotonicity:**

\[
\alpha \not\models \beta, \quad \alpha \not\models \gamma \\
\frac{}{\alpha \land \beta \not\models \gamma}
\]
Desirable Properties of Nonmonotonic Reasoning (4)

5. **Cautious Monotonicity:**

\[
\frac{\alpha \models \neg \beta, \alpha \models \neg \gamma}{\alpha \land \beta \models \neg \gamma}
\]

- **Rationale:** In general, we do not want monotonicity. However, if all new premises are already plausible conclusions, we want monotonicity in the conclusion!
5. **Cautious Monotonicity:**

\[
\alpha \not\models \beta, \alpha \not\models \gamma \\
\alpha \land \beta \not\models \gamma
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- **Rationale:** In general, we do not want monotonicity. However, if all new premises are already plausible conclusions, we want monotonicity in the conclusion!

- **Example:** Assume that Mary goes to the party *normally implies* that Cleve goes to the party and assume that Mary goes to the party *normally implies* that John goes to the party.
5. **Cautious Monotonicity:**

\[ \alpha \vdash \beta, \alpha \vdash \gamma \]

\[ \alpha \land \beta \vdash \gamma \]

- **Rationale:** In general, we do not want monotonicity. However, if all new premises are already plausible conclusions, we want monotonicity in the conclusion!

- **Example:** Assume that Mary goes to the party *normally implies* that Cleve goes to the party and assume that Mary goes to the party *normally implies* that John goes to the party. Then Mary goes to the party *and* Jack goes to the party might not *normally imply* that John goes to the party.
Desirable Properties of Nonmonotonic Reasoning (4)

5. **Cautious Monotonicity:**

\[
\alpha \models \sim \beta, \; \alpha \models \sim \gamma \\
\alpha \land \beta \models \sim \gamma
\]

- **Rationale:** In general, we do not want monotonicity. However, if all new premises are already plausible conclusions, we want monotonicity in the conclusion!

- **Example:** Assume that Mary goes to the party *normally implies* that Cleve goes to the party and assume that Mary goes to the party *normally implies* that John goes to the party. Then Mary goes to the party *and* Jack goes to the party might not *normally imply* that John goes to the party. However, Mary goes to the party *and* Cleve goes to the party *should normally imply* that John goes to the party.

- **Note:** DL does not satisfy this principle, but sceptical reasoning in Theorist does.
Lemma. The rules 4 & 5 can be equivalently characterized by

If $\alpha \not\models \beta$, then the sets of plausible conclusions from $\alpha$ and $\alpha \land \beta$ are identical.
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- This property is also called **cumulativity**.
Desirable Properties of Nonmonotonic Reasoning (5)

**Lemma.** The rules 4 & 5 can be equivalently characterized by

If $\alpha \models \beta$, then the sets of plausible conclusions from $\alpha$ and $\alpha \land \beta$ are identical.

- This property is also called **cumulativity**.
- Each consequence relation satisfying rules 1–5 will be called **cumulative consequence relation**.
Desirable Properties of Nonmonotonic Reasoning (5)

Lemma. The rules 4 & 5 can be equivalently characterized by

If $\alpha \not\models \beta$, then the sets of plausible conclusions from $\alpha$ and $\alpha \land \beta$ are identical.

- This property is also called **cumulativity**.
- Each consequence relation satisfying rules 1–5 will be called **cumulative consequence relation**.
- The system of rules 1–5 is denoted by $C$. 
The System C

1. **Reflexivity**

   \[ \alpha \vdash \alpha \]

2. **Left Logical Equivalence**

   \[ \models \alpha \leftrightarrow \beta, \ \alpha \vdash \gamma \]

   \[ \beta \vdash \gamma \]

3. **Right Weakening**

   \[ \models \alpha \to \beta, \ \gamma \vdash \alpha \]

   \[ \gamma \vdash \beta \]

4. **Cut**

   \[ \alpha \land \beta \vdash \gamma, \ \alpha \vdash \beta \]

   \[ \alpha \vdash \gamma \]

5. **Cautious Monotonicity**

   \[ \alpha \vdash \beta, \ \alpha \vdash \gamma \]

   \[ \alpha \land \beta \vdash \gamma \]
Derived Rules in C

• Equivalence:

\[
\begin{align*}
\alpha &\sim \beta, \quad \beta \sim \alpha, \quad \alpha \sim \gamma \\
\beta &\sim \gamma
\end{align*}
\]
Derived Rules in C

- **Equivalence:**
  \[
  \alpha \sim \beta, \beta \sim \alpha, \alpha \sim \gamma
  \]
  \[
  \beta \sim \gamma
  \]

