3.6 Spatial Representation and Reasoning: RCC8 and Topology

Bernhard Nebel

- Motivation
- RCC8 - A formalism for qualitative spatial descriptions
- Topology
- Topological set constraints
- From set constraints to modal logic
Motivation

We may want to state qualitative relationships between regions in space.
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We may want to state qualitative relationships between regions in space, for example:

- “Region $X$ touches region $Y$”
- “Germany and Switzerland have a common border”
- “Freiburg is located in Baden-Württemberg”
Possible Applications

This can be useful when only partial information is available:

- We may know that region $X$ is not connected with region $Y$ without knowing the shape and location of $X$ and $Y$. 
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- Show me all countries **bordering** the Mediterranean!
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We may want to *query* a database:

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We may want to state *integrity constraints*:

- An island has to be located *in the interior of* a sea.
Qualitative Relations Between Regions: RCC-8

Eight relations between regions:

- \( DC(X,Y) \)
- \( PO(X,Y) \)
- \( TPP(X,Y) \)
- \( NTPP(X,Y) \)
- \( EC(X,Y) \)
- \( EO(X,Y) \)
- \( TPP^u(X,Y) \)
- \( NTPP^u(X,Y) \)
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- **EC** means that they only share borders
- **PO** means that the two regions share interior points
- **TPP** means that one region is a subset of the other sharing some points on the borders
- **NTPP** same, but without sharing any bordering points
Questions

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• How can we formalize these relations?

• Are they disjoint and exhaustive?

• Can we come up with a composition table?

• What is the computational complexity of reasoning with these relations?

• Can we identify a tractable fragment?
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Topology

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- If $(O_i)_{i \in I}$ is a (possibly infinite) family of elements from $\mathcal{O}$, then
  $$\bigcup_{i \in I} O_i \in \mathcal{O}.$$
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\[
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**Example:** In Euclidian space, a set \( O \) is open if for each point \( x \in O \) there is a ball surrounding \( x \) that is contained in \( O \).
A set $N \subseteq \mathcal{U}$ is a **neighborhood** of a point $x$ if there is an open set $O \in \mathcal{O}$ such that $x \in O \subseteq N$. 
Terminology & Notation

A set $N \subseteq U$ is a **neighborhood** of a point $x$ if there is an open set $O \in \mathcal{O}$ such that $x \in O \subseteq N$. Let $X \subseteq U$ and $x \in U$. Then $x$ is

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**Notation:**

- $i(X)$ is the set of interior points of $X$ (the **interior** of $X$)
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- \( i(X) \) is the set of **interior points** of \( X \) (the **interior** of \( X \))
- \( i(X) \) is the largest open set contained in \( X \).
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A set is **closed** if $X = cl(X)$. 
The interior operator $i(\cdot)$ has the following properties:

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- $X$ is open iff $X = i(X)$
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Theorem. Let $\mathcal{U}$ be a set and $i : 2^\mathcal{U} \to 2^\mathcal{U}$ be a function that maps subsets of $\mathcal{U}$ to subsets of $\mathcal{U}$ in the same way an interior operator does.
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From Interior Operators to Topologies and Back

**Theorem.** Let $\mathcal{U}$ be a set and $i: 2^\mathcal{U} \to 2^\mathcal{U}$ be a function that maps subsets of $\mathcal{U}$ to subsets of $\mathcal{U}$ in the same way an interior operator does. Define $\mathcal{O} := \{O \subseteq \mathcal{U} \mid O = i(O)\}$. Then $\mathcal{T} = (\mathcal{U}, \mathcal{O})$ is a topological space.

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**Theorem.** Let $\mathcal{U}$ be a set and $i: 2^\mathcal{U} \to 2^\mathcal{U}$ be a function that maps subsets of $\mathcal{U}$ to subsets of $\mathcal{U}$ in the same way an interior operator does. Define $\mathcal{O} := \{ O \subseteq \mathcal{U} \mid O = i(O) \}$. Then $\mathcal{T} = (\mathcal{U}, \mathcal{O})$ is a topological space.

