

# Principles of AI Planning

## 10. Planning as search: abstractions

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## General procedure for obtaining a heuristic

Solve an easier version of the problem.

Two common methods:

- **relaxation**: consider **less constrained** version of the problem
- **abstraction**: consider **smaller** version of real problem

In previous chapters, we have studied **relaxation**, which has been very successfully applied to **satisficing planning**.

Now, we study **abstraction**, which is one of the most prominent techniques for **optimal planning**.

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Abstracting a transition system means **dropping some distinctions** between states, while **preserving the transition behaviour** as much as possible.

- An abstraction of a transition system  $\mathcal{T}$  is defined by an **abstraction mapping**  $\alpha$  that defines which states of  $\mathcal{T}$  should be distinguished and which ones should not.
- From  $\mathcal{T}$  and  $\alpha$ , we compute an **abstract transition system**  $\mathcal{T}'$  which is similar to  $\mathcal{T}$ , but smaller.
- The **abstract goal distances** (goal distances in  $\mathcal{T}'$ ) are used as heuristic estimates for goal distances in  $\mathcal{T}$ .

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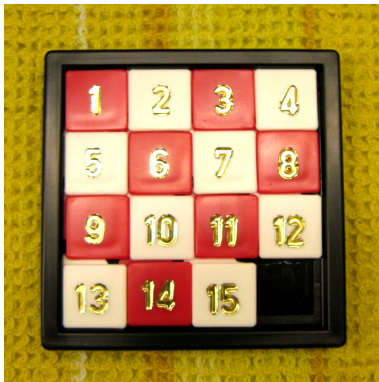
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# Abstracting a transition system: example

## Example (15-puzzle)



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## Example (15-puzzle)

A **15-puzzle** state is given by a permutation  $\langle b, t_1, \dots, t_{15} \rangle$  of  $\{1, \dots, 16\}$ , where  $b$  denotes the blank position and the other components denote the positions of the 15 tiles.

One possible **abstraction mapping** ignores the precise location of tiles 8–15, i. e., two states are distinguished iff they differ in the position of the blank or one of the tiles 1–7:

$$\alpha(\langle b, t_1, \dots, t_{15} \rangle) = \langle b, t_1, \dots, t_7 \rangle$$

The heuristic values for this abstraction correspond to the cost of moving tiles 1–7 to their goal positions.

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# Abstraction example: 15-puzzle



9	2	12	6
5	7	14	13
3	4	1	11
15	10	8	



1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	

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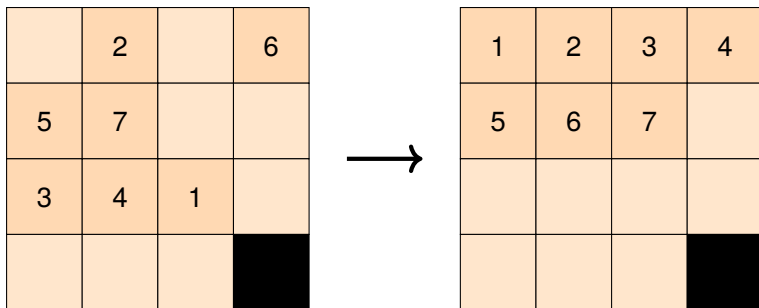
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## real state space

- $16! = 20922789888000 \approx 2 \cdot 10^{13}$  states
- $\frac{16!}{2} = 10461394944000 \approx 10^{13}$  reachable states

# Abstraction example: 15-puzzle



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## abstract state space

- $16 \cdot 15 \cdot \dots \cdot 9 = 518918400 \approx 5 \cdot 10^8$  states
- $16 \cdot 15 \cdot \dots \cdot 9 = 518918400 \approx 5 \cdot 10^8$  reachable states



Given  $\mathcal{T}$  and  $\alpha$ , how do we compute  $\mathcal{T}'$ ?

## Requirement

We want to obtain an **admissible heuristic**.

Hence,  $h^*(\alpha(s))$  (in the abstract state space  $\mathcal{T}'$ ) should never overestimate  $h^*(s)$  (in the concrete state space  $\mathcal{T}$ ).

An easy way to achieve this is to ensure that **all solutions in  $\mathcal{T}$  also exist in  $\mathcal{T}'$** :

- If  $s$  is a goal state in  $\mathcal{T}$ , then  $\alpha(s)$  is a goal state in  $\mathcal{T}'$ .
- If  $\mathcal{T}$  has a transition from  $s$  to  $t$ , then  $\mathcal{T}'$  has a transition from  $\alpha(s)$  to  $\alpha(t)$ .

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## Example (15-puzzle)

In the running example:

- $\mathcal{T}$  has the unique goal state  $\langle 16, 1, 2, \dots, 15 \rangle$ .
  - $\rightsquigarrow \mathcal{T}'$  has the unique goal state  $\langle 16, 1, 2, \dots, 7 \rangle$ .
- Let  $x$  and  $y$  be neighboring positions in the  $4 \times 4$  grid.
  - $\mathcal{T}$  has a transition from  $\langle x, t_1, \dots, t_{i-1}, y, t_{i+1}, \dots, t_{15} \rangle$  to  $\langle y, t_1, \dots, t_{i-1}, x, t_{i+1}, \dots, t_{15} \rangle$  for all  $i \in \{1, \dots, 15\}$ .
    - $\rightsquigarrow \mathcal{T}'$  has a transition from  $\langle x, t_1, \dots, t_{i-1}, y, t_{i+1}, \dots, t_7 \rangle$  to  $\langle y, t_1, \dots, t_{i-1}, x, t_{i+1}, \dots, t_7 \rangle$  for all  $i \in \{1, \dots, 7\}$ .
    - $\rightsquigarrow$  Moreover,  $\mathcal{T}'$  has a transition from  $\langle x, t_1, \dots, t_7 \rangle$  to  $\langle y, t_1, \dots, t_7 \rangle$  if  $y \notin \{t_1, \dots, t_7\}$ .

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To be useful in practice, an abstraction heuristic must be efficiently computable. This gives us two requirements for  $\alpha$ :

- For a given state  $s$ , the **abstract state**  $\alpha(s)$  must be efficiently computable.
- For a given abstract state  $\alpha(s)$ , the **abstract goal distance**  $h^*(\alpha(s))$  must be efficiently computable.

