Multi-Agent Systems

Modal Logic for Multi-Agent Systems, Tableaux

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Recap

- Last time
  - Kripke models represent specific situations involving Knowledge, Desires, Obligations, ...
  - The language of modal logic can be used to formally talk about Kripke models.
  - Model Checking: Given a formula, is it true in possible world w in Kripke model M?
- Today
  - Beyond specific situations: Automated satisfiability checking.

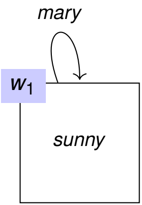
Motivating Example

Consider the personal-assistant robot Alfred. Alfred maintains knowledge about the people he cares for. E.g., Alfred can represent that Mary knows that the sun is shining (and therefore there is no need to tell her about the weather conditions).

Modeling Alfred's Knowledge: Model Checking vs. Theorem Proving

- Traditionally, two approaches can be distinguished (cf., [3] for a discussion):
  - What the agent knows is represented as a Kripke model. Reasoning is modeled as deleting/adding nodes/edges, and model checking.
  - What the agent knows is represented as a set of formulae. Reasoning is modeled as deleting/adding formulae, and theorem proving.

M_0:



$KB = \{K_{mary} sunny\}$

Absence of Knowledge I

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- Observations
  1. While each of M_0, M_1, M_2 agrees about Mary’s knowledge (of which Alfred is sure), they disagree about Tom’s knowledge (of which Alfred has no information).
  2. Why make a choice? Alfred’s answer should be “Maybe, depends on how the world actually looks like...”
     $\Rightarrow$Consider all possible models.

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  - $KB \models_{\mathbf{S5}_n} K_{tom} sunny$? Answer: No, because there are models in which KB is true and $K_{tom} sunny$ is false (e.g., M_2).
  - $KB \models_{\mathbf{S5}_n} \neg K_{tom} sunny$? Answer: No, because there are models in which KB is true and $\neg K_{tom} sunny$ is false (e.g., M_1).

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$\Rightarrow$It is possible that both a formula and its negation are satisfiable. In this case, none of them is valid, and the agent may answer "Maybe, depends on how the world actually looks like..."

Conclusions and Outlook

- Wanted: A procedure to check satisfiability of a modal-logic formula.
  - Can then be used to check validity of a formula by proving its negation unsatisfiable.
- Good news: Satisfiability is decidable for all the modal logics we consider.
- Approach: For a given formula, we will try to construct a Kripke model. If we succeed, the input formula is satisfiable. If we fail, the input formula is unsatisfiable (and thus its negation is valid).
  - Next: Sound, Complete, and Terminating procedure described in [1, 2].

Premodels

Def. Premodel

Given a set of labels L, a premodel is a labelled graph $M = (W, R, V)$ where: W is a non-empty set, $R : L \to 2^{W \times W}$, $V : L \to 2^{W}$.

- Idea
  - First, a premodel is initialized with an input formula whose satisfiability should be proven.
  - Then, rules transform the premodel to other premodels by systematically adding nodes, edges, and formulae.
  - Finally, if no more rules are applicable, a Kripke model can be derived from a premodel iff the input formula is satisfiable.

Rules for Boolean Connectives

- And: If node contains formula $(\varphi \wedge \psi)$ then add φ and ψ.
- NotAnd: If node contains formula $\neg(\varphi \wedge \psi)$ then add $(\neg\varphi \vee \neg\psi)$.
- NotNot: If node contains formula $\neg\neg\varphi$ then add φ.
- NotOr: If node contains formula $\neg(\varphi \vee \psi)$ then add $(\neg\varphi \wedge \neg\psi)$.
- Or: If node contains formula $(\varphi \vee \psi)$ then copy the graph g to g' and add φ to the node in g and ψ to the node in g'.
- Impl: If node contains formula $(\varphi \rightarrow \psi)$ then add $(\neg\varphi \vee \psi)$.
- NotImpl: If node contains formula $\neg(\varphi \rightarrow \psi)$ then add $(\varphi \wedge \neg\psi)$.
- $\perp$: If node contains φ and $\neg\varphi$ then add $\perp$.

Saturation and Openness

- The rules for rewriting the graphs are applied as often as possible.
- A premodel is saturated when no more rule can be applied.
- Premodels with a node containing $\bot$ are called closed; otherwise they are called open.

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$\{(rain \rightarrow wet), \neg wet\} \models \neg rain$?

- to show: $\models ((rain \rightarrow wet) \land \neg wet) \rightarrow \neg rain)$
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$\Rightarrow$No open premodel found. Kripke model cannot be constructed. Formula is unsatisfiable. Hence, it’s negation is valid (q.e.d).