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Let $O := \bigcup_{i \in I} O_i$, $O_i = i(O_i)$ for all $i$. Of course, $i(O) \subseteq O$. Clearly, $O_i \subseteq O$ for all $i$.

Then $O_i = i(O_i) \subseteq i(O)$. Therefore $O = \bigcup_{i \in I} O_i \subseteq i(O)$. Hence, $O = i(O)$, i.e., $O \in \mathcal{O}$. Thus, $\mathcal{O}$ is closed under arbitrary unions.
Theorem. Let $\mathcal{U}$ be a set and $i: 2^{\mathcal{U}} \to 2^{\mathcal{U}}$ be a function that maps subsets of $\mathcal{U}$ to subsets of $\mathcal{U}$ in the same way an interior operator does. Define $\mathcal{O} := \{ O \subseteq \mathcal{U} \mid O = i(O) \}$. Then $\mathcal{T} = (\mathcal{U}, \mathcal{O})$ is a topological space.

Proof. Since $i(\mathcal{U}) = \mathcal{U}$ by (1), we have $\mathcal{U} \in \mathcal{O}$.

Since $i(\emptyset) \subseteq \emptyset$ by (3), we have $i(\emptyset) = \emptyset$, and therefore $\emptyset \in \mathcal{O}$.

By (2), $\mathcal{O}$ is closed under pairwise intersection.

From (2), it follows that $X \subseteq Y$ implies $i(X) \subseteq i(Y)$ (which we need below).

Let $O := \bigcup_{i \in I} O_i$, $O_i = i(O_i)$ for all $i$. Of course, $i(O) \subseteq O$. Clearly, $O_i \subseteq O$ for all $i$.

Then $O_i = i(O_i) \subseteq i(O)$. Therefore $O = \bigcup_{i \in I} O_i \subseteq i(O)$. Hence, $O = i(O)$, i.e., $O \in \mathcal{O}$. Thus, $\mathcal{O}$ is closed under arbitrary unions.

Note: One can show that the interior operator of the constructed topology is identical to our original $i$. 
From Interior Operators to Topologies and Back

**Theorem.** Let $\mathcal{U}$ be a set and $i: 2^\mathcal{U} \to 2^\mathcal{U}$ be a function that maps subsets of $\mathcal{U}$ to subsets of $\mathcal{U}$ in the same way an interior operator does. Define

$$O := \{O \subseteq \mathcal{U} \mid O = i(O)\}.$$  

Then $\mathcal{T} = (\mathcal{U}, O)$ is a topological space.

**Proof.** Since $i(\mathcal{U}) = \mathcal{U}$ by (1), we have $\mathcal{U} \in O$.

Since $i(\emptyset) \subseteq \emptyset$ by (3), we have $i(\emptyset) = \emptyset$, and therefore $\emptyset \in O$.

By (2), $O$ is closed under pairwise intersection.

From (2), it follows that $X \subseteq Y$ implies $i(X) \subseteq i(Y)$ (which we need below).

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Then $O_i = i(O_i) \subseteq i(O)$. Therefore $O = \bigcup_{i \in I} O_i \subseteq i(O)$. Hence, $O = i(O)$, i.e., $O \in O$. Thus, $O$ is closed under arbitrary unions.

**Note:** One can show that the interior operator of the constructed topology is identical to our original $i$. We can play the same game starting with a topology $\mathcal{T}$, constructing $i$ and then defining a new topology $\mathcal{T}'$. 
**Theorem.** Let $\mathcal{U}$ be a set and $i: 2^\mathcal{U} \to 2^\mathcal{U}$ be a function that maps subsets of $\mathcal{U}$ to subsets of $\mathcal{U}$ in the same way an interior operator does. Define $\mathcal{O} := \{ O \subseteq \mathcal{U} \mid O = i(O) \}$. Then $\mathcal{T} = (\mathcal{U}, \mathcal{O})$ is a topological space.

