Multi-Agent Systems

(Classical) Multi-Agent Path Finding

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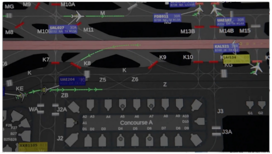
Agents moving in a spatial environment

A central problem in many applications is the coordinated movement of agents/robots/vehicles in a given spatial environment.

Logistic robots (KARIS)



Airport ground traffic control (atrics)

Multi-agent path finding

Definition (Multi-agent path finding (MAPF) problem)

Given a set of *agents* A, a (perhaps directed) *graph* $G = (V, E)$, an *initial state* modelled by an injective function $\alpha_0 : A \to V$, and a *goal state* modelled by another injective function α_*, can α_0 be *transformed* into α_* by *movements of single agents* without collisions?

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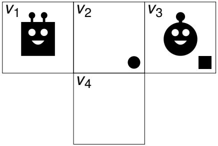
- *Optimal plan generation problem*: Generate a shortest plan.

Example

Can we find a (central) plan to move the square robot S to v_3 and the circle robot C to v_2?



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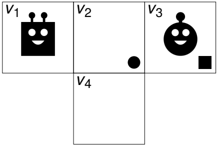

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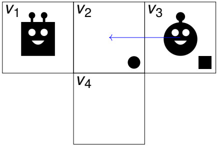

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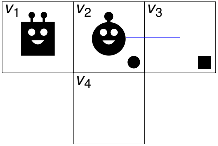

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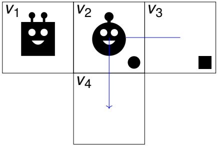

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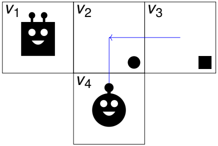

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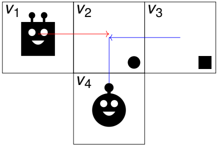

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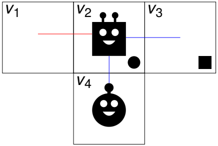

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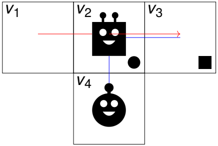

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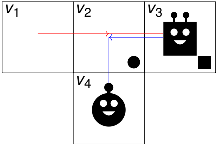

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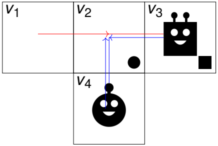

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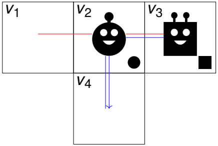

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A special case: 15-puzzle

Pictures from Wikipedia article on 15-Puzzle

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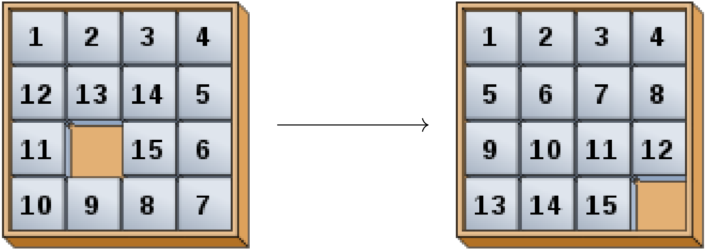

Pictures from Wikipedia article on 15-Puzzle

Lecture plan

- MAPF: variations, algorithms, complexity
- Distributed MAPF (each agent plans on it own): DMAPF
- Distributed MAPF with destination uncertainty: MAPF/DU

Sequential MAPF

- *Sequential MAPF* (or pebble motion on a graph) allows only one agent to move per time step.
- An agent $a \in A$ can move in one step from $s \in V$ to $t \in V$ transforming α to α', if
  - $\alpha(a) = s$,
  - $\langle s,t \rangle \in E$,
  - there is no agent b such that $\alpha(b) = t$.
- In this case, α' is determined as follows:
  - $\alpha'(a) = t$,
  - for all agents $b \neq a : \alpha(b) = \alpha'(b)$,
- One usually wants to minimize the number of single movements (= *sum-of-cost* over all agents)

Parallel MAPF

- *Parallel MAPF* allows many agents to move in parallel, provided they do not collide.
- Two models:
  - *Parallel*: A chain of agents can move provided the first agent can move on a an unoccupied vertex.
  - *Parallel with rotations*: A closed cycle in move synchronously.
- In both cases, one is usually interested in the number of parallel steps (= *make-span*).
- However, also the sum-of-cost is sometimes considered.