There are different ways of achieving these requirements:

- **pattern database heuristics** (Culberson & Schaeffer, 1996)
- merge-and-shrink abstractions (Dräger, Finkbeiner & Podelski, 2006)
- structural patterns (Katz & Domshlak, 2008)
- Cartesian abstractions (Ball, Podelski & Rajamani, 2001; Seipp & Helmert, 2013)

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# Practical requirements for abstractions: example



## Example (15-puzzle)

In our running example,  $\alpha$  can be very efficiently computed: just project the given 16-tuple to its first 8 components.

To compute abstract goal distances efficiently during search, most common algorithms precompute **all abstract goal distances** prior to search by performing a backward breadth-first search from the goal state(s). The distances are then stored in a table (requires about 495 MB of RAM). During search, computing  $h^*(\alpha(s))$  is just a table lookup.

This heuristic is an example of a **pattern database heuristic**.

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- One important practical question is how to come up with a suitable abstraction mapping  $\alpha$ .
- Indeed, there is usually a **huge number of possibilities**, and it is important to pick good abstractions (i. e., ones that lead to informative heuristics).
- However, it is generally **not necessary to commit to a single abstraction**.

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## Maximizing several abstractions:

- Each abstraction mapping gives rise to an admissible heuristic.
- By computing the **maximum** of several admissible heuristics, we obtain another admissible heuristic which **dominates** the component heuristics.
- Thus, we can always compute several abstractions and maximize over the individual abstract goal distances.

## Adding several abstractions:

- In some cases, we can even compute the **sum** of individual estimates and still stay admissible.
- Summation often leads to **much higher estimates** than maximization, so it is **important to understand when it is admissible**.

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## Example (15-puzzle)

- mapping to tiles 1–7 was arbitrary  
   $\rightsquigarrow$  can use **any subset** of tiles
- with the same amount of memory required for the tables for the mapping to tiles 1–7, we could store the tables for **nine different abstractions** to six tiles and the blank
- use **maximum** of individual estimates

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# Adding several abstractions: example



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15	10	8	

9	2	12	6
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■ **1st abstraction:** ignore precise location of 8–15

■ **2nd abstraction:** ignore precise location of 1–7

⇒ Is the **sum** of the abstraction heuristics **admissible**?



# Adding several abstractions: example



	2		6
5	7		
3	4	1	

9		12	
		14	13
			11
15	10	8	

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Summary

■ **1st abstraction:** ignore precise location of 8–15

■ **2nd abstraction:** ignore precise location of 1–7

⇒ The **sum** of the abstraction heuristics is **not admissible**.

# Adding several abstractions: example



	2		6
5	7		
3	4	1	

9		12	
		14	13
			11
15	10	8	

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■ **1st abstraction:** ignore precise location of 8–15 and blank

■ **2nd abstraction:** ignore precise location of 1–7 and blank

⇒ The **sum** of the abstraction heuristics is **admissible**.

# Our plan for the next lectures



In the following, we take a deeper look at abstractions and their use for admissible heuristics.

- In the rest of **this chapter**, we **formally introduce** abstractions and abstraction heuristics and study some of their most important properties.
- In the **following chapter**, we discuss one particular class of abstraction heuristics in detail, namely **pattern database heuristics**.

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Reminder from Chapter 2:

## Definition (transition system)

A **transition system** is a 5-tuple  $\mathcal{T} = \langle S, L, T, s_0, S_\star \rangle$  where

- $S$  is a finite set of **states**,
- $L$  is a finite set of (transition) **labels**,
- $T \subseteq S \times L \times S$  is the **transition relation**,
- $s_0 \in S$  is the **initial state**, and
- $S_\star \subseteq S$  is the set of **goal states**.

We say that  $\mathcal{T}$  **has the transition**  $\langle s, l, s' \rangle$  if  $\langle s, l, s' \rangle \in T$ .

We also write this  $s \xrightarrow{l} s'$ , or  $s \rightarrow s'$  when not interested in  $l$ .

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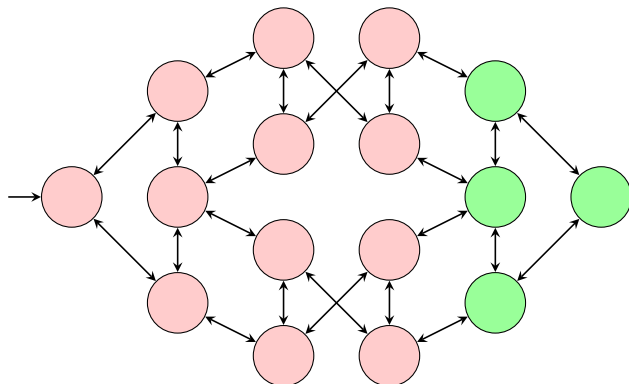
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**Note:** To reduce clutter, our figures usually omit arc labels and collapse transitions between identical states. However, these are important for the formal definition of the transition system.

## Definition (induced transition system of an FDR planning task)

Let  $\Pi = \langle V, I, O, \gamma \rangle$  be an FDR planning task.

The **induced transition system of  $\Pi$** , in symbols  $\mathcal{T}(\Pi)$ , is the transition system  $\mathcal{T}(\Pi) = \langle S, L, T, s_0, S_\star \rangle$ , where

- $S$  is the set of states over  $V$ ,
- $L = O$ ,
- $T = \{ \langle s, o, t \rangle \in S \times L \times S \mid \text{app}_o(s) = t \}$ ,
- $s_0 = I$ , and
- $S_\star = \{ s \in S \mid s \models \gamma \}$ .