- **And:**
  \[
  \alpha \sim \beta, \alpha \sim \gamma
  \]
  \[
  \alpha \sim \beta \land \gamma
  \]
Derived Rules in C

- **Equivalence:**
  \[
  \alpha \models \sim \beta, \ \beta \models \sim \alpha, \ \alpha \models \sim \gamma \\
  \beta \models \sim \gamma
  \]

- **And:**
  \[
  \alpha \models \sim \beta, \ \alpha \models \sim \gamma \\
  \alpha \models \sim \beta \land \gamma
  \]

- **MPC:**
  \[
  \alpha \models \sim \beta \rightarrow \gamma, \ \alpha \models \sim \beta \\
  \alpha \models \sim \gamma
  \]
Proofs

Equivalence:

Assumption: \( \alpha \sim \beta, \beta \sim \alpha, \alpha \sim \gamma \)
Proofs

Equivalence:

Assumption: \( \alpha \vdash \beta, \beta \vdash \alpha, \alpha \vdash \gamma \) (\& we want: \( \beta \vdash \gamma \))
Proofs

Equivalence:

Assumption: \( \alpha \models \sim \beta, \beta \models \sim \alpha, \alpha \models \sim \gamma \) (& we want: \( \beta \models \sim \gamma \))

Cautious monotonicity: \( \alpha \land \beta \models \sim \gamma \)
Proofs

Equivalence:

Assumption: \( \alpha \not\sim \beta, \beta \not\sim \alpha, \alpha \not\sim \gamma \) (& we want: \( \beta \not\sim \gamma \))

Cautious monotonicity: \( \alpha \land \beta \not\sim \gamma \)

Left equivalence: \( \beta \land \alpha \not\sim \gamma \)
Proofs

Equivalence:

Assumption: \( \alpha \not\models \beta, \beta \not\models \alpha, \alpha \not\models \gamma \) (\& we want: \( \beta \not\models \gamma \))

Cautious monotonicity: \( \alpha \land \beta \not\models \gamma \)

Left equivalence: \( \beta \land \alpha \not\models \gamma \)

Cut: \( \beta \not\models \gamma \)
Proofs

Equivalence:

Assumption: $\alpha \not\sim \beta, \beta \not\sim \alpha, \alpha \not\sim \gamma$ (& we want: $\beta \not\sim \gamma$)

Cautious monotonicity: $\alpha \land \beta \not\sim \gamma$

Left equivalence: $\beta \land \alpha \not\sim \gamma$

Cut: $\beta \not\sim \gamma$

And:

Assumption: $\alpha \not\sim \beta, \alpha \not\sim \gamma$
Proofs

Equivalence:

Assumption: $\alpha \not\sim \beta, \beta \not\sim \alpha, \alpha \not\sim \gamma$ (\& we want: $\beta \not\sim \gamma$)

Cautious monotonicity: $\alpha \land \beta \not\sim \gamma$

Left equivalence: $\beta \land \alpha \not\sim \gamma$

Cut: $\beta \not\sim \gamma$

And:

Assumption: $\alpha \not\sim \beta, \alpha \not\sim \gamma$ (\& we want: $\alpha \not\sim \beta \land \gamma$)
Proofs

**Equivalence:**

Assumption: \( \alpha \not\sim \beta, \beta \not\sim \alpha, \alpha \not\sim \gamma \) (& we want: \( \beta \not\sim \gamma \))

Cautious monotonicity: \( \alpha \land \beta \not\sim \gamma \)

Left equivalence: \( \beta \land \alpha \not\sim \gamma \)

Cut: \( \beta \not\sim \gamma \)

**And:**

Assumption: \( \alpha \not\sim \beta, \alpha \not\sim \gamma \) (& we want: \( \alpha \not\sim \beta \land \gamma \))

Cautious monotonicity: \( \alpha \land \beta \not\sim \gamma \)
Proofs

Equivalence:

Assumption: $\alpha \vdash \beta$, $\beta \vdash \alpha$, $\alpha \vdash \gamma$ (& we want: $\beta \vdash \gamma$)
Cautious monotonicity: $\alpha \land \beta \vdash \gamma$
Left equivalence: $\beta \land \alpha \vdash \gamma$
Cut: $\beta \vdash \gamma$

And:

Assumption: $\alpha \vdash \beta$, $\alpha \vdash \gamma$ (& we want: $\alpha \vdash \beta \land \gamma$)
Cautious monotonicity: $\alpha \land \beta \vdash \gamma$
Because $\alpha \land \beta \land \gamma \models \beta \land \gamma$
Proofs

Equivalence:

Assumption: $\alpha \vdash \beta, \beta \vdash \alpha, \alpha \vdash \gamma$ (& we want: $\beta \vdash \gamma$)
Cautious monotonicity: $\alpha \land \beta \vdash \gamma$
Left equivalence: $\beta \land \alpha \vdash \gamma$
Cut: $\beta \vdash \gamma$

And:

Assumption: $\alpha \vdash \beta, \alpha \vdash \gamma$ (& we want: $\alpha \vdash \beta \land \gamma$)
Cautious monotonicity: $\alpha \land \beta \vdash \gamma$
Because $\alpha \land \beta \land \gamma \models \beta \land \gamma$, we have:
Super classicality: $\alpha \land \beta \land \gamma \vdash \beta \land \gamma$
Proofs

Equivalence:

Assumption: \( \alpha \vdash \beta, \beta \vdash \alpha, \alpha \vdash \gamma \) (\& we want: \( \beta \vdash \gamma \))

Cautious monotonicity: \( \alpha \land \beta \vdash \gamma \)

Left equivalence: \( \beta \land \alpha \vdash \gamma \)

Cut: \( \beta \vdash \gamma \)

And:

Assumption: \( \alpha \vdash \beta, \alpha \vdash \gamma \) (\& we want: \( \alpha \vdash \beta \land \gamma \))

Cautious monotonicity: \( \alpha \land \beta \vdash \gamma \)

Because \( \alpha \land \beta \land \gamma \models \beta \land \gamma \), we have:

Super classicality: \( \alpha \land \beta \land \gamma \vdash \beta \land \gamma \)

Cut: \( \alpha \land \beta \vdash \beta \land \gamma \)
Proofs

**Equivalence:**

**Assumption:** \( \alpha \not\vdash \beta, \beta \not\vdash \alpha, \alpha \not\vdash \gamma \) (\& we want: \( \beta \not\vdash \gamma \))

**Cautious monotonicity:** \( \alpha \wedge \beta \not\vdash \gamma \)

**Left equivalence:** \( \beta \wedge \alpha \not\vdash \gamma \)

**Cut:** \( \beta \not\vdash \gamma \)

**And:**

**Assumption:** \( \alpha \not\vdash \beta, \alpha \not\vdash \gamma \) (\& we want: \( \alpha \not\vdash \beta \wedge \gamma \))

**Cautious monotonicity:** \( \alpha \wedge \beta \not\vdash \gamma \)

**Because** \( \alpha \wedge \beta \wedge \gamma \models \beta \wedge \gamma \), **we have:**

**Super classicality:** \( \alpha \wedge \beta \wedge \gamma \not\vdash \beta \wedge \gamma \)

**Cut:** \( \alpha \wedge \beta \not\vdash \beta \wedge \gamma \)

**Cut:** \( \alpha \not\vdash \beta \wedge \gamma \)
Proofs

**Equivalence:**

Assumption: \(\alpha \vdash \beta, \beta \vdash \alpha, \alpha \vdash \gamma\) (\& we want: \(\beta \vdash \gamma\))

Cautious monotonicity: \(\alpha \land \beta \vdash \gamma\)

Left equivalence: \(\beta \land \alpha \vdash \gamma\)

Cut: \(\beta \vdash \gamma\)

**And:**

Assumption: \(\alpha \vdash \beta, \alpha \vdash \gamma\) (\& we want: \(\alpha \vdash \beta \land \gamma\))

Cautious monotonicity: \(\alpha \land \beta \vdash \gamma\)

Because \(\alpha \land \beta \land \gamma \models \beta \land \gamma\), we have:

Super classicality: \(\alpha \land \beta \land \gamma \vdash \beta \land \gamma\)

Cut: \(\alpha \land \beta \vdash \beta \land \gamma\)

Cut: \(\alpha \vdash \beta \land \gamma\)

MPC
Proofs

Equivalence:

Assumption: \( \alpha \vdash \beta, \beta \vdash \alpha, \alpha \vdash \gamma \) (\& we want: \( \beta \vdash \gamma \))

Cautious monotonicity: \( \alpha \land \beta \vdash \gamma \)

Left equivalence: \( \beta \land \alpha \vdash \gamma \)

Cut: \( \beta \vdash \gamma \)

And:

Assumption: \( \alpha \vdash \beta, \alpha \vdash \gamma \) (\& we want: \( \alpha \vdash \beta \land \gamma \))

Cautious monotonicity: \( \alpha \land \beta \vdash \gamma \)

Because \( \alpha \land \beta \land \gamma \models \beta \land \gamma \), \textit{we have:}

Super classicality: \( \alpha \land \beta \land \gamma \vdash \beta \land \gamma \)

Cut: \( \alpha \land \beta \vdash \beta \land \gamma \)

Cut: \( \alpha \vdash \beta \land \gamma \)

\textbf{MPC:} Exercise