Principles of Knowledge Representation and Reasoning

3. Qualitative Representation and Reasoning

3.7 Spatial Representation and Reasoning: Reasoning with RCC8

Bernhard Nebel

- Reminder
- Upper Bounds
- Lower Bound - Proving NP-Hardness
- Constraint Reasoning
- Some Empirical Results
- Outlook & Open Problems
Reminder

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- Validity of **topological set expressions** is equivalent to S4-validity of translations of these set expressions
- Using an **additional K-modality**, satisfiability of the topological set constraints can be tested

~> Reduction of spatial reasoning problems to modal logic reasoning problems.
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  - If we allow 3 dimensions, then the issue whether regions are internally connected is not crucial anymore.
Upper Bounds – Using Results From Modal Logic

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- Deciding the satisfiability of **topological set constraints** is in PSPACE
A Better Upper Bound for RCC8

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**Proposition.** RCC8 satisfiability is in NP.
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- . . . and ask ourselves for which fragment of RCC8 it is complete
## Composition Table

<table>
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<th>PO</th>
<th>TPP</th>
<th>NTPP</th>
<th>TPP~</th>
<th>NTPP~</th>
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The Reduction

- **Relations:** \( R_t = NTPP, \; R_f = EQ \)

- **Polarity constraints:**

- **Clause constraints:**

- RCC8 sat. \( \Rightarrow \) 3-SAT: follows from reduction

- 3-SAT \( \Rightarrow \) RCC8 sat.: Construction of model for \( \Theta \phi \) for each positive 3-SAT instance \( \phi \)
Tractable Fragments?

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  - There are **2 additional maximal subsets** that allow for poly. satisfiability testing!
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• Can $\mathcal{H}_8$ be used to speed up the satisfiability testing?
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    - $H(n, d, l)$: only relations out of RCC8 - $\mathcal{H}_8$
Phase Transition for \( A(n, d, 4) \)

- Phase transition for \( A(n, d, 4) \) between \( d = 8 \) and \( d = 10 \) for \( 10 \leq n \leq 100 \).
Phase Transition for $H(n, d, 4)$

- Phase transition for $H(n, d, 4)$ between $d = 10$ and $d = 15$ for $10 \leq n \leq 80$. 

500 instances per data point
Hard Instances …

… using more than 10,000 search nodes

500 instances per data point
Quality of Path Consistency…

…measured as the percentage of path consistent but unsatisfiable CSPs

Percentage points of incorrect PCA answers for A(n,d,4.0)

Percentage points of incorrect PCA answers for H(n,d,4.0)

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- Combining RCC8 and Allen’s interval calculus to form a temporal-spatial calculus
- Are there other interesting spatial calculi?
Literature


(*) Werner Nutt, On the Translation of Qualitative Spatial Reasoning Problems into Modal Logics, *Advances in Artificial Intelligence, Proc. 23rd Annual German Conference on Artificial Intelligence, KI’99*, Bonn (Germany), September 1999.