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# Example task: one package, two trucks

## Example (one package, two trucks)

Consider the following FDR planning task  $\langle V, I, O, \gamma \rangle$ :

- $V = \{p, t_A, t_B\}$  with
  - $\mathcal{D}_p = \{L, R, A, B\}$
  - $\mathcal{D}_{t_A} = \mathcal{D}_{t_B} = \{L, R\}$
- $I = \{p \mapsto L, t_A \mapsto R, t_B \mapsto R\}$
- $O = \{\text{pickup}_{i,j} \mid i \in \{A, B\}, j \in \{L, R\}\}$   
 $\cup \{\text{drop}_{i,j} \mid i \in \{A, B\}, j \in \{L, R\}\}$   
 $\cup \{\text{move}_{i,j,j'} \mid i \in \{A, B\}, j, j' \in \{L, R\}, j \neq j'\}$ , where
  - $\text{pickup}_{i,j} = \langle t_i = j \wedge p = j, p := i \rangle$
  - $\text{drop}_{i,j} = \langle t_i = j \wedge p = i, p := j \rangle$
  - $\text{move}_{i,j,j'} = \langle t_i = j, t_i := j' \rangle$
- $\gamma = (p = R)$

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## Definition (abstraction, abstraction mapping)

Let  $\mathcal{T} = \langle S, L, T, s_0, S_\star \rangle$  and  $\mathcal{T}' = \langle S', L', T', s'_0, S'_\star \rangle$  be transition systems with the same label set  $L = L'$ , and let  $\alpha : S \rightarrow S'$  be a **surjective** function.

We say that  $\mathcal{T}'$  is **an abstraction of  $\mathcal{T}$  with abstraction mapping  $\alpha$**  (or: **abstraction function  $\alpha$** ) if

- $\alpha(s_0) = s'_0$ ,
- for all  $s \in S_\star$ , we have  $\alpha(s) \in S'_\star$ , and
- for all  $\langle s, \ell, t \rangle \in T$ , we have  $\langle \alpha(s), \ell, \alpha(t) \rangle \in T'$ .

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Let  $\mathcal{T}$  and  $\mathcal{T}'$  be transition systems and  $\alpha$  a function such that  $\mathcal{T}'$  is an abstraction of  $\mathcal{T}$  with abstraction mapping  $\alpha$ .

- $\mathcal{T}$  is called the **concrete transition system**.
- $\mathcal{T}'$  is called the **abstract transition system**.
- Similarly: **concrete/abstract state space**, **concrete/abstract transition**, etc.

We say that:

- $\mathcal{T}'$  is an **abstraction of  $\mathcal{T}$**  (without mentioning  $\alpha$ )
- $\alpha$  is an **abstraction mapping on  $\mathcal{T}$**  (without mentioning  $\mathcal{T}'$ )

**Note:** For a given  $\mathcal{T}$  and  $\alpha$ , there can be multiple abstractions  $\mathcal{T}'$ , and for a given  $\mathcal{T}$  and  $\mathcal{T}'$ , there can be multiple abstraction mappings  $\alpha$ .

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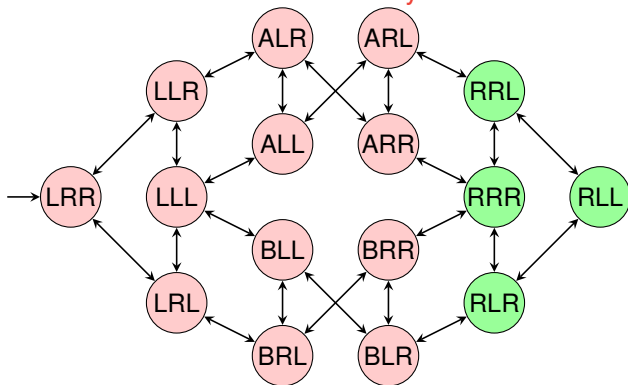
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concrete transition system



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## Definition (induced abstractions)

Let  $\mathcal{T} = \langle S, L, T, s_0, S_\star \rangle$  be a transition system, and let  $\alpha : S \rightarrow S'$  be a surjective function.

The **abstraction (of  $\mathcal{T}$ ) induced by  $\alpha$** , in symbols  $\mathcal{T}^\alpha$ , is the transition system  $\mathcal{T}^\alpha = \langle S', L, T', s'_0, S'_\star \rangle$  defined by:

- $T' = \{ \langle \alpha(s), \ell, \alpha(t) \rangle \mid \langle s, \ell, t \rangle \in T \}$
- $s'_0 = \alpha(s_0)$
- $S'_\star = \{ \alpha(s) \mid s \in S_\star \}$

**Note:** It is easy to see that  $\mathcal{T}^\alpha$  is an abstraction of  $\mathcal{T}$ . It is the “**smallest**” abstraction of  $\mathcal{T}$  with abstraction mapping  $\alpha$ .

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Let  $\mathcal{T}$  and  $\mathcal{T}'$  be transition systems and  $\alpha$  be a function such that  $\mathcal{T}' = \mathcal{T}^\alpha$  (i. e.,  $\mathcal{T}'$  is the abstraction of  $\mathcal{T}$  induced by  $\alpha$ ).

- $\alpha$  is called a **strict homomorphism** from  $\mathcal{T}$  to  $\mathcal{T}'$ , and  $\mathcal{T}'$  is called a **strictly homomorphic abstraction** of  $\mathcal{T}$ .
- If  $\alpha$  is bijective, it is called an **isomorphism** between  $\mathcal{T}$  and  $\mathcal{T}'$ , and the two transition systems are called **isomorphic**.

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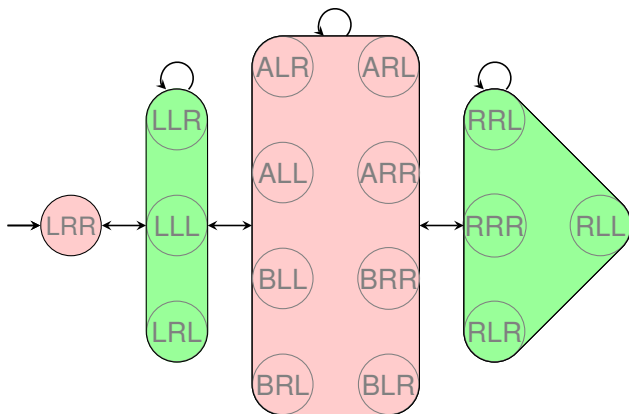
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# Strictly homomorphic abstractions: example



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If we add any goal states or transitions, it is still an abstraction of  $\mathcal{T}$ , but no longer a strictly homomorphic one.

## Definition (abstr. heur. induced by an abstraction)

Let  $\Pi$  be an FDR planning task with state space  $S$ , and let  $\mathcal{A}$  be an abstraction of  $\mathcal{T}(\Pi)$  with abstraction mapping  $\alpha$ .