**Proof.** Since $i(\mathcal{U}) = \mathcal{U}$ by (1), we have $\mathcal{U} \in \mathcal{O}$.

Since $i(\emptyset) \subseteq \emptyset$ by (3), we have $i(\emptyset) = \emptyset$, and therefore $\emptyset \in \mathcal{O}$.

By (2), $\mathcal{O}$ is closed under pairwise intersection.

From (2), it follows that $X \subseteq Y$ implies $i(X) \subseteq i(Y)$ (which we need below).

Let $O := \bigcup_{i \in I} O_i$, $O_i = i(O_i)$ for all $i$. Of course, $i(O) \subseteq O$. Clearly, $O_i \subseteq O$ for all $i$.

Then $O_i = i(O_i) \subseteq i(O)$. Therefore $O = \bigcup_{i \in I} O_i \subseteq i(O)$. Hence, $O = i(O)$, i.e., $O \in \mathcal{O}$. Thus, $\mathcal{O}$ is closed under arbitrary unions.

**Note:** One can show that the interior operator of the constructed topology is identical to our original $i$. We can play the same game starting with a topology $\mathcal{T}$, constructing $i$ and then defining a new topology $\mathcal{T}'$. It turns out that $\mathcal{T} = \mathcal{T}'$. 
Topological Set Expressions and Their Interpretations

Topological set expressions describe subsets of a topological space.
Topological Set Expressions and Their Interpretations

**Topological set expressions** describe subsets of a topological space:

\[ s \rightarrow X \]
Topological Set Expressions and Their Interpretations

Topological set expressions describe subsets of a topological space:

\[ s \rightarrow X \mid \top \]
Topological set expressions describe subsets of a topological space:

\[ s \rightarrow X | \top | \bot \]
Topological set expressions describe subsets of a topological space:

\[ s \rightarrow X \mid \top \mid \bot \mid s' \sqcap s'' \]
Topological Set Expressions and Their Interpretations

Topological set expressions describe subsets of a topological space:

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Topological set expressions describe subsets of a topological space:

$$ s \rightarrow X \mid \top \mid \bot \mid s' \cap s'' \mid s' \cup s'' \mid \overline{s} \mid \mathbb{I}s' $$
Topological set expressions describe subsets of a topological space:

\[ s \rightarrow X | \top | \bot | s' \cap s'' | s' \cup s'' | \overline{s} | I s' , \]

with set variables \( X, Y, Z \)
Topological Set Expressions and Their Interpretations

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A **topological interpretation** is a tuple \( I = (T, d) \).
Topological set expressions describe subsets of a topological space:

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A topological interpretation is a tuple \( \mathcal{I} = (\mathcal{T}, d) \), where \( \mathcal{T} = (\mathcal{U}, \mathcal{O}) \) is a topological space with an associated interior operator \( i \).
Topological set expressions describe subsets of a topological space:

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A topological interpretation is a tuple \( \mathcal{I} = (\mathcal{T}, d) \), where \( \mathcal{T} = (\mathcal{U}, \mathcal{O}) \) is a topological space with an associated interior operator \( \mathbb{I} \) and \( d \) is a function from set variables to subsets of \( \mathcal{U} \).
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\( d \) is extended to topological set expressions as follows:

\[ d(\bot) = \emptyset \]
Topological Set Expressions and Their Interpretations

Topological set expressions describe subsets of a topological space:

\[ s \rightarrow X \mid \top \mid \bot \mid s' \cap s'' \mid s' \cup s'' \mid \overline{s} \mid Is', \]

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    d(\bot) & = \emptyset \\
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\end{align*}
\]
Topological Set Expressions and Their Interpretations

**Topological set expressions** describe subsets of a topological space:

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\( d \) is extended to **topological set expressions** as follows:

\[
\begin{align*}
d(\bot) &= \emptyset \\
d(\top) &= \mathcal{U} \\
d(s \cap s') &= d(s) \cap d(s')
\end{align*}
\]
Topological set expressions describe subsets of a topological space:

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d(\top) & = \mathcal{U} \\
d(s \cap s') & = d(s) \cap d(s') \\
d(s \cup s') & = d(s) \cup d(s') \\
d(\overline{s}) & = \mathcal{U} - d(s)
\end{align*}
\]
Topological Set Expressions and Their Interpretations

**Topological set expressions** describe subsets of a topological space:

\[ s \rightarrow X \mid \top \mid \bot \mid s' \cap s'' \mid s' \cup s'' \mid s^c \mid \text{Is'}, \]

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A **topological interpretation** is a tuple \( \mathcal{I} = (\mathcal{T}, d) \), where \( \mathcal{T} = (\mathcal{U}, \mathcal{O}) \) is a topological space with an associated interior operator \( i \) and \( d \) is a function from **set variables** to subsets of \( \mathcal{U} \).

\( d \) is extended to **topological set expressions** as follows:

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\begin{align*}
d(\bot) & = \emptyset & d(\top) & = \mathcal{U} \\
d(s \cap s') & = d(s) \cap d(s') & d(s \cup s') & = d(s) \cup d(s') \\
d(s^c) & = \mathcal{U} - d(s) & d(\text{Is}) & = i(d(s))
\end{align*}
\]
Topological Set Constraints

Elementary set constraints:

\[ s \equiv t \quad \text{or} \quad s \not\equiv t \]
Topological Set Constraints

Elementary set constraints:

\[ s = t \quad \text{or} \quad s \neq t \]

Complex set constraints: combinations using \( \land, \lor, \) and \( \neg \).
Topological Set Constraints

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A topological interpretation \( \mathcal{I} = (\mathcal{T}, d) \) satisfies a constraint
Topological Set Constraints

Elementary set constraints:

\[ s = t \text{ or } s \neq t \]

Complex set constraints: combinations using \(\land\), \(\lor\), and \(\neg\).

A topological interpretation \(\mathcal{I} = (T, d)\) satisfies a constraint:

\[ \mathcal{I} \models s = t \iff d(s) = d(t) \]
Topological Set Constraints

Elementary set constraints:

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Topological Set Constraints

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Complex set constraints: combinations using \( \land, \lor, \) and \( \neg \).

A topological interpretation \( \mathcal{I} = (\mathcal{T}, d) \) satisfies a constraint:

\[ \mathcal{I} \models s = t \iff d(s) = d(t) \]
\[ \mathcal{I} \models s \neq t \iff d(s) \neq d(t) \]

As usual: model, satisfiability, equivalence, entailment, \ldots
What Kind of Regions Do We Want to Consider?
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A and D are reasonable regions
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A and D are **reasonable** regions, B, C, and E are not
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In other words, \( X \) is a region iff it is non-empty.

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\[
X \neq \bot
\]

and “regular”, i.e., the closure of an open set:

\[
X \doteq \overline{I(IX)}.
\]
What Kind of Regions Do We Want to Consider?

A and D are reasonable regions, B, C, and E are not.

In other words, $X$ is a region iff it is non-empty

$$X \neq \bot$$

and “regular”, i.e., the closure of an open set:

$$X = \overline{\text{I}(\text{I}X)}.$$

It is not necessary that a region is internally connected.
Applying the Topological Set Constraints to RCC8

The RCC8 relations are short hands for topological set constraints:

\[ \text{DC}(X, Y) := X \cap Y = \bot \]
Applying the Topological Set Constraints to RCC8

The RCC8 relations are short hands for topological set constraints:

\[
\text{DC}(X, Y) := X \cap Y \sqsubseteq \bot
\]
\[
\text{EC}(X, Y) := X \cap Y \not\subseteq \bot \land IX \cap IY \sqsubseteq \bot
\]
Applying the Topological Set Constraints to RCC8

The RCC8 relations are short hands for topological set constraints:

$$DC(X, Y) := X \cap Y \doteq \bot$$

$$EC(X, Y) := X \cap Y \neq \bot \land IX \cap IY \doteq \bot$$

$$PO(X, Y) := IX \cap IY \neq \bot \land X \cap \overline{Y} \neq \bot \land \overline{X} \cap Y \neq \bot$$
Applying the Topological Set Constraints to RCC8