Rules for Modalities

- $\langle \mathbf{l} \rangle$: If node contains formula $\langle \mathbf{l} \rangle \varphi$ and so far no l-successor contains φ then add an l-labeled edge to a new node that contains φ.
- $[\mathbf{l}]$: If node contains formula $[\mathbf{l}]\varphi$ then add φ to all l-connected nodes (that do not already contain φ).
- $\neg\langle \mathbf{l} \rangle$: If node contains formula $\neg\langle \mathbf{l} \rangle \varphi$ then add $\neg\varphi$ to all l-connected nodes (that do not already contain $\neg\varphi$).
- $\neg[\mathbf{l}]$: If node contains formula $\neg[\mathbf{l}]\varphi$ and so far no l-successor contains $\neg\varphi$ then add an l-labeled edge to a new node that contains $\neg\varphi$.

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- to show: ¬*brown_eyes* ∧ **<sibling>[sibling]***brown_eyes* is **K**-satisfiable.

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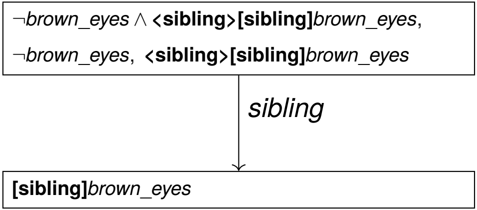

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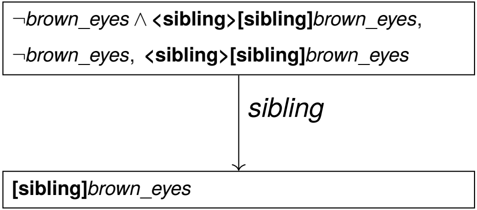

- A Kripke model can be derived: $M = (W, R, V)$ with $W = \{w_1, w_2\}$, $R(sibling) = \{(w_1, w_2)\}$, $V(brown\_eyes) = \{\}$.
  Indeed
  $M, w_1 \models$ ¬*brown_eyes* ∧ **<sibling>[sibling]***brown_eyes*.
- Problem: The relation **sibling** should be symmetric.

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  - First attempt: If node has no *I*-successor then add one. (Problem: Won't terminate.)
  - Better: If node has no *I*-successor and contains **[I]** then add a new *I*-successor. After the premodel is built, add reflexive edges to leaf nodes.

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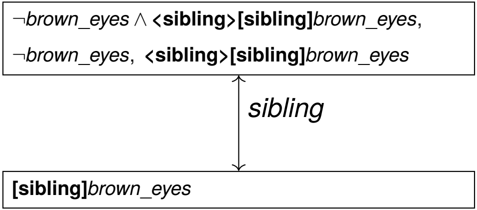

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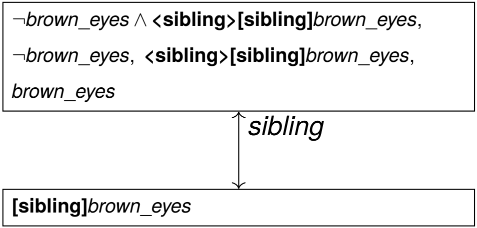

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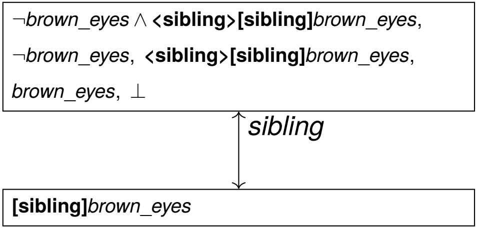

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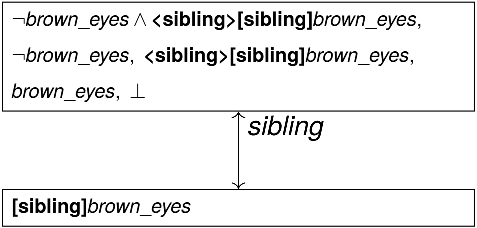

- No Kripke model can be derived. ⇒The formula is unsatisfiable in **KB**, hence its negation is **KB**-valid (q.e.d).

Remark: Multiple Modalities

- Different modalities can be mixed. E.g., the approach also works for $\mathbf{S5}_n$ (multi-agent knowledge), which we will have a closer look on next time. E.g., also $K_{mary}K_{tom}p \rightarrow K_{mary}p$ is valid in $\mathbf{S5}_n$.
- However, in general one has to mind undesired interactions. E.g., mixing the epistemic modality K (**S5**) and the deontic modality O (**KD**) yields the validity $\models_{\mathbf{S5} \otimes \mathbf{KD}} OKp \rightarrow Op$, which says that only obligatory facts must be known.

Literature I

B. Said, **Graph rewriting for model construction in modal logic**, PhD Thesis, Université Paul Sabatier – Toulouse III, 2010. Available Online: `https://tel.archives-ouvertes.fr/tel-00466115/`

O. Gasquet, A. Herzig, B. Said, F. Schwarzentruber, **Kripke's Worlds — An Introduction to Modal Logics via Tableaux**, Springer, ISBN 978-3-7643-8503-3, 2014.

J. Y. Halpern and M. Y. Vardi. **Model checking vs. theorem proving: A manifesto**. Artificial Intelligence and Mathematical Theory of Computation, 212:151–176, 1991.