Multi-Agent Systems

Cooperative Game Theory

Albert-Ludwigs-Universität Freiburg

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Bernhard Nebel, Rolf Bergdoll, and Thorsten Engesser

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Motivation

The famous prisoner's dilemma with the following payoff matrix:

	Silent	*Betray*
Silent	$-1,-1$	$-3,0$
Betray	$0,-3$	$-2,-2$

In games like this one cooperation is prevented, because:

- Binding agreements are not possible
- Pay-off is given directly to individuals as the result of individual action

Cooperative Game Theory

- In many situations...
  - Contracts can form binding agreements
  - Pay-off is given to groups of agents rather than to individuals
- Hence, cooperation is both possible and rational.
- Cooperative game theory asks which contracts are meaningful solutions among self-interested agents.

Characterization (Shoham, Keyton-Brown, 2009, Ch. 12)

[Cooperative game theory is about] how self-interested agents can combine to form effective teams.

Real-world examples

- Political parties form coalitions to ensure majorities. Division of power (ministry posts).
- Companies cooperate to save ressources.
- People buy expensive things together they could not afford to buy alone.
- Buildings are built by several people with different capabilities (craftsmen, electricians, architects, . . .). Who should earn how much?
- People share a taxi. How to split the fare?

Key Questions

- Which coalition should/will form?
- We assume usually the grand coalition, provided this is rational.
- How should the value be divided among the members?

Cooperative Game Theory: Terminology

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Cooperative Game (with transferable utility)

A cooperative game with transferable utility is a pair (N, v):

- N: Set of agents
- Any subset $S \subseteq N$ is called a coalition
- N is the grand coalition
- $v : 2^N \to \mathbb{R}$: characteristic function that assigns a value $v(S)$ to each $S \subseteq N$, $v(\emptyset) = 0$, also called the payoff of S.

- Transferable value assumption:
  - Value of a coalition can be (arbitrarily) redistributed among the coalition's members
  - I.e., value is dispensed in some universal currency
  - Each coalition can be assigned a single value

Division of Value

- $\Psi(N,v) = (\Psi_1(N,v),\ldots,\Psi_k(N,v))$ is a distribution of value to members $1,\ldots,k$ of N.

Feasible distribution

A distribution $\Psi(N,v)$ is feasible iff

$$\sum_{i \in N} \Psi_i(N,v) \leq v(N)$$

Efficient distribution

A distribution $\Psi(N,v)$ is efficient iff

$$\sum_{i \in N} \Psi_i(N,v) \geq v(N)$$

Example: Gloves

Mr A and Mr B are knitting gloves. The gloves are one-size-fits-all, and two gloves make a pair that they sell for 5 EUR. They have each made three gloves. How to share the proceeds from the sale?

- $(N,v), N = 1,2, v(\{1\}) = 5, v(\{2\}) = 5, v(\{1,2\}) = 15$
- Assume they form a coalition. What about these feasible and efficient divisions of value?
  - $\Psi^a(N,v) = (7.5, 7.5)$
  - $\Psi^b(N,v) = (5, 10)$
  - $\Psi^c(N,v) = (4, 11)$

Example: Gloves Extended

Mr A and Mr B and Mr C are knitting gloves. The gloves are one-size-fits-all, and two gloves make a pair that they sell for 5 EUR. They have each made three gloves. How to share the proceeds from the sale?

- $(N, v), N = 1, 2, 3$
  - $v(\{1\}) = 5, v(\{2\}) = 5, v(\{3\}) = 5, v(\{1,2\}) = 15, v(\{1,3\}) = 15, v(\{2,3\}) = 15, v(\{1,2,3\}) = 20$
- Assume they form a coalition. What about these feasible and efficient divisions of value?
  - $\Psi^d(N, v) = (6.6, 6.6, 6.6)$
  - $\Psi^e(N, v) = (7.5, 7.5, 5)$

Is the grand coalition stable: The Core

Core

The core of a cooperative game (N, v) is the set of feasible and efficient distributions of value Ψ, such that no $S \subseteq N$ can do better by splitting off, i.e., Ψ satisfies:

$$\begin{aligned} \sum_{i \in N} \Psi_i(N, v) &= v(N) \\ \sum_{i \in S} \Psi_i(N, v) &\geq v(S) \quad \forall S \subseteq N \end{aligned}$$

Existence, uniqueness, and fairness of the core

- Is the core always nonempty? No.
  - In the extended glove example, the core is empty.
- Is the core always unique? No.
  - In the original glove example, all $\Psi = (\Psi_1, \Psi_2)$ such that $\Psi_1 > 5, \Psi_2 > 5, \Psi_1 + \Psi_2 = 15$ are in the core.
- Is the core fair?
  - No. In the example, people could get different shares, although they contributed the same!