Anonymous MAPF

- There is a set of agents and a set of targets (of the same cardinality as the agent set).
- Each target must be reached by one agent.
- This means one first has to assign a target and then to solve the original MAPF problem.
- Interestingly, the problem as a whole is easier to solve (using flow-based techniques).

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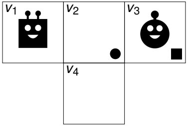
- *Rule-based algorithms*: *Kornhauser's algorithm*, *Push-and-Rotate*, **BIBOX**, …(complete on a given class of graphs, but suboptimal)

A*-based algorithm

- Define state space:
  - A state is an assignment of agents to vertices (modelled by a function α)
  - There is a transition from one state α to α' iff there is a legal move from α to α' according to the appropriate semantics (sequential, parallel, or parallel with rotations)
- Search in this state space using the A* algorithm.
- Possible *heuristic estimator*: Sum or maximum over the length of the individual movement plans (ignoring other agents).
- **Problem**: Large *branching factor* because of many agents that can move.

Example: State space for A* algorithm



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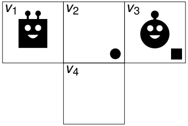

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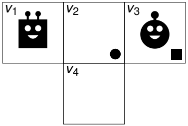

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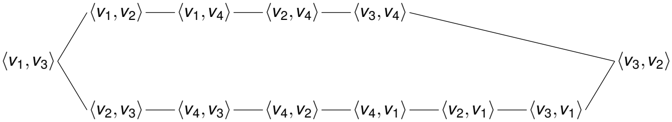
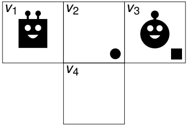


Example: State space for A^* algorithm



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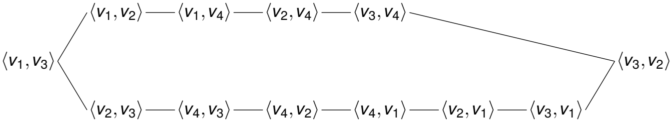

Question: Heuristic value for states $\langle v_1, v_2 \rangle$ and $\langle v_2, v_3 \rangle$ under the sum-aggregation?

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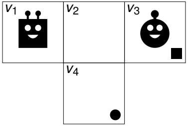
  - Solvability depends on chosen order.
  - Our small example is not solvable (shortest paths lead to head on collision), but small modification works.

Example CA* run



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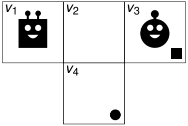

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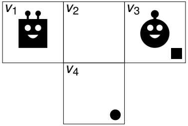

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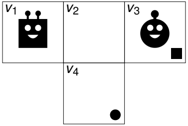

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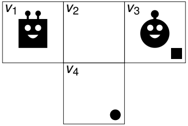

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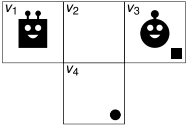

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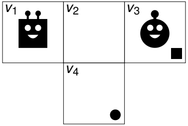

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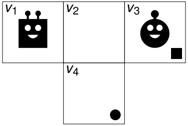

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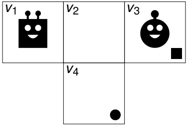

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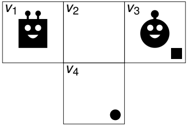

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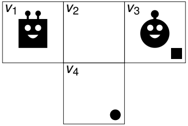

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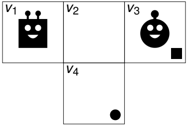

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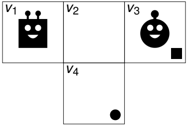

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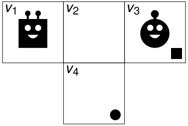

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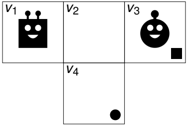

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- Not solvable with different order!