The RCC8 relations are short hands for topological set constraints:

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\begin{align*}
\text{DC}(X, Y) & := X \cap Y \doteq \bot \\
\text{EC}(X, Y) & := X \cap Y \not\doteq \bot \land \overline{I} X \cap \overline{I} Y \doteq \bot \\
\text{PO}(X, Y) & := \overline{I} X \cap \overline{I} Y \not\doteq \bot \land X \cap \overline{Y} \not\doteq \bot \land \overline{X} \cap Y \not\doteq \bot \\
\text{EQ}(X, Y) & := X \doteq Y
\end{align*}
\]
Applying the Topological Set Constraints to RCC8

The **RCC8** relations are short hands for topological set constraints:

\[
\begin{align*}
\text{DC}(X, Y) & : = \quad X \cap Y \models \bot \\
\text{EC}(X, Y) & : = \quad X \cap Y \not\models \bot \land IX \cap IY \models \bot \\
\text{PO}(X, Y) & : = \quad IX \cap IY \not\models \bot \land X \cap \overline{Y} \not\models \bot \land \overline{X} \cap Y \not\models \bot \\
\text{EQ}(X, Y) & : = \quad X \models Y \\
\text{TPP}(X, Y) & : = \quad X \cap \overline{Y} \models \bot \land X \cap \overline{IY} \not\models \bot
\end{align*}
\]
Applying the Topological Set Constraints to RCC8

The RCC8 relations are short hands for topological set constraints:

\[
\begin{align*}
DC(X, Y) & := X \cap Y \doteqdot \perp \\
EC(X, Y) & := X \cap Y \neq \perp \land IX \cap IY \doteqdot \perp \\
PO(X, Y) & := IX \cap IY \neq \perp \land X \cap \overline{Y} \neq \perp \land \overline{X} \cap Y \neq \perp \\
EQ(X, Y) & := X \doteqdot Y \\
TPP(X, Y) & := X \cap \overline{Y} \doteqdot \perp \land X \cap \overline{IY} \neq \perp \\
NTPP(X, Y) & := X \cap \overline{IY} \doteqdot \perp
\end{align*}
\]

In addition, each named region must satisfy non-emptiness and regularity.

\[\rightarrow\] It follows that the relations are disjoint and exhaustive.
Normal Form Constraints

• A topological set constraint is in normal form if it is $s \models T$ or $s \not\models T$. 
Normal Form Constraints

- A topological set constraint is in normal form if it is \( s \models \top \) or \( s \not\models \top \).

- Every set constraint can be translated into normal form.
Normal Form Constraints

- A topological set constraint is in **normal form** if is is $s \triangleq \top$ or $s \not\triangleq \top$.

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- $s \triangleq t$ is equivalent to $(\overline{s} \sqcup t) \sqcap (\overline{t} \sqcup s) \triangleq \top$

- $\text{DC}(X, Y) = \overline{X} \sqcup \overline{Y} \triangleq \top$
Normal Form Constraints

- A topological set constraint is in **normal form** if is is $s \vdash T$ or $s \nvdash T$.

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Normal Form Constraints

• A topological set constraint is in normal form if it is $s \models \top$ or $s \not\models \top$.

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  ○ $\text{DC}(X,Y) = \overline{X} \cup \overline{Y} \models \top$

  ○ $\text{EC}(X,Y) = \overline{X} \cup \overline{Y} \not\models \top \land \overline{X} \cup \overline{Y} \models \top$

  ○ …
A Deduction Theorem for Set Constraints and Convexity

We may want to know when a set constraint follows (or does not follow) from another set constraint.
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We may want to know when a set constraint follows (or does not follow) from another set constraint. **Notation**: \( s \sqsubseteq t \) stands for \( \overline{s} \cup t \vdash \top \).