The **abstraction heuristic induced by  $\mathcal{A}$  and  $\alpha$** ,  $h^{\mathcal{A},\alpha}$ , is the heuristic function  $h^{\mathcal{A},\alpha} : S \rightarrow \mathbb{N}_0 \cup \{\infty\}$  which maps each state  $s \in S$  to  $h_{\mathcal{A}}^*(\alpha(s))$  (the goal distance of  $\alpha(s)$  in  $\mathcal{A}$ ).

**Note:**  $h^{\mathcal{A},\alpha}(s) = \infty$  if no goal state of  $\mathcal{A}$  is reachable from  $\alpha(s)$

## Definition (abstr. heur. induced by strict homomorphism)

Let  $\Pi$  be an FDR planning task and  $\alpha$  a strict homomorphism on  $\mathcal{T}(\Pi)$ . The **abstraction heuristic induced by  $\alpha$** ,  $h^\alpha$ , is the abstraction heuristic induced by  $\mathcal{T}(\Pi)^\alpha$  and  $\alpha$ , i. e.,  
 $h^\alpha := h^{\mathcal{T}(\Pi)^\alpha, \alpha}$ .

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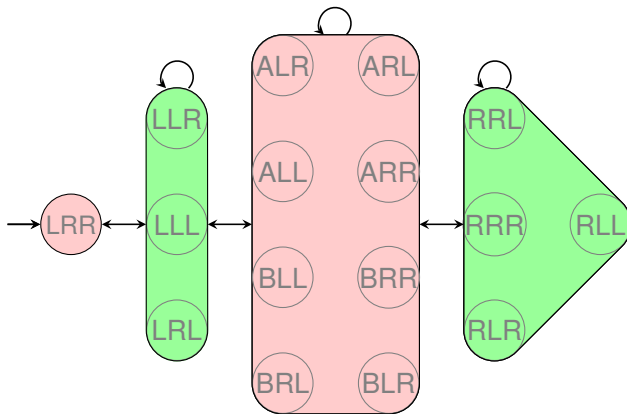
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# Abstraction heuristics: example



$$h^{\mathcal{A}, \alpha}(\{p \mapsto L, t_A \mapsto R, t_B \mapsto R\}) = 1$$

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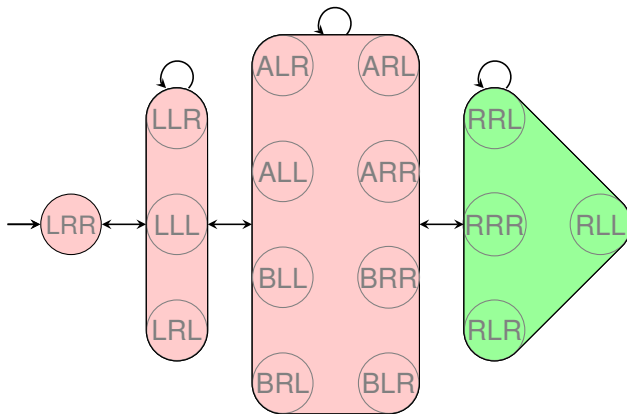
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# Abstraction heuristics: example



$$h^\alpha(\{p \mapsto L, t_A \mapsto R, t_B \mapsto R\}) = 3$$

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## Theorem (consistency and admissibility of $h^{\mathcal{A}, \alpha}$ )

Let  $\Pi$  be an FDR planning task, and let  $\mathcal{A}$  be an abstraction of  $\mathcal{T}(\Pi)$  with abstraction mapping  $\alpha$ .

Then  $h^{\mathcal{A}, \alpha}$  is safe, goal-aware, admissible and consistent.

## Proof.

We prove goal-awareness and consistency;  
the other properties follow from these two.

Let  $\mathcal{T} = \mathcal{T}(\Pi) = \langle S, L, T, s_0, S_\star \rangle$  and  $\mathcal{A} = \langle S', L', T', s'_0, S'_\star \rangle$ .

**Goal-awareness:** We need to show that  $h^{\mathcal{A}, \alpha}(s) = 0$  for all  $s \in S_\star$ , so let  $s \in S_\star$ . Then  $\alpha(s) \in S'_\star$  by the definition of abstractions and abstraction mappings, and hence  $h^{\mathcal{A}, \alpha}(s) = h^*_\mathcal{A}(\alpha(s)) = 0$ .

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## Proof (ctd.)

**Consistency:** Let  $s, t \in S$  such that  $t$  is a successor of  $s$ . We need to prove that  $h^{\mathcal{A}, \alpha}(s) \leq h^{\mathcal{A}, \alpha}(t) + 1$ .

Since  $t$  is a successor of  $s$ , there exists an operator  $o$  with  $app_o(s) = t$  and hence  $\langle s, o, t \rangle \in T$ .

By the definition of abstractions and abstraction mappings, we get  $\langle \alpha(s), o, \alpha(t) \rangle \in T' \rightsquigarrow \alpha(t)$  is a successor of  $\alpha(s)$  in  $\mathcal{A}$ .

Therefore,  $h^{\mathcal{A}, \alpha}(s) = h_{\mathcal{A}}^*(\alpha(s)) \leq h_{\mathcal{A}}^*(\alpha(t)) + 1 = h^{\mathcal{A}, \alpha}(t) + 1$ , where the inequality holds because the shortest path from  $\alpha(s)$  to the goal in  $\mathcal{A}$  cannot be longer than the shortest path from  $\alpha(s)$  to the goal via  $\alpha(t)$ . □

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## Definition (orthogonal abstraction mappings)

Let  $\alpha_1$  and  $\alpha_2$  be abstraction mappings on  $\mathcal{T}$ .

We say that  $\alpha_1$  and  $\alpha_2$  are **orthogonal** if for all transitions  $\langle s, \ell, t \rangle$  of  $\mathcal{T}$ , we have  $\alpha_i(s) \neq \alpha_i(t)$  for at most one  $i \in \{1, 2\}$ .

## Definition (affecting transition labels)

Let  $\mathcal{A}$  be a transition system, and let  $l$  be one of its labels.  
We say that  $l$  **affects**  $\mathcal{A}$  if  $\mathcal{A}$  has a transition  $\langle s, l, t \rangle$  with  $s \neq t$ .