BIBOX

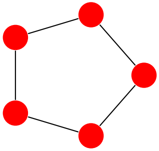
BIBOX is a rule-based algorithm that is complete on all *bi-connected* graphs with at least two unoccupied nodes in the graph.

Definition

A graph $G = (V, E)$ is *connected* iff $|V| \geq 2$ and there is *path* between each pair of nodes $s, t \in V$. A graph is *bi-connected* iff $|V| \geq 3$ and for each $v \in V$, the graph $(V - \{v\}, E')$ with $E' = \left\{ \{x, y\} \in E \,|\, x, y \neq v \right\}$ is connected.

Loop decomposition

Every bi-connected graph can be constructed from a *cycle* by adding *loops* iteratively.



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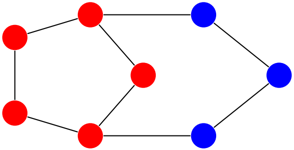

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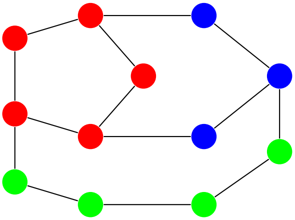

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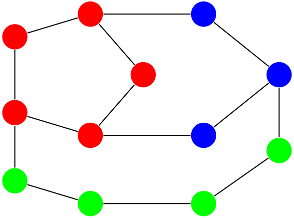

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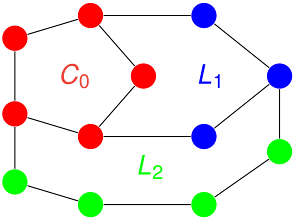
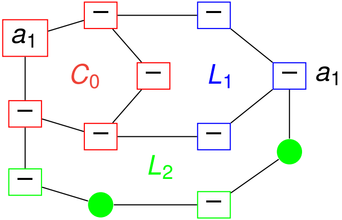


A *loop decomposition* into a basic cycle and additional loops can be done in time $O(|V|^2)$.
Let us name them C_0, L_1, L_2, ..., where the index depends on the time when the loop is added.

Moving unoccupied nodes and agents around



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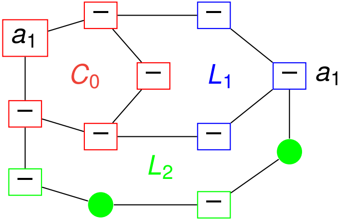

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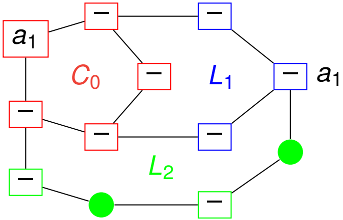

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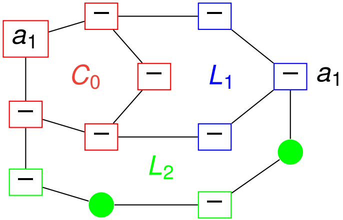

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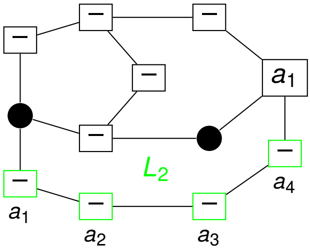
- This can be done without disturbing loops with a higher index than the one the agent starts and finishes in.

Filling loops

- Starting with highest-index loop: Move agents to destination loop, then shift agents to their destinations.
- Special case: When agents are already in the destination loop, they have to be rotated out of the loop.