Simple game, Veto agent

Simple game

A game (N, v) is a simple game iff for all $S \subseteq N, v(S) \in \{0, 1\}$

Veto agent

An agent i is a veto agent iff $v(N \setminus \{i\}) = 0$.

Simple Game: Example

Consider the coaltion of three parties A, B, C with 30, 25, and 15 votes, respectively.

- Case 1: 55 votes necessary to win the election.
  - $v(A) = 0, v(B) = 0, v(AB) = 1, v(AC) = 0, v(BC) = 0, v(ABC) = 1$
  - Who is a veto agent? How does the core look like?
- Case 2: 15 votes necessary to win the election.
  - $v(A) = 1, v(B) = 1, v(AB) = 1, v(AC) = 1, v(BC) = 1, v(ABC) = 1$
  - Who is a veto agent? How does the core look like?

Simple Games: Core Existence

Theorem

In a simple game the core is empty iff there is no veto agent. If there are veto agents, the core consists of all value distributions in which the nonveto agents get 0.

Fair Division

- Goal: Grand coalition is to divide its value 'fair'.
- Shapley's idea: Members should receive value proportional to their contributions.
- However:
  - Consider $v(N) = 1$ and $v(S) = 0$ for all $S \neq N$.
  - Thus, $v(N) - v(N \setminus \{i\}) = 1$ for every agent i: everybody's contribution is 1 (everybody is indeed likewise essential).
  - Clearly, one cannot pay 1 to everybody
  - Needed: Some way of weighing. How to design it?
  - Next: Axiomatic characterization of properties of a fair value division (due to Shapley).

Symmetry

Definition Interchangeability

Agents i and j are interchangeable relative to v iff they always contribute the same amount to every coalition of the other agents, i.e., for all S that contain neither i nor j, $v(S \cup \{i\}) = v(S \cup \{j\})$.

Axiom Symmetry

For any v, if i and j are interchangeable then $\Psi_i(N, v) = \Psi_j(N, v)$.

- Agents who contribute the same to every possible coalition should get the same.

Dummy Player

Definition Dummy Player

Agent i is a dummy player iff the amount that i contributes to any coalition is $v(\{i\})$, i.e., for all S with $i \notin S$:,
$v(S \cup \{i\}) - v(S) = v(\{i\})$.

Axiom Dummy Player

For any $S \subseteq N, v$ if i is a dummy player then $\Psi_i(S, v) = v(\{i\})$.

- Dummy players should receive the amount they contribute.

Axiom Additivity

For any two v_1, v_2, it holds that
$\Psi_i(N, v_1 + v_2) = \Psi_i(N, v_1) + \Psi_i(N, v_2)$ for each i, where the game $(N, v_1 + v_2)$ is defined by $(v_1 + v_2)(S) = v_1(S) + v_2(S)$.

Shapley Value Theorem

Theorem

Given a coalitional game (N, v), there is a unique payoff division $\Psi(N, v)$ that divides the full payoff of the grand coalition and that satisfies Symmetry, Dummy Player, and Additivity: The Shapley Value.

Marginal value

Marginal value of agent i

The marginal value of an agent i to any coalition $S \subseteq N$ is defined by $\mu_i : 2^N \to \mathbb{R}$:

$$\mu_i(S) := \begin{cases} v(S \cup \{i\}) - v(S), & i \notin S \\ v(S) - v(S \setminus \{i\}), & i \in S \end{cases} .$$

Shapley Value

Definition Shapley Value

Given a cooperative game (N, v), the Shapley Value divides value according to:

$$\Psi_i(N,v) = \frac{1}{N!} \sum_{o \in \Pi(N)} \mu_i(C_i(o))$$

- $\Pi_n = \{(x_1, \ldots, x_n) | x_i \in N, \forall i, j [i \neq j \Rightarrow x_i \neq x_j]\}$
- $C_i(o)$: set containing only those agents that appear before agent i in o, e.g., $o = (3,1,2)$, then $C_3(o) = \emptyset$, $C_2(o) = \{1,3\}$