## Theorem (affecting labels vs. orthogonality)

*Let  $\mathcal{A}_1$  be an abstraction of  $\mathcal{T}$  with abstraction mapping  $\alpha_1$ .  
Let  $\mathcal{A}_2$  be an abstraction of  $\mathcal{T}$  with abstraction mapping  $\alpha_2$ .  
If no label of  $\mathcal{T}$  affects both  $\mathcal{A}_1$  and  $\mathcal{A}_2$ , then  $\alpha_1$  and  $\alpha_2$  are orthogonal.*

(Easy proof omitted.)

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# Orthogonal abstraction mappings: example



	2		6
5	7		
3	4	1	

9		12	
		14	13
			11
15	10	8	

Abstractions:  
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Are the abstraction mappings orthogonal?

# Orthogonal abstraction mappings: example



	2		6
5	7		
3	4	1	

9		12	
		14	13
			11
15	10	8	

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Are the abstraction mappings orthogonal?



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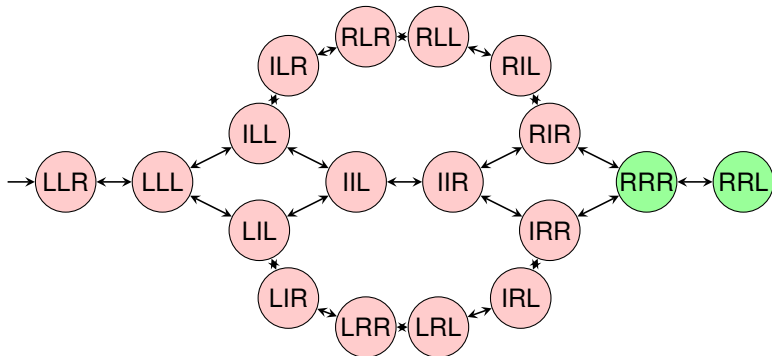
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## Theorem (additivity for orthogonal abstraction mappings)

Let  $h^{\mathcal{A}_1, \alpha_1}, \dots, h^{\mathcal{A}_n, \alpha_n}$  be abstraction heuristics for the same planning task  $\Pi$  such that  $\alpha_i$  and  $\alpha_j$  are orthogonal for all  $i \neq j$ . Then  $\sum_{i=1}^n h^{\mathcal{A}_i, \alpha_i}$  is a safe, goal-aware, admissible and consistent heuristic for  $\Pi$ .

# Orthogonality and additivity: example



transition system  $\mathcal{T}$

state variables: first package, second package, truck

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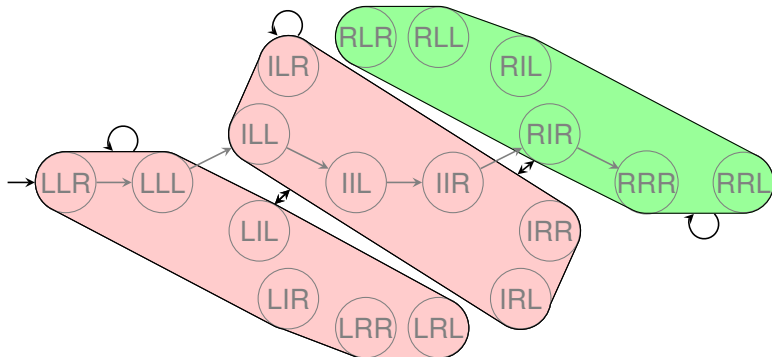
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# Orthogonality and additivity: example



abstraction  $\mathcal{A}_1$

mapping: only consider state of first package

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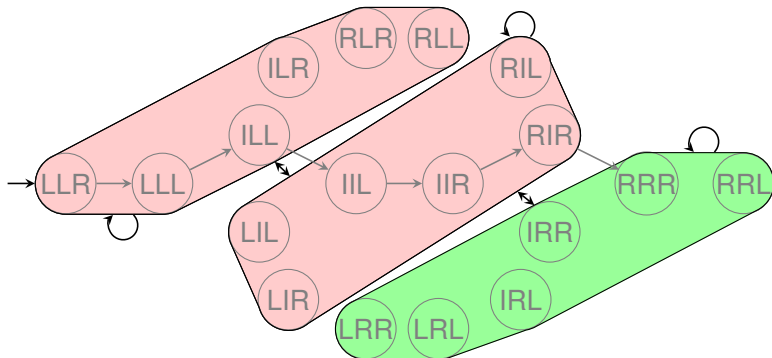
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# Orthogonality and additivity: example



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**abstraction  $\mathcal{A}_2$**  (orthogonal to  $\mathcal{A}_1$ )  
mapping: only consider state of second package

## Proof.

We prove goal-awareness and consistency;  
the other properties follow from these two.

Let  $\mathcal{T} = \mathcal{T}(\Pi) = \langle S, L, T, s_0, S_\star \rangle$ .

**Goal-awareness:** For goal states  $s \in S_\star$ ,

$\sum_{i=1}^n h^{\mathcal{A}_i, \alpha_i}(s) = \sum_{i=1}^n 0 = 0$  because all individual abstractions  
are goal-aware.

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## Proof (ctd.)

**Consistency:** Let  $s, t \in S$  such that  $t$  is a successor of  $s$ .

Let  $L := \sum_{i=1}^n h^{\mathcal{A}_i, \alpha_i}(s)$  and  $R := \sum_{i=1}^n h^{\mathcal{A}_i, \alpha_i}(t)$ .

We need to prove that  $L \leq R + 1$ .

Since  $t$  is a successor of  $s$ , there exists an operator  $o$  with  $app_o(s) = t$  and hence  $\langle s, o, t \rangle \in T$ .

Because the abstraction mappings are orthogonal,  $\alpha_i(s) \neq \alpha_j(t)$  for **at most one**  $i \in \{1, \dots, n\}$ .

