Introduction to search algorithms for planning
Our plan for the next lectures

Choices to make:

1. search direction: progression/regression/both
   ⇔ previous chapter

2. search space representation: states/sets of states
   ⇔ previous chapter

3. search algorithm: uninformed/heuristic; systematic/local
   ⇔ this chapter

4. search control: heuristics, pruning techniques
   ⇔ next chapters
Search algorithms are used to find solutions (plans) for transition systems in general, not just for planning tasks. Planning is one application of search among many. In this chapter, we describe some popular and/or representative search algorithms, and (the basics of) how they apply to planning. Most of this is review of material that should be known (details: Russell and Norvig’s textbook).
Search states vs. search nodes

In search, one distinguishes:

- **search states** $s \mapsto$ states (vertices) of the transition system
- **search nodes** $\sigma \mapsto$ search states plus information on where/when/how they are encountered during search

What is in a search node?

Different search algorithms store different information in a search node $\sigma$, but typical information includes:

- $\text{state}(\sigma)$: associated search state
- $\text{parent}(\sigma)$: pointer to search node from which $\sigma$ is reached
- $\text{action}(\sigma)$: action leading from $\text{state}(\text{parent}(\sigma))$ to $\text{state}(\sigma)$
- $g(\sigma)$: cost of $\sigma$ (length of path from the root node)

For the root node, $\text{parent}(\sigma)$ and $\text{action}(\sigma)$ are undefined.
Search states vs. planning states

Search states $\neq$ (planning) states:

- **Search states** don’t have to correspond to **states** in the planning sense.
  - progression: search states $\approx$ (planning) states
  - regression: search states $\approx$ sets of states (formulae)

- Search algorithms for planning where search states are planning states are called **state-space search** algorithms.

- Strictly speaking, regression is **not** an example of state-space search, although the term is often used loosely.

- However, we will put the emphasis on progression, which is almost always state-space search.
Required ingredients for search

A general search algorithm can be applied to any transition system for which we can define the following three operations:

- **init()**: generate the **initial state**
- **is-goal(s)**: test if a given state is a **goal state**
- **succ(s)**: generate the set of **successor states** of state s, along with the **operators** through which they are reached (represented as pairs \(\langle o, s' \rangle\) of operators and states)

Together, these three functions form a **search space** (a very similar notion to a transition system).
Search for planning: progression

Let $\Pi = \langle A, I, O, \gamma \rangle$ be a planning task.

**Search space for progression search**

**states:** all states of $\Pi$ (assignments to $A$)

- $\text{init}() = I$
- $\text{is-goal}(s) = \begin{cases} \text{true} & \text{if } s \models \gamma \\ \text{false} & \text{otherwise} \end{cases}$
- $\text{succ}(s) = \{ \langle o, s' \rangle \mid \text{applicable } o \in O, s' = \text{app}_o(s) \}$
Let \( \Pi = \langle A, I, O, \gamma \rangle \) be a planning task.

### Search space for regression search

**states:** all formulae over \( A \) (how many?)

- \( \text{init}() = \gamma \)
- \( \text{is-goal}(\varphi) = \begin{cases} \text{true} & \text{if } I \models \varphi \\ \text{false} & \text{otherwise} \end{cases} \)
- \( \text{succ}(\varphi) = \{ \langle o, \varphi' \rangle \mid o \in O, \varphi' = \text{regr}_o(\varphi), \varphi' \text{ is satisfiable} \} \)

(modified if splitting is used)
Classification of search algorithms

uninformed search vs. heuristic search:

- **uninformed search algorithms** only use the basic ingredients for general search algorithms
- **heuristic search algorithms** additionally use heuristic functions which estimate how close a node is to the goal

systematic search vs. local search:

- **systematic algorithms** consider a large number of search nodes simultaneously
- **local search algorithms** work with one (or a few) candidate solutions (search nodes) at a time
- not a black-and-white distinction; there are crossbreeds (e.g., enforced hill-climbing)
uninformed vs. heuristic search:

- For **satisficing** planning, heuristic search vastly outperforms uninformed algorithms on most domains.
- For **optimal** planning, the difference is less pronounced.

systematic search vs. local search:

- For **satisficing** planning, the most successful algorithms are somewhere between the two extremes.
- For **optimal** planning, systematic algorithms are required.
Before we describe the different search algorithms, we introduce three procedures used by all of them:

- **make-root-node**: Create a search node without parent.
- **make-node**: Create a search node for a state generated as the successor of another state.
- **extract-solution**: Extract a solution from a search node representing a goal state.
**Procedure make-root-node**

```python
def make-root-node(s):
    σ := new node
    state(σ) := s
    parent(σ) := undefined
    action(σ) := undefined
    g(σ) := 0
    return σ
```

**make-root-node**: Create a search node without parent.
**Procedure make-node**

**make-node**: Create a search node for a state generated as the successor of another state.

```python
def make-node(σ, o, s):
    σ′ := new node
    state(σ′) := s
    parent(σ′) := σ
    action(σ′) := o
    g(σ′) := g(σ) + 1
    return σ'
```
**Procedure extract-solution**

`extract-solution`: Extract a solution from a search node representing a goal state.

```python
def extract-solution(σ):
    solution := new list
    while parent(σ) is defined:
        solution.push-front(action(σ))
        σ := parent(σ)
    return solution
```
Uninformed search algorithms
Uninformed search algorithms

- Uninformed algorithms are less relevant for planning than heuristic ones, so we keep their discussion brief.
- Uninformed algorithms are mostly interesting to us because we can compare and contrast them to related heuristic search algorithms.

Popular uninformed systematic search algorithms:
- breadth-first search
- depth-first search
- iterated depth-first search

Popular uninformed local search algorithms:
- random walk
Breadth-first search without duplicate detection

Breadth-first search

\[\text{queue} := \textbf{new} \ \text{fifo-queue}\]
\[\text{queue}.\text{push-back}(\text{make-root-node}(\text{init}()))\]
\[\textbf{while not} \ \text{queue}.\text{empty}():\]
\[\quad \sigma = \text{queue}.\text{pop-front}();\]
\[\quad \textbf{if} \ \text{is-goal}(\text{state}(\sigma)):\]
\[\quad\quad \text{return} \ \text{extract-solution}(\sigma)\]
\[\quad \textbf{for each} \ \langle o, s \rangle \in \text{succ}(\text{state}(\sigma)):\]
\[\quad\quad \sigma' := \text{make-node}(\sigma, o, s)\]
\[\quad \text{queue}.\text{push-back}(\sigma')\]
\[\text{return} \ \text{unsolvable}\]

- Possible improvement: \textit{duplicate detection} (see next slide).
- Another possible improvement: test if \(\sigma'\) is a goal node; if so, terminate immediately. (We don’t do this because it obscures the similarity to some of the later algorithms.)
Breadth-first search with duplicate detection

```plaintext
queue := new fifo-queue
queue.push-back(make-root-node(init()))
closed := ∅
while not queue.empty():
    ŝ = queue.pop-front()
    if state(ŝ) /∈ closed:
        closed := closed ∪ {state(ŝ)}
        if is-goal(state(ŝ)):
            return extract-solution(ŝ)
        for each ⟨o, s⟩ ∈ succ(state(ŝ)):
            ŝ′ := make-node(ŝ, o, s)
            queue.push-back(ŝ′)
return unsolvable
```
Breadth-first search with duplicate detection

```
queue := new fifo-queue
queue.push-back(make-root-node(init()))
closed := Ø

while not queue.empty():
    σ = queue.pop-front()
    if state(σ) ∉ closed:
        closed := closed ∪ {state(σ)}
        if is-goal(state(σ)):
            return extract-solution(σ)
    for each ⟨o, s⟩ ∈ succ(state(σ)):
        σ′ := make-node(σ, o, s)
        queue.push-back(σ′)

return unsolvable
```
Random walk

\[ \sigma := \text{make-root-node}(\text{init}()) \]

forever:

\[ \text{if } \text{is-goal}(\text{state}(\sigma)):\]

\[ \text{return } \text{extract-solution}(\sigma) \]

Choose a random element \( \langle o, s \rangle \) from \( \text{succ}(\text{state}(\sigma)) \).

\[ \sigma := \text{make-node}(\sigma, o, s) \]

- The algorithm usually does not find any solutions, unless almost every sequence of actions is a plan.
- Often, it runs indefinitely without making progress.
- It can also fail by reaching a dead end, a state with no successors. This is a weakness of many local search approaches.
Heuristic search algorithms
Heuristic search algorithms are the most common and overall most successful algorithms for classical planning.