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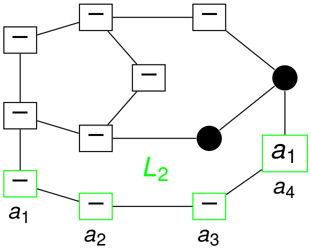

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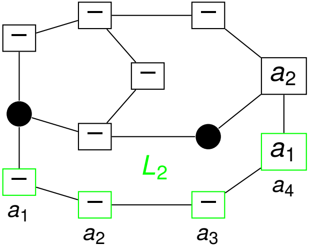

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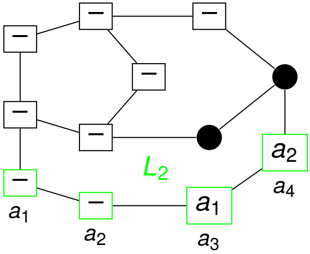

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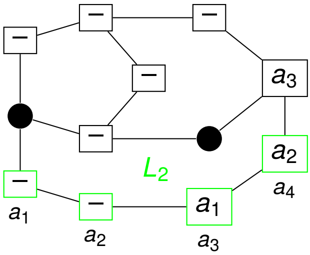

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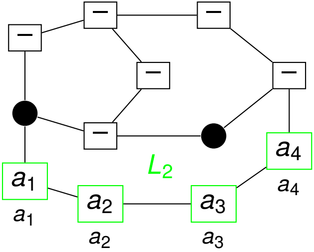

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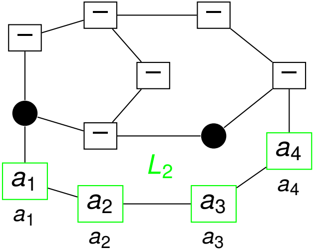

- When done with one loop, repeat for next one with next lower index.

Reordering agents in the cycle

- Assumption: The destinations for the empty places are in the cycle C_0 (can be relaxed).
- If the agents are in the right order, just rotate them to their destinations.
- Otherwise reorder by successively take one out and re-insert.

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- Existence: For arbitrary graphs with at least one empty place, the problem is polynomial ($O(|V|^3)$ using Kornhauser's algorithm). For BIBOX on bi-connected with at least two empty places also cubic, but smaller constant.

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- Question: Is the problem also NP-hard?

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Definition (Exact Cover By 3-Sets (X3C) Problem)

Given a set of elements U and a collection of subsets $C = \{s_j\}$ with $s_j \subseteq U$ and $|s_j| = 3$. Is there a sub-collection of subsets $C' \subseteq C$ such that $\bigcup_{s \in C'} s = U$ and all subsets in C' are pairwise disjoint, i.e., $s_a \cap s_b = \emptyset$ for each $s_a, s_b \in C'$ with $s_a \neq s_b$?

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NP-hardness of MAPF: Reduction from X3C

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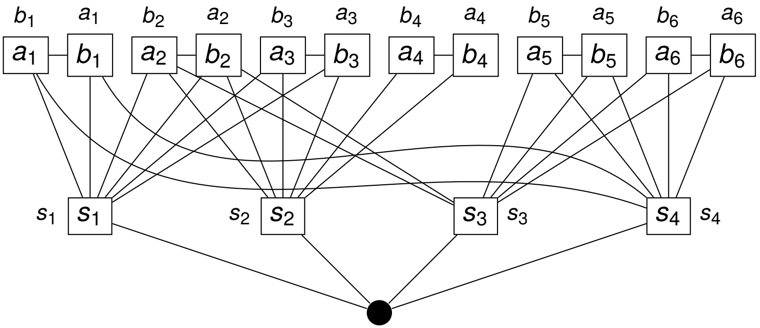