Examples: Shapley Value

- Original Glove Example: Shapley-Value $(7.5, 7.5)$
  - Permutation AB
    - Marginal Contribution of A: 5
    - Marginal Contribution of B: 10
  - Permutation BA
    - Marginal Contribution of A: 10
    - Marginal Contribution of B: 5

Examples: Shapley Value

- Extended Glove Example: Shapley-Value $(6.6, 6.6, 6.6)$
  - Permutation ABC
    - Marginal Contributions of A, B, C: 5, 10, 5
  - Permutation ACB
    - Marginal Contributions of A, B, C: 5, 5, 10
  - Permutation BAC
    - Marginal Contributions of A, B, C: 10, 5, 5
  - Permutation BCA
    - Marginal Contributions of A, B, C: 5, 5, 10
  - Permutation CBA
    - Marginal Contributions of A, B, C: 5, 10, 5
  - Permutation CAB
    - Marginal Contributions of A, B, C: 10, 5, 5

$\Rightarrow$Not in the core!

Convex Games

Convex game

A game (N, v) is convex, iff the value of a coalition increases no slower when these coalitions grow in size, i.e., $v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T)$ for all $S \subseteq T \subseteq N, i \in N \setminus T$.
For cost games, the inequalities are reversed.

Theorem

In every convex game, the Shapley value is in the core.

Corollary

Every convex game has a nonempty core.

$\Rightarrow$ Fair and stable distributions exist!

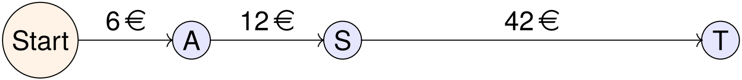
Interesting Example I: Bankruptcy game instance

- Claimants: $N = \{A, B\}$
- Claims: $c_A = 80, c_B = 40$
- Estate: $E = 100$
- $v(C) = \max\{0, E - \sum_{i \in N \setminus C} c_i\}$
  - $v(\emptyset) = 0, v(\{A\}) = 60, v(\{B\}) = 20, v(\{A, B\}) = 100$

Properties

- This game is convex $\Rightarrow$the Shapley value is in the core.
- Shapley value: $\Psi = (\Psi_A, \Psi_B) = \frac{(60,40)+(80,20)}{2} = (70, 30)$
- In core indeed, because:
  - $\Psi_A = 70 \geq v(\{A\}) = 60$ ☺
  - $\Psi_B = 30 \geq v(\{B\}) = 20$ ☺
  - $\Psi_A + \Psi_B = 70 + 30 \geq v(\{A, B\}) = 100$ ☺

Interesting Example II: Taxi Share



- characteristic function v
  - $v(\{A\}) = 6$
  - $v(\{S\}) = 12$
  - $v(\{T\}) = 42$
  - $v(\{A, S\}) = 12$
  - $v(\{A, T\}) = 42$
  - $v(\{S, T\}) = 42$
  - $v(\{A, S, T\}) = 42$
- Shapley value computation
  - $(A, S, T) \to (6, 6, 30)$
  - $(A, T, S) \to (6, 0, 36)$
  - $(S, A, T) \to (0, 12, 30)$
  - $(S, T, A) \to (0, 12, 30)$
  - $(T, A, S) \to (0, 0, 42)$
  - $(T, S, A) \to (0, 0, 42)$
  - $\Psi(N, v) = (2, 5, 35)$

Intermediate Summary

- Cooperative game theory is concerned with what agents can achieve if they form coalitions, viz., binding agreements.
  - Values are given to coalitions first
  - Coalitions redistribute value to their members
- Solution concepts for cooperative games
  - Core: stability; sometimes exists; not unique
  - Shapley value: fairness; always exists; unique
  - For convex games, the Shapley value is in the Core
- Next
  - Computational aspects
  - Coalition structure formation

Computational aspects

Remember: Given a cooperative game (N, v), the Shapley Value divides value according to:

$$\Psi_i(N, v) = \frac{1}{N!} \sum_{o \in \Pi(N)} \mu_i(C_i(o))$$

- Imagine you wanted to compute the Shapley value of an agent i of a cooperative game (N, v)

  def shapleyValue(N, v, i):

  ...
- How many entries are in v?
- How many steps are necessary to compute Shapley value?