**Theorem (Nutt 99)**. (Deduction Theorem) Let \( s, t \) be set expressions. Then

\[
s \vdash \top \models t \vdash \top \iff \models Is \sqsubseteq It.
\]
A Deduction Theorem for Set Constraints and Convexity

We may want to know when a set constraint follows (or does not follow) from another set constraint. **Notation**: $s \sqsubseteq t$ stands for $\overline{s} \sqcup t \models \top$.

**Theorem (Nutt 99)**. (Deduction Theorem) Let $s, t$ be set expressions. Then

$$s \models \top \models t \models \top \iff \models I_s \sqsubseteq I_t.$$  

(without proof)
A Deduction Theorem for Set Constraints and Convexity

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**Theorem.** (Convexity) The conjunctive set constraint

$$s_1 \models \top \land \ldots \land s_m \models \top \land t_1 \neq \top \land \ldots \land t_n \neq \top$$

is satisfiable iff and only if the following constraints are satisfiable for each $j \in \{1, \ldots, n\}$

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A Deduction Theorem for Set Constraints and Convexity

We may want to know when a set constraint follows (or does not follow) from another set constraint. **Notation:** \( s \sqsubseteq t \) stands for \( \overline{s} \cup t = \top \).

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\[
s_1 \models \top \land \ldots \land s_m \models \top \land t_j \neq \top.
\]

**Proof Idea.** (\( \Leftarrow \)) Construct models for each \( j \) and create a common model by taking disjoint union.
The modal logic S4 can be characterized by the following axiom schemata (with I instead of □ as the modal box operator)

- $I\top \leftrightarrow \top$ (valid in all frames)
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4. $II\varphi \leftrightarrow I\varphi$ (valid in $T4$-frames: reflexivity and transitivity)
The modal logic S4 can be characterized by the following axiom schemata (with $I$ instead of $\Box$ as the modal box operator)

- $I\top \leftrightarrow \top$ \textit{(valid in all frames)}
- $I\varphi \rightarrow \varphi$ \textit{(valid in $T$-frames: reflexivity)}
- $I\varphi \land I\psi \leftrightarrow I(\varphi \land \psi)$ \textit{(valid in all frames)}
- $II\varphi \leftrightarrow I\varphi$ \textit{(valid in $T4$-frames: reflexivity and transitivity)}

**Reminder**: Interior operator

- $i(U) = U$
- $i(X) \cap i(Y) = i(X \cap Y)$
- $i(X) \subseteq X$
- $i(i(X)) = i(X)$
Define translation function $\pi$ from set expressions to S4 formulae as follows

- $\pi(X) = X$
- $\pi(\overline{s}) = \neg \pi(s)$
- $\pi(s \cap t) = \pi(s) \land \pi(t)$
- $\pi(s \cup t) = \pi(s) \lor \pi(t)$
- $\pi(I_s) = I\pi(s)$
Define translation function $\pi$ from set expressions to S4 formulae as follows

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- $\pi(s \cap t) = \pi(s) \land \pi(t)$
- $\pi(s \cup t) = \pi(s) \lor \pi(t)$
- $\pi(\mathcal{I}s) = \mathcal{I}\pi(s)$

A set expression $s$ is called a **topological tautology** if $d(s) = \mathcal{U}$ for all topological interpretations $\mathcal{I} = ((\mathcal{U}, \mathcal{O}), d)$. 
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A set expression $s$ is called a topological tautology if $d(s) = \mathcal{U}$ for all topological interpretations $\mathcal{I} = ((\mathcal{U}, \mathcal{O}), d)$.

**Theorem (McKinsey & Tarski 48).** $s$ is a topological tautology iff $\pi(s)$ is S4-valid.
Define translation function $\pi$ from set expressions to S4 formulae as follows

- $\pi(X) = X$
- $\pi(\neg s) = \neg \pi(s)$
- $\pi(s \cap t) = \pi(s) \land \pi(t)$
- $\pi(s \cup t) = \pi(s) \lor \pi(t)$
- $\pi(I \cdot s) = I\pi(s)$

A set expression $s$ is called a **topological tautology** if $d(s) = \mathcal{U}$ for all topological interpretations $\mathcal{I} = ((\mathcal{U}, \mathcal{O}), d)$.