**Case 1:**  $\alpha_i(s) = \alpha_i(t)$  for all  $i \in \{1, \dots, n\}$ .

$$\begin{aligned} \text{Then } L &= \sum_{i=1}^n h^{\mathcal{A}_i, \alpha_i}(s) \\ &= \sum_{i=1}^n h^*_{\mathcal{A}_i}(\alpha_i(s)) \\ &= \sum_{i=1}^n h^*_{\mathcal{A}_i}(\alpha_i(t)) \\ &= \sum_{i=1}^n h^{\mathcal{A}_i, \alpha_i}(t) \\ &= R \leq R + 1. \end{aligned}$$

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## Proof (ctd.)

Case 2:  $\alpha_i(s) \neq \alpha_i(t)$  for exactly one  $i \in \{1, \dots, n\}$ .

Let  $k \in \{1, \dots, n\}$  such that  $\alpha_k(s) \neq \alpha_k(t)$ .

$$\begin{aligned} \text{Then } L &= \sum_{i=1}^n h^{\mathcal{A}_i, \alpha_i}(s) \\ &= \sum_{i \in \{1, \dots, n\} \setminus \{k\}} h_{\mathcal{A}_i}^*(\alpha_i(s)) + h^{\mathcal{A}_k, \alpha_k}(s) \\ &\leq \sum_{i \in \{1, \dots, n\} \setminus \{k\}} h_{\mathcal{A}_i}^*(\alpha_i(t)) + h^{\mathcal{A}_k, \alpha_k}(t) + 1 \\ &= \sum_{i=1}^n h^{\mathcal{A}_i, \alpha_i}(t) + 1 \\ &= R + 1, \end{aligned}$$

where the inequality holds because  $\alpha_i(s) = \alpha_i(t)$  for all  $i \neq k$  and  $h^{\mathcal{A}_k, \alpha_k}$  is consistent. □

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## Theorem (transitivity of abstractions)

Let  $\mathcal{T}$ ,  $\mathcal{T}'$  and  $\mathcal{T}''$  be transition systems.

- If  $\mathcal{T}'$  is an abstraction of  $\mathcal{T}$  and  $\mathcal{T}''$  is an abstraction of  $\mathcal{T}'$ , then  $\mathcal{T}''$  is an abstraction of  $\mathcal{T}$ .
- If  $\mathcal{T}'$  is a strictly homomorphic abstraction of  $\mathcal{T}$  and  $\mathcal{T}''$  is a strictly homomorphic abstraction of  $\mathcal{T}'$ , then  $\mathcal{T}''$  is a strictly homomorphic abstraction of  $\mathcal{T}$ .

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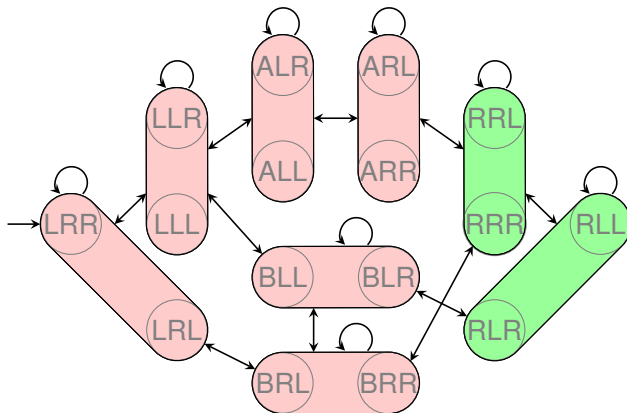
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# Abstractions of abstractions: example



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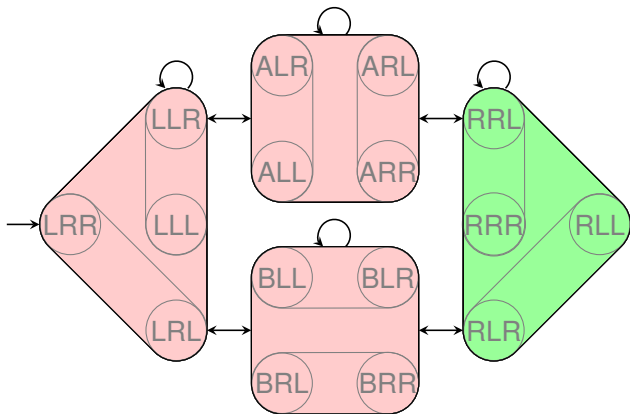
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Transition system  $\mathcal{T}'$  as an abstraction of  $\mathcal{T}$

# Abstractions of abstractions: example



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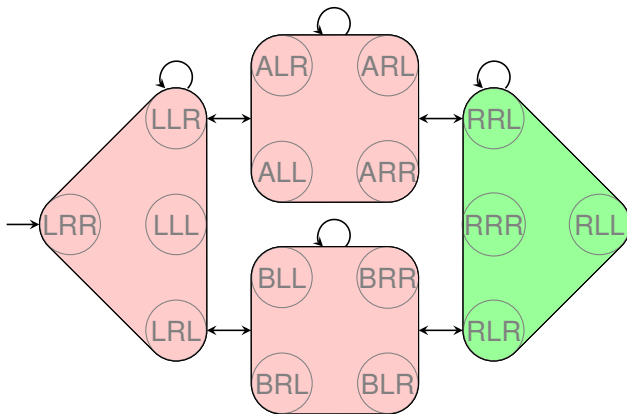
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Transition system  $\mathcal{T}''$  as an abstraction of  $\mathcal{T}'$

# Abstractions of abstractions: example



Transition system  $\mathcal{I}''$  as an abstraction of  $\mathcal{I}$

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## Proof.

Let  $\mathcal{T} = \langle S, L, T, s_0, S_\star \rangle$ , let  $\mathcal{T}' = \langle S', L, T', s'_0, S'_\star \rangle$  be an abstraction of  $\mathcal{T}$  with abstraction mapping  $\alpha$ , and let  $\mathcal{T}'' = \langle S'', L, T'', s''_0, S''_\star \rangle$  be an abstraction of  $\mathcal{T}'$  with abstraction mapping  $\alpha'$ .

We show that  $\mathcal{T}''$  is an abstraction of  $\mathcal{T}$  with abstraction mapping  $\beta := \alpha' \circ \alpha$ , i. e., that

- 1  $\beta(s_0) = s''_0$ ,
- 2 for all  $s \in S_\star$ , we have  $\beta(s) \in S''_\star$ , and
- 3 for all  $\langle s, \ell, t \rangle \in T$ , we have  $\langle \beta(s), \ell, \beta(t) \rangle \in T''$ .