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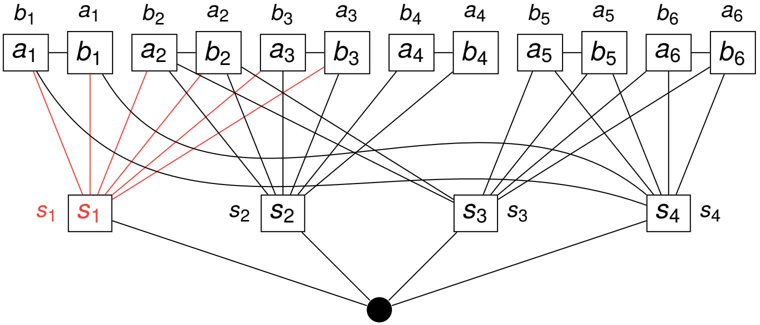
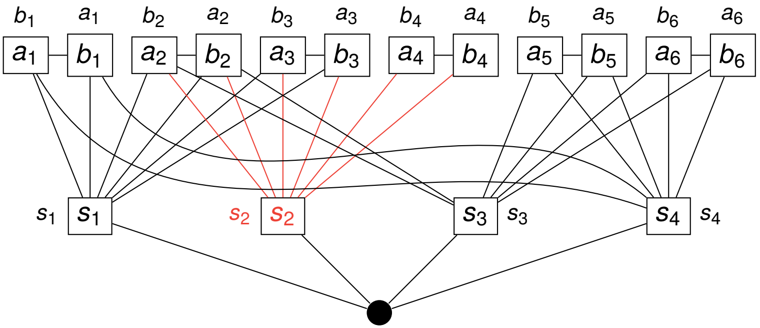


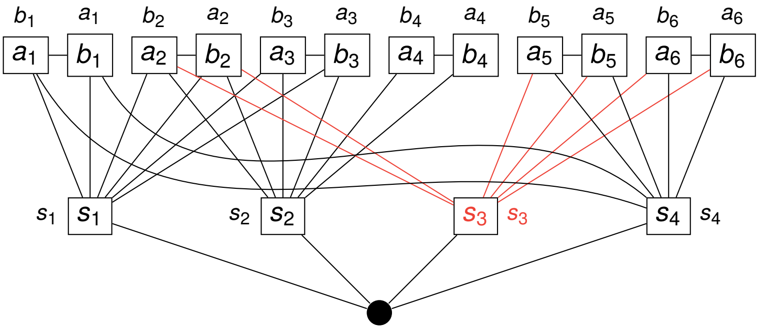
NP-hardness of MAPF: Reduction from X3C

$C = \{\{1,2,3\}, \{2,3,4\}, \{2,5,6\}, \{1,5,6\}\}$



NP-hardness of MAPF: Reduction from X3C

$C = \{\{1,2,3\},\{2,3,4\},\{2,5,6\},\{1,5,6\}\}$



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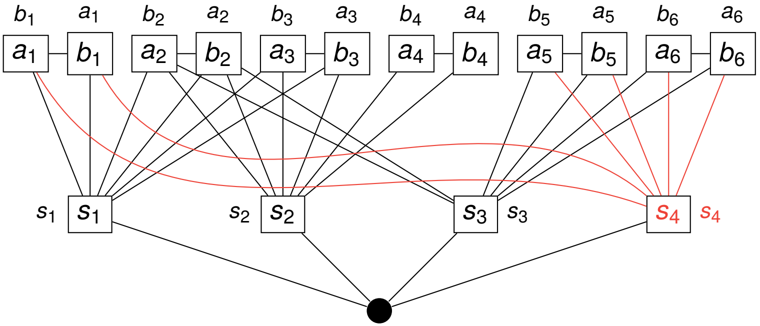

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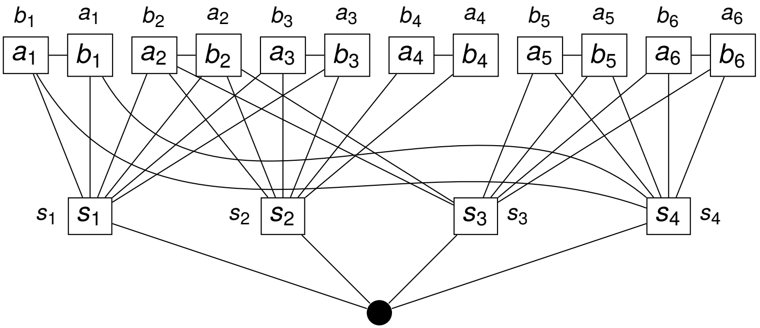

Claim: There is an exact cover by 3-sets iff the constructed MAPF instance can be solved in at most $k = 11/3|U|$ moves.

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