Feasible Game Types

- Some cooperative games can be treated more efficiently
  - Weighted graph games
  - Weighted voting games
- Centralized algorithm for coalition structure generation

Weighted graph game: Definition

Assumption

The value of a coalition is the sum of the pairwise synergies among agents.

Definition

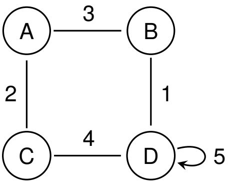
Let (V, W) denote an undirected weighted graph, where V is the set of vertices and $W \in \mathbb{R}^{V \times V}$ is the set of edge weights; denote the weight of the edge between vertices i and j as $w_{\{i,j\}}$. This graph defines a weighted graph game, where the cooperative game is constructed as follows:

- $N = V$
- $v(S) = \sum_{\{i,j\} \subseteq S} w_{\{i,j\}}$

Weighted graph game: Example

Revenue Sharing game

Consider the problem of dividing the revenues from toll highways between the cities that the highways connect. The pair of cities connected by a highway get to share in the revenues only when they form an agreement on revenue splitting; otherwise, the tolls go to the state.



$$
\begin{aligned}
v(\{A,B,C\}) &= 3+2=5 \\
v(\{D\}) &= 5 \\
v(\{B,D\}) &= 1+5=6 \\
v(\{A,C\}) &= 2
\end{aligned}
$$

Weighted graph game: Shapley Value

1. Only N^2 many values to store (adjacency matrix).
2. Shapley-Value sh_i of agent i: $sh_i = w_{\{i,i\}} + \frac{1}{2}\sum_{i \neq j} w_{\{i,j\}}$

Each pair of agents plays a game, in which they are interchangeable. Thus, they get the same value (Symmetry).

Axiom Symmetry

For any $S \subseteq N, v$, if i and j are interchangeable then $\Psi_i(S, v) = \Psi_j(S, v)$.

Value adds up in "bigger" games due to Additivity.

Axiom Additivity

For any two v_1, v_2, it holds that $\Psi_i(N, v_1 + v_2) = \Psi_i(N, v_1) + \Psi_i(N, v_2)$ for each i, where the game $(N, v_1 + v_2)$ is defined by $(v_1 + v_2)(S) = v_1(S) + v_2(S)$.

Weighted graph game: Properties

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Theorem

If all the weights are nonnegative then the game is convex.

Remember:

Theorem

Every convex game has a nonempty core.

Theorem

In every convex game, the Shapley value is in the core.

$\Rightarrow$A fair and stable value distribution exists and can be computed in polynomial time w.r.t. to number of agents.

Weighted graph game: Incompleteness

- Not every game can be represented as a weighted graph game. For example, consider this voting game:
- Parties A, B, C, D have 45, 25, 15, and 15 representatives, respectively. 51 votes are required to pass the bill.
  - Agents: $N = \{A, B, C, D\}$
  - Coalitions: $\{A\}, \ldots, \{A, B, C, D\} \in 2^N$
  - Characteristic function $v : 2^N \to \mathbb{R}$
    - $v(\{A\}) = v(\{B\}) = v(\{C\}) = v(\{D\}) = v(\{B, C\}) = v(\{B, D\}) = v(\{C, D\}) = 0$
    - $v(\{A, B\}) = v(\{A, C\}) = v(\{A, D\}) = v(\{B, C, D\}) = 1$
  - E.g., $v(\{B, C\}) + v(\{B, D\}) + v(\{C, D\}) \neq v(\{B, C, D\})$ ☹, i.e., the overall payoff is not the result of the local coalitions.

Weighted voting game: Definition

Definition

A weighted voting game $(q; w_1, \ldots, w_n)$ consists of a set of agents $Ag = \{1, \ldots, n\}$ and a quota q. The cooperative game (N, v) is then given by:

- $N = Ag$
- $v(C) = \begin{cases} 1, & \sum_{i \in C} w_i \geq q \\ 0, & else \end{cases}$

Weighted voting game: Example

- Parties A, B, C, D have 45, 25, 15, and 15 representatives, respectively. 51 votes are required to pass the bill.
  - Weighted voting game: $(51; 45, 25, 15, 15)$

Weighted voting game: Properties

- Computing the Shapley value is NP-hard
- But checking if core is non-empty is easy

Remember:

Theorem

In a simple game the core is empty iff there is no veto agent. If there are veto agents, the core consists of all value distributions in which the nonveto agents get 0.

