3. Qualitative Representation and Reasoning

3.4 A Maximal Tractable Sub-algebra

Bernhard Nebel

- The Endpoint-Class
- The OH-Class
- Complexity of the OH-Class
- Maximality
- Relevance
The EP-Subclass

**End-Point Subclass:** $\mathcal{P} \subseteq \mathcal{A}$ is the subclass that permits a clause form containing only unit clauses ($a \neq b$ is allowed).
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Example: all basic relations
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Example: all basic relations and \( \{d, o\} \) since

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\pi(X \{d, o\} Y) = \{ X^- < X^+, Y^- < Y^+, \\
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X^+ < Y^+ \}
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\[
\begin{array}{c}
\cdots \ X \cdots \\
\hline
X
\end{array}
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\[
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**Theorem** (Vilain, Kautz 86, Ladkin, Maddux 88). The path-consistency method decides CSAT($\mathcal{P}$).
The ORD-Horn Subclass

ORD-Horn Subclass: $\mathcal{H} \subseteq \mathcal{A}$ is the subclass that permits a clause form containing only Horn clauses, where only the following literals are allowed:

$$(a \leq b), (a = b), (a \neq b)$$
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$$\pi(X\{o, s, f^-\}Y) = \left\{ (X^- \leq X^+), (X^- \neq X^+), (Y^- \leq Y^+), (Y^- \neq Y^+), (X^- \leq Y^-), (X^- \leq Y^+), (X^- \neq Y^+), (Y^- \leq X^+), (X^+ \neq Y^-), (X^+ \leq Y^+), (X^- \neq Y^- \lor X^+ \neq Y^+) \right\}.$$
Partial Orders: The $\textit{ORD}$ Theory

Let $\textit{ORD}$ be the following theory:

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- \textit{ORD} is a \textbf{Horn theory}
- What is missing wrt to \textit{dense} and \textit{linear} orders?
Satisfiability over Partial Orders

**Proposition:** Let $\Theta$ be a CSP over $\mathcal{H}$. $\Theta$ is satisfiable over interval interpretations iff $\pi(\Theta) \cup ORD$ is satisfiable over arbitrary interpretations.
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$\Leftarrow$: Each extension of a partial order to a linear order satisfies all formulae of the form $(a \leq b)$, $(a = b)$, and $(a \neq b)$ which have been satisfied over the original partial order.
Complexity of CSAT($\mathcal{H}$)

Let $ORD_{\pi(\Theta)}$ be the propositional theory resulting from instantiating all axioms with the endpoints occurring in $\pi(\Theta)$. 
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$\mathcal{C} \subset \mathcal{P} \subset \mathcal{H}$,

$|\mathcal{C}|=83$, $|\mathcal{P}|=188$, $|\mathcal{H}|=868$
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**Proof Idea.** One can show that \( ORD_{\pi(\hat{\Theta})} \cup \pi(\hat{\Theta}) \) is closed wrt positive unit resolution. Since this inference rule is refutation complete for Horn theories, the claim follows.
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\( \leadsto \) Maximality of \( \mathcal{H} \)?

\( \leadsto \) Do we have to check all 8192 - 868 extensions?
Complexity of Sub-algebras

Let $\hat{S}$ be the closure of $S \subseteq A$ under converse, intersection, and composition (i.e., the carrier of the least sub-algebra generated by $S$)
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**Theorem.** CSAT($\hat{S}$) can be polynomially transformed to CSAT($S$).

**Proof Idea:** All relations in $\hat{S} - S$ can be modeled by a finite, fixed number of compositions, intersections, and conversions of relations in $S$, introducing perhaps some fresh variables (prerequisite: the universal relation is in $S$).
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\[ \leadsto \text{polynomiality of } S \text{ extends to } \hat{S}. \]

\[ \leadsto \text{NP-hardness of } \hat{S} \text{ is inherited by all generating sets } S. \]

**Note:** \( \mathcal{H} = \hat{\mathcal{H}}. \)
Minimal Extensions of the $\mathcal{H}$-Subclass

A computer-aided case analysis leads to the following result:

There are two minimal sub-algebras that strictly contain $\mathcal{H}$: $\mathcal{X}_1, \mathcal{X}_2$
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$N_1 = \{d, d^{-}, o^{-}, s^{-}, f\} \in \mathcal{X}_1$

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The clause form of these relations contain “proper” disjunctions!

**Theorem:** $\text{CSAT}(\mathcal{H} \cup \{N_i\})$ is NP-complete.

**Question:** Are there other maximal tractable subclasses?
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$\leadsto \mathcal{H}$ is the only maximal tractable subclass that is interesting.
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Meanwhile, there is a complete classification of all sub-algebras containing at least one basic relation [IJCAI 2001]
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\( \sim \) \( \mathcal{H} \) is the only maximal tractable subclass that is interesting.

Meanwhile, there is a complete classification of all sub-algebras containing at least one basic relation [IJCAI 2001] . . . but the question for sub-algebras not containing a basic relation is open
Relevance?

Theoretical:
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⊕ We now know the boundary between polynomial and NP-hard reasoning problems along the dimension *expressiveness*. 
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We now know the boundary between polynomial and NP-hard reasoning problems along the dimension expressiveness.

All known applications either need only $\mathcal{P}$ or they need more than $\mathcal{H}$!
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? Backtracking methods might profit from the result because the branching factor is lower.
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∼ How difficult is CSAT($\mathcal{A}$) in practice?
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∽ How difficult is CSAT($\mathcal{A}$) in practice?

∽ What are the relevant branching factors?
Solving General Allen CSPs

- Backtracking algorithm using path-consistency as a forward-checking method
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• Relies on tractable fragments of Allen’s calculus: Split relations into relations of a tractable fragment, and backtrack over these.
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- Refinements and evaluation of different heuristics
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- Relies on tractable fragments of Allen’s calculus: Split relations into relations of a tractable fragment, and backtrack over these.
- Refinements and evaluation of different heuristics

Which tractable fragment should one use?
Branching Factors

- If the labels are split into base relations, then on average a label is split into
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\(\sim\) a difference of 0.422

\(\sim\) Does it make a difference in practice? Yes . . . for “hard” instances
Summary for Allen’s Interval Calculus

- Allen’s interval calculus is often adequate for describing relative orders of events that have duration
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- Can be used in practice for backtracking algorithms.
Literature


