Motivation: Applications of Possible World Semantics

The set of interpretations of a set of propositional variables can be viewed as possible worlds. Relations between possible worlds can be used to express more interesting concepts:

- Temporal concepts like *always*, *next*, ... can be modeled as relations between world states (Prior, 1957).
- Execution of computer program can be modeled as transitions between world states (Pratt, 1976).
- Knowledge and belief of an agent can be modeled as truth in all worlds states that the agent considers possible (Hintikka, 1962).
- Obligations and permissions can be modeled as truth in all (resp. some) ideal world states (Kanger, 1957; Hintikka 1957).
- Desires and intentions can be modeled as truth in all world states an agent prefers (Cohen & Levesque, 1990).
Graphical Model

A graphical model is made up of nodes and edges between nodes. Both nodes and edges may have labels.
If the light is on then it is true that after toggling the light is off. If the light is off then it is true that after toggling the light is on.
If the light is on then it is true that Mary considers possible both that the light is on or off. If the light is off then it is true that Mary considers possible both that the light is on or off.
If the light is on it is true that John only considers possible that the light is on. If the light is off it is true that John only considers possible that the light is off.

In either world it is true that Mary is uncertain about the state of the switch and John knows about the state of the switch.
If the light is on it is true that it is permissible to bring about that the light is off and it is not permissible to leave the light on.

If the light is off it is true that it is permissible leave the light off and it is not permissible to bring about that the light is on.

⇒ In both worlds it is obligatory to bring about/maintain that the light is off.
Kripke Models

Kripke Frame

Given a countable set of edge labels $\mathcal{I}$, a Kripke Frame is a tuple $(W, R)$ such that:

- $W$ is a non-empty set of possible worlds, and
- $R : I \rightarrow 2^{W \times W}$ maps each $I \in \mathcal{I}$ to a binary relation $R(I)$ on $W$ (called the accessibility relation of $I$).

Kripke Model

$M = (W, R, V)$ is a Kripke Model where:

- $(W, R)$ is a Kripke frame, and
- $V : \mathcal{P} \rightarrow 2^W$ is called the valuation of a set of node labels $\mathcal{P}$. 
Kripke Model: Example

- **Kripke Frame** \((W, R)\)
  - Possible worlds \(W = \{w_l, w_r\}\)
  - Edge labels \(I = \{mary\}\)
  - \(R(mary) = \{(w_l, w_l), (w_l, w_r), (w_r, w_r), (w_r, w_l)\}\)

- **Kripke Model** \((W, R, V)\)
  - \(W, R\) as before.
  - Node labels \(P = \{light\_on, light\_off\}\)
  - \(V(light\_on) = \{w_l\}, V(light\_off) = \{w_r\}\)
Classes of Kripke Models

- Besides being able to model concrete situations, we are interested in the study of the general properties of concepts like knowledge, intention, obligation etc.
- Identify particular classes of Kripke models as representations of the concept under consideration.
- Classes of Kripke models can be distinguished based on the properties of their respective frames.
  - **K**: All Kripke frames
  - **T**: Kripke frames with reflexive accessibility relation
  - **D**: Kripke frames with serial accessibility relation
  - **4**: Kripke frames with transitive accessibility relation
  - **5**: Kripke frames with Euclidean accessibility relation
- Can be combined:
  - **K, KD, K4, K5, KT = KDT, K45, KD5, KD4, KT4 = KDT4, KD45, KT5 = KT45 = KDT5 = KDT45**
  - Some abbreviations often used: **KT** is called **T**, **KT4** is called **S4**, **KD45** is weak-**S5**, **KT5** called **S5**.
Discussion: Which Class of Models for which Concept?

- Programs:
- Knowledge:
- Belief:
- Desire:
- Obligation:

**Hint:** ask yourself for each of these concepts, decide whether or not:

- If $[I]x$ then $x$? **reflexive**
- Is it impossible that $[I]x$ and $[I]\neg x$? **serial**
- If $[I]x$ then $[I][I]x$? **transitive**
- If $\neg[I]x$ then $[I]\neg[I]x$? **Euclidean**
Kripke models can be described and reasoned about using modal logics.

- Does a given Kripke model satisfy some given property?
  - E.g., is it currently true that Mary does not know whether the light is on?

- Do all Kripke models of a class satisfying property A also satisfy property B?
  - E.g., is it always true that if some agent X knows that some agent Y knows Z that agent X knows Z, too?

⇒ We will learn how to check formulae against given Kripke models, and how to automatically build Kripke models to (dis-)prove a formula’s (un-)satisfiability.
Literature