Moreover, we show that if  $\alpha$  and  $\alpha'$  are strict homomorphisms, then  $\beta$  is also a strict homomorphism.

...

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## Proof (ctd.)

1.  $\beta(s_0) = s''_0$

Because  $\mathcal{T}'$  is an abstraction of  $\mathcal{T}$  with mapping  $\alpha$ , we have  $\alpha(s_0) = s'_0$ . Because  $\mathcal{T}''$  is an abstraction of  $\mathcal{T}'$  with mapping  $\alpha'$ , we have  $\alpha'(s'_0) = s''_0$ .

Hence  $\beta(s_0) = \alpha'(\alpha(s_0)) = \alpha'(s'_0) = s''_0$ .

...

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## Proof (ctd.)

2. For all  $s \in S_*$ , we have  $\beta(s) \in S''_*$ :

Let  $s \in S_*$ . Because  $\mathcal{T}'$  is an abstraction of  $\mathcal{T}$  with mapping  $\alpha$ , we have  $\alpha(s) \in S'_*$ . Because  $\mathcal{T}''$  is an abstraction of  $\mathcal{T}'$  with mapping  $\alpha'$  and  $\alpha(s) \in S'_*$ , we have  $\alpha'(\alpha(s)) \in S''_*$ .

Hence  $\beta(s) = \alpha'(\alpha(s)) \in S''_*$ .

**Strict homomorphism if  $\alpha$  and  $\alpha'$  strict homomorphisms:**

Let  $s'' \in S''_*$ . Because  $\alpha'$  is a strict homomorphism, there exists a state  $s' \in S'_*$  such that  $\alpha'(s') = s''$ . Because  $\alpha$  is a strict homomorphism, there exists a state  $s \in S_*$  such that  $\alpha(s) = s'$ . Thus  $s'' = \alpha'(\alpha(s)) = \beta(s)$  for some  $s \in S_*$ .

...

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## Proof (ctd.)

3. For all  $\langle s, l, t \rangle \in T$ , we have  $\langle \beta(s), l, \beta(t) \rangle \in T''$

Let  $\langle s, l, t \rangle \in T$ . Because  $\mathcal{T}'$  is an abstraction of  $\mathcal{T}$  with mapping  $\alpha$ , we have  $\langle \alpha(s), l, \alpha(t) \rangle \in T'$ . Because  $\mathcal{T}''$  is an abstraction of  $\mathcal{T}'$  with mapping  $\alpha'$  and  $\langle \alpha(s), l, \alpha(t) \rangle \in T'$ , we have  $\langle \alpha'(\alpha(s)), l, \alpha'(\alpha(t)) \rangle \in T''$ .

Hence  $\langle \beta(s), l, \beta(t) \rangle = \langle \alpha'(\alpha(s)), l, \alpha'(\alpha(t)) \rangle \in T''$ .

## Strict homomorphism if $\alpha$ and $\alpha'$ strict homomorphisms:

Let  $\langle s'', l, t'' \rangle \in T''$ . Because  $\alpha'$  is a strict homomorphism, there exists a transition  $\langle s', l, t' \rangle \in T'$  such that  $\alpha'(s') = s''$  and  $\alpha'(t') = t''$ . Because  $\alpha$  is a strict homomorphism, there exists a transition  $\langle s, l, t \rangle \in T$  such that  $\alpha(s) = s'$  and  $\alpha(t) = t'$ .

Thus  $\langle s'', l, t'' \rangle = \langle \alpha'(\alpha(s)), l, \alpha'(\alpha(t)) \rangle = \langle \beta(s), l, \beta(t) \rangle$  for some  $\langle s, l, t \rangle \in T$ . □

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**Terminology:** Let  $\mathcal{T}$  be a transition system, let  $\mathcal{T}'$  be an abstraction of  $\mathcal{T}$  with abstraction mapping  $\alpha$ , and let  $\mathcal{T}''$  be an abstraction of  $\mathcal{T}'$  with abstraction mapping  $\alpha'$ .

Then:

- $\langle \mathcal{T}'', \alpha' \circ \alpha \rangle$  is called a **coarsening** of  $\langle \mathcal{T}', \alpha \rangle$ , and
- $\langle \mathcal{T}', \alpha \rangle$  is called a **refinement** of  $\langle \mathcal{T}'', \alpha' \circ \alpha \rangle$ .



## Theorem (heuristic quality of refinements)

Let  $h^{\mathcal{A},\alpha}$  and  $h^{\mathcal{B},\beta}$  be abstraction heuristics for the same planning task  $\Pi$  such that  $\langle \mathcal{A}, \alpha \rangle$  is a refinement of  $\langle \mathcal{B}, \beta \rangle$ . Then  $h^{\mathcal{A},\alpha}$  dominates  $h^{\mathcal{B},\beta}$ .

In other words,  $h^{\mathcal{A},\alpha}(s) \geq h^{\mathcal{B},\beta}(s)$  for all states  $s$  of  $\Pi$ .

### Proof.

Since  $\langle \mathcal{A}, \alpha \rangle$  is a refinement of  $\langle \mathcal{B}, \beta \rangle$ , there exists a mapping  $\alpha'$  such that  $\beta = \alpha' \circ \alpha$  and  $\mathcal{B}$  is an abstraction of  $\mathcal{A}$  with abstraction mapping  $\alpha'$ .

For any state  $s$  of  $\Pi$ , we get  $h^{\mathcal{B},\beta}(s) = h_{\mathcal{B}}^*(\beta(s)) = h_{\mathcal{B}}^*(\alpha'(\alpha(s))) = h^{\mathcal{B},\alpha'}(\alpha(s)) \leq h_{\mathcal{A}}^*(\alpha(s)) = h^{\mathcal{A},\alpha}(s)$ , where the inequality holds because  $h^{\mathcal{B},\alpha'}$  is an admissible heuristic in the transition system  $\mathcal{A}$ .

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## Definition (isomorphic transition systems)

Let  $\mathcal{T} = \langle S, L, T, s_0, S_\star \rangle$  and  $\mathcal{T}' = \langle S', L', T', s'_0, S'_\star \rangle$  be transition systems.