Popular systematic heuristic search algorithms:

- greedy best-first search
- A*
- weighted A*
- IDA*
- depth-first branch-and-bound search
- ...
Heuristic search algorithms are the most common and overall most successful algorithms for classical planning.

Popular heuristic local search algorithms:

- hill-climbing
- enforced hill-climbing
- beam search
- tabu search
- genetic algorithms
- simulated annealing
- ...
Heuristic search: idea
A heuristic search algorithm requires one more operation in addition to the definition of a search space.

**Definition (heuristic function)**

Let $\Sigma$ be the set of nodes of a given search space. A **heuristic function** or **heuristic** (for that search space) is a function $h : \Sigma \rightarrow \mathbb{N}_0 \cup \{\infty\}$.

The value $h(\sigma)$ is called the **heuristic estimate** or **heuristic value** of heuristic $h$ for node $\sigma$. It is supposed to estimate the distance from $\sigma$ to the nearest goal node.
What exactly is a heuristic estimate?

What does it mean that $h$ “estimates the goal distance”?

- For most heuristic search algorithms, $h$ does not need to have any strong properties for the algorithm to work (= be correct and complete).

- However, the **efficiency** of the algorithm closely relates to how accurately $h$ reflects the actual goal distance.

- For some algorithms, like A*, we can prove strong formal relationships between properties of $h$ and properties of the algorithm (optimality, dominance, run-time for bounded error, . . .)

- For other search algorithms, “it works well in practice” is often as good an analysis as one gets.
Heuristics applied to nodes or states?

- Most texts apply heuristic functions to states, not nodes.
- This is slightly less general than our definition:
  - Given a state heuristic $h$, we can define an equivalent node heuristic as $h'(\sigma) := h(state(\sigma))$.
  - The opposite is not possible. (Why not?)
- There is good justification for only allowing state-defined heuristics: why should the estimated distance to the goal depend on how we ended up in a given state $s$?
- We call heuristics which don’t just depend on $state(\sigma)$ pseudo-heuristics.
- In practice there are sometimes good reasons to have the heuristic value depend on the generating path of $\sigma$ (e.g., landmark pseudo-heuristic, Richter et al. 2008).
Let $\Sigma$ be the set of nodes of a given search space.

**Definition (optimal/perfect heuristic)**

The **optimal** or **perfect heuristic** of a search space is the heuristic $h^*$ which maps each search node $\sigma$ to the length of a shortest path from $\text{state}(\sigma)$ to any goal state.

**Note:** $h^*(\sigma) = \infty$ iff no goal state is reachable from $\sigma$. 
Properties of heuristics

A heuristic $h$ is called

- safe if $h^*(\sigma) = \infty$ for all $\sigma \in \Sigma$ with $h(\sigma) = \infty$
- goal-aware if $h(\sigma) = 0$ for all goal nodes $\sigma \in \Sigma$
- admissible if $h(\sigma) \leq h^*(\sigma)$ for all nodes $\sigma \in \Sigma$
- consistent if $h(\sigma) \leq h(\sigma') + 1$ for all nodes $\sigma, \sigma' \in \Sigma$ such that $\sigma'$ is a successor of $\sigma$.

Relationships?

---

1 or: $h(\sigma) \leq h(\sigma') + \text{cost}(\sigma, \sigma')$ for non-unit costs, where $\text{cost}(\sigma, \sigma')$ is the cost of the transition from $\sigma$ to $\sigma'$. 
Greedy best-first search