- Check if agent i is veto agent:
  1. Draw up $C = N \setminus \{i\}$
  2. Check that both hold:
     - $\sum_{j \in C} w_j < q$, i.e., no winner without i
     - $\sum_{j \in C \cup \{i\}} w_j \geq q$, i.e., winner with i

Next

- Some cooperative games can be treated more efficiently
  - Weighted graph games
  - Weighted voting games
- Centralized algorithm for coalition structure generation

Coalition Structure Formation

- Agents can use their capacity to compute Shapley values to try to optimize their local payoff.
- If, however, there is a central component that knows of all the agents, this component can attempt to maximize social welfare of the whole system.

Coalition Structure

A coalition structure is a partition of the overall set of agents N into mutually disjoint coalitions.

Example, with $N = \{1,2,3\}$:

- Seven possible coalitions:

$$\{1\},\{2\},\{3\},\{1,2\},\{2,3\},\{3,1\},\{1,2,3\}$$

- Five possible coalition structures:

$$\{\{1\},\{2\},\{3\}\},\{\{1\},\{2,3\}\},\{\{2\},\{1,3\}\},$$

$$\{\{3\},\{1,2\}\},\{\{1,2,3\}\}$$

Coalition Structure Formation

Given game $G = (N, v)$, the socially optimal coalition structure CS^* is defined as:

$$CS^* = \text{argmax}_{CS \in \text{ partitions of } N} V(CS)$$

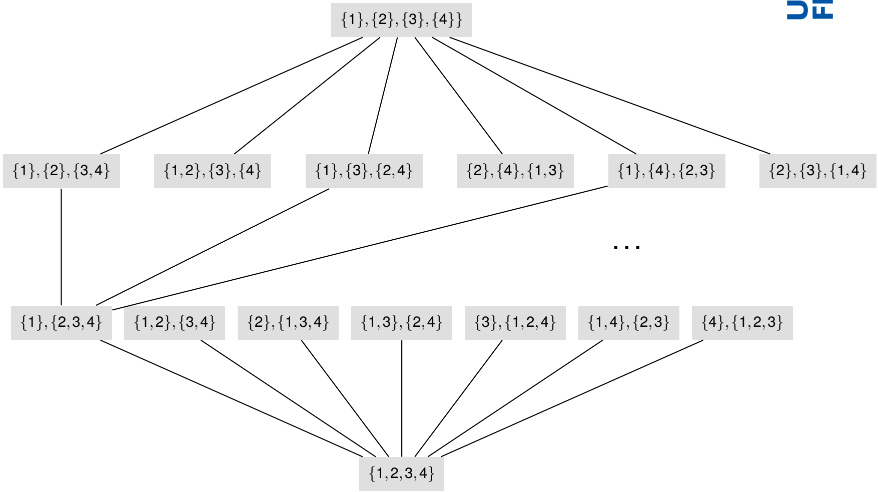
where

$$V(CS) = \sum_{C \in CS} v(C)$$

Unfortunately, there are exponentially many coalition structures over the sets of agents N

$\Rightarrow$ Exhaustive search is infeasible!

Coalition Structure Graph



Coalition Structure Graph: Search

- Observation: At the first two levels every coalition is present.
- Let CS' be the best structure we find in these levels.
- Let CS^* be the best structure overall (as defined earlier).
- Let $C^* = \text{argmax}_{C \subseteq N} v(C)$ the coalition with highest possible value.

Then:

- $V(CS^*) \leq |N| v(C^*) \leq |N| V(CS')$
- $\Rightarrow$in worst case, $V(CS') = \frac{V(CS^*)}{|N|}$

Algorithm:

1. Search first two bottom levels, keep track of best one.
2. Continue with breadth-first search beginning with top level.

Summary

- The Core: Stability / rationality of a coalition
- Shapley Value: Fairness
- Simple Games: Core easy to determine
- Convex Games: Shapley Value is in the core
- Weighted graph games
  - Compact representation of games with additive values
  - Efficient compuation of Shapley values
- Weighted voting games
  - Compact representation of certain simple games
  - Efficient computation of a core value distribution
- Coalition Structure Formation
  - Centralized search-based algorithm to find a partition of agents into coalitions maximizing overall value.
  - Provable bounds of solution quality.

Literature

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