We say that  $\mathcal{T}$  is isomorphic to  $\mathcal{T}'$ , in symbols  $\mathcal{T} \sim \mathcal{T}'$ , if there exist bijective functions  $\varphi : S \rightarrow S'$  and  $\psi : L \rightarrow L'$  such that:

- $\varphi(s_0) = s'_0$ ,
- $s \in S_\star$  iff  $\varphi(s) \in S'_\star$ , and
- $\langle s, l, t \rangle \in T$  iff  $\langle \varphi(s), \psi(l), \varphi(t) \rangle \in T'$ .

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## Definition (graph-equivalent transition systems)

Let  $\mathcal{T} = \langle S, L, T, s_0, S_\star \rangle$  and  $\mathcal{T}' = \langle S', L', T', s'_0, S'_\star \rangle$  be transition systems.

We say that  $\mathcal{T}$  is **graph-equivalent to**  $\mathcal{T}'$ , in symbols  $\mathcal{T} \stackrel{G}{\sim} \mathcal{T}'$ , if there exists a bijective function  $\varphi : S \rightarrow S'$  such that:

- $\varphi(s_0) = s'_0$ ,
- $s \in S_\star$  iff  $\varphi(s) \in S'_\star$ , and
- $\langle s, \ell, t \rangle \in T$  for some  $\ell \in L$  iff  $\langle \varphi(s), \ell', \varphi(t) \rangle \in T'$  for some  $\ell' \in L'$ .

**Note:** There is no requirement that the labels of  $\mathcal{T}$  and  $\mathcal{T}'$  correspond in any way. For example, it is permitted that all transitions of  $\mathcal{T}$  have different labels and all transitions of  $\mathcal{T}'$  have the same label.

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# Isomorphism vs. graph equivalence



- $(\sim)$  and  $(\overset{G}{\sim})$  are equivalence relations.
- Two isomorphic transition systems are interchangeable for all practical intents and purposes.
- Two graph-equivalent transition systems are interchangeable for most intents and purposes.  
In particular, their state distances are identical, so they define the same abstraction heuristic for corresponding abstraction functions.
- Isomorphism implies graph equivalence, but not vice versa.

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Summary

In practice, there are conflicting goals for abstractions:

- we want to obtain an **informative heuristic**, but
- want to keep its **representation small**.

Abstractions have small representations if they have

- **few abstract states** and
- a **succinct encoding for  $\alpha$** .

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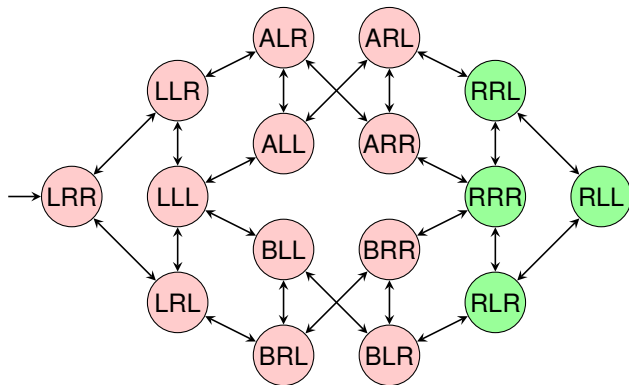
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# Counterexample: identity abstraction



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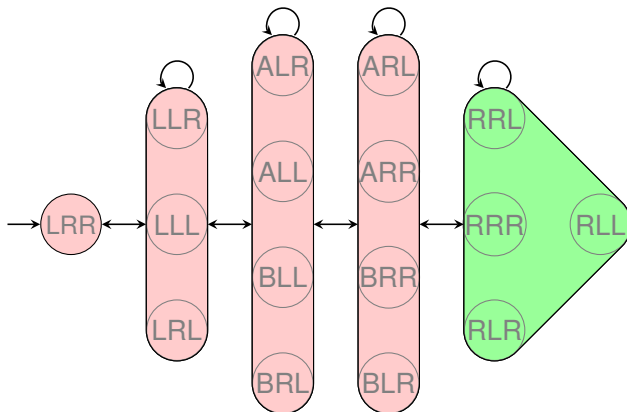
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Summary

Identity abstraction:  $\alpha(s) := s$ .

- + perfect heuristic and succinct encoding for  $\alpha$
- too many abstract states

# Counterexample: perfect abstraction



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Summary

Perfect abstraction:  $\alpha(s) := h^*(s)$ .

- + perfect heuristic and usually few abstract states
- usually no succinct encoding for  $\alpha$

# Automatically deriving good abstraction heuristics



## Abstraction heuristics for planning: main research problem

Automatically derive effective abstraction heuristics for planning tasks.

~> we will study one state-of-the-art approach in the next chapter.

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- An **abstraction** relates a transition system  $\mathcal{T}$  (e. g. of a planning task) to another (usually smaller) transition system  $\mathcal{T}'$  via an **abstraction mapping**  $\alpha$ .
- Abstraction **preserves all important aspects** of  $\mathcal{T}$ : initial state, goal states and (labeled) transitions.
- Hence, they can be used to define **heuristics** for the original system  $\mathcal{T}$ : estimate the goal distance of  $s$  in  $\mathcal{T}$  by the optimal goal distance of  $\alpha(s)$  in  $\mathcal{T}'$ .
- Such **abstraction heuristics** are **safe**, **goal-aware**, **admissible** and **consistent**.

Abstractions:  
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Summary

- **Strictly homomorphic abstractions** are desirable as they do not include “unnecessary” abstract goal states or transitions (which could lower heuristic values).
- Any surjection from the states of  $\mathcal{I}$  to any set induces a strictly homomorphic abstraction in a natural way.
- Multiple abstraction heuristics can be added without losing properties like admissibility if the underlying abstraction mappings are **orthogonal**.
- One sufficient condition for orthogonality is that abstractions are **affected** by disjoint sets of labels.

Abstractions:  
informally

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Summary

- The process of abstraction is **transitive**: an abstraction can be abstracted further to yield another abstraction.
- Based on this notion, we can define abstractions that are **coarsenings** or **refinements** of others.
- A refinement can never lead to a worse heuristic.
- Practically useful abstractions are those which give **informative heuristics**, yet have a **small representation**.

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