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**Corollary.** $s$ is topologically satisfiable iff $\pi(s)$ is S4-satisfiable.
Topological Set Constraints and Modal Logic (1)

How can we use this result for conjunctive topological set constraints?
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**Note:** Using the convexity theorem, we only have to test the satisfiability of constraints of the form $C_J = (s_1 \models \top \land \ldots \land s_m \models \top \land t_j \not\models \top)$. 
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Using the deduction theorem, it suffices to check $\not\vdash Is \sqsubseteq It_j$
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**Note:** Using the convexity theorem, we only have to test the satisfiability of constraints of the form $C_j = (s_1 \models T \land \ldots \land s_m \models T \land t_j \neq T)$. $C_j$ is satisfiable iff $s_1 \models T \land \ldots \land s_m \models T \not\models t_j \models T$. Equivalently, we can test $s \models T \not\models t_j \models T$, with $s = s_1 \cap \ldots \cap s_m$.

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**Theorem.** (Translation) $(s_1 \models \top \land \ldots \land s_m \models \top \land t_1 \not\models \top \land \ldots \land t_n \not\models \top)$ is satisfiable if the following formulae are S4-satisfiable for all $j \in \{1, \ldots, n\}$:
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Using the deduction theorem, it suffices to check $\not\vdash I_{s} \sqsubseteq I_{t_j}$, i.e., whether $I_{s} \sqcup I_{t_j}$ is not a tautology, i.e., whether $I_{s} \sqcap \overline{I_{t_j}}$ is satisfiable. Using the Mckinsey/Tarski theorem, this amounts to test for S4-satisfiability of $\pi(I_{s} \sqcap \overline{I_{t_j}})$.

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**Note:** Using the convexity theorem, we only have to test the satisfiability of constraints of the form $C_J = (s_1 \models \top \land \ldots \land s_m \models \top \land t_j \not\models \top)$. $C_j$ is satisfiable iff $s_1 \models \top \land \ldots \land s_m \models \top \not\models t_j \models \top$. Equivalently, we can test $s \models \top \not\models t_j \models \top$, with $s = s_1 \sqcap \ldots \sqcap s_m$.

Using the deduction theorem, it suffices to check $\not\models Is \subseteq It_j$, i.e., whether $Is \cup \overline{It_j}$ is not a tautology, i.e., whether $Is \sqcap \overline{It_j}$ is satisfiable. Using the Mckinsey/Tarski theorem, this amounts to test for S4-satisfiability of $\pi(Is \sqcap \overline{It_j})$.

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$$\mathbf{I}\pi(s_1) \land \ldots \land \mathbf{I}\pi(s_m) \land \neg \mathbf{I}\pi(t_j).$$
Let □ and ◇ be K-modalities.
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**Proposition.** Let \( \varphi_1, \ldots, \varphi_m, \psi_1, \ldots, \psi_n \) be multi-modal formulae not containing the K-operators □ and ◊. Then

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\]

is satisfiable iff for all \( j \in \{1, \ldots, n\} \) the formulae

\[
\varphi_1 \land \ldots \land \varphi_m \land \psi_j
\]

are satisfiable.

**Proof idea.** Create from models satisfying the later formula a modal interpretation for the former formula.
New Translation

Use a multi-modal logic for the translation.
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This leads to the following translation of RCC8 constraints:

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This leads to the following translation of RCC8 constraints:

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This leads to the following translation of RCC8 constraints:

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- $\pi(\text{EC}(X,Y)) = \square I\neg(I\!X \land I\!Y) \land \Diamond \neg I\neg(X \land Y)$
- $\ldots$

**Theorem.** (Translation) Let $C$ be an arbitrary topological set constraint. Then $C$ is satisfiable iff $\pi(C)$ is satisfiable.
Outlook

- We wanted to state qualitative relationships between spatial regions
- **Semantics**: Topology
- **Language for describing relations**: Topological set constraints
  - can be translated to modal logic (McKinsey & Tarski)
  - Combination can be handled with another modality
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