Greedy best-first search (with duplicate detection)

\[
\begin{align*}
\text{open} & := \textbf{new} \text{ min-heap ordered by } (\sigma \mapsto h(\sigma)) \\
\text{open}.\text{insert}(\text{make-root-node}(|\text{init}|())) \\
\text{closed} & := \emptyset \\
\textbf{while not} \ \text{open}.\text{empty}(): & \\
& \quad \sigma = \text{open}.\text{pop-min}() \\
& \quad \textbf{if } \text{state}(\sigma) \notin \text{closed}: \\
& \quad \quad \text{closed} := \text{closed} \cup \{\text{state}(\sigma)\} \\
& \quad \quad \textbf{if } \text{is-goal}(\text{state}(\sigma)): \\
& \quad \quad \quad \text{return } \text{extract-solution}(\sigma) \\
& \quad \quad \textbf{for each } \langle o, s \rangle \in \text{succ}(\text{state}(\sigma)): \\
& \quad \quad \quad \sigma' := \text{make-node}(\sigma, o, s) \\
& \quad \quad \quad \textbf{if } h(\sigma') < \infty: \\
& \quad \quad \quad \quad \text{open}.\text{insert}(\sigma') \\
& \textbf{return } \text{unsolvable}
\end{align*}
\]
Properties of greedy best-first search

- one of the three most commonly used algorithms for satisficing planning
- **complete** for safe heuristics (due to duplicate detection)
- **suboptimal** unless $h$ satisfies some very strong assumptions (similar to being perfect)
- invariant under all strictly monotonic transformations of $h$ (e.g., scaling with a positive constant or adding a constant)
A* (with duplicate detection and reopening)

\[ \text{open} := \textbf{new} \text{ min-heap ordered by } (\sigma \mapsto g(\sigma) + h(\sigma)) \]
\[ \text{open}.\text{insert}:\text{make-root-node}(\text{init}) \]
\[ \text{closed} := \emptyset \]
\[ \text{distance} := \emptyset \]
\[ \text{while not open}.\text{empty}: \]
\[ \quad \sigma = \text{open}.\text{pop-min} \]
\[ \quad \text{if state}(\sigma) \notin \text{closed} \text{ or } g(\sigma) < \text{distance}(\text{state}(\sigma)):\]
\[ \quad \quad \text{closed} := \text{closed} \cup \{ \text{state}(\sigma) \} \]
\[ \quad \quad \text{distance}(\text{state}(\sigma)) := g(\sigma) \]
\[ \quad \text{if is-goal}(\text{state}(\sigma)):\]
\[ \quad \quad \text{return} \text{ extract-solution}(\sigma) \]
\[ \quad \text{for each } \langle o, s \rangle \in \text{succ}(\text{state}(\sigma)):\]
\[ \quad \quad \sigma' := \text{make-node}(\sigma, o, s) \]
\[ \quad \quad \text{if } h(\sigma') < \infty: \text{open}.\text{insert}(\sigma') \]
\[ \text{return unsolvable} \]
A* example

Example

\[ 0 + 3 \]

\[ 3 \]

\[ \gamma \]
A* example

Example
Introduction

Uninformed search

Heuristic search

Heuristics

Systematic search

Local search

Summary

A* example

Example
A* example

Example

- Uninformed search
- Heuristic search
- Heuristics
- Systematic search
- Local search
- Summary
A* example

Example
Terminology for A* 

- **f value** of a node: defined by \( f(\sigma) := g(\sigma) + h(\sigma) \)
- **generated nodes**: nodes inserted into open at some point
- **expanded nodes**: nodes \( \sigma \) popped from open for which the test against closed and distance succeeds
- **reexpanded nodes**: expanded nodes for which \( state(\sigma) \in closed \) upon expansion (also called reexpanded nodes)
Properties of A*:

- the most commonly used algorithm for optimal planning
- rarely used for satisficing planning
- complete for safe heuristics (even without duplicate detection)
- optimal if $h$ is admissible (even without duplicate detection)
- never reopens nodes if $h$ is consistent

Implementation notes:

- in the heap-ordering procedure, it is considered a good idea to break ties in favour of lower $h$ values
- can simplify algorithm if we know that we only have to deal with consistent heuristics
- common, hard to spot bug: test membership in closed at the wrong time
Weighted A* (with duplicate detection and reopening)

\[
\text{open} := \text{new min-heap ordered by } (\sigma \mapsto g(\sigma) + W \cdot h(\sigma)) \\
\text{open}.\text{insert}(\text{make-root-node}(\text{init}())) \\
\text{closed} := \emptyset \\
\text{distance} := \emptyset \\
\text{while not } \text{open}.\text{empty}(): \\
\quad \sigma = \text{open}.\text{pop-min}() \\
\quad \text{if state}(\sigma) \notin \text{closed} \text{ or } g(\sigma) < \text{distance(state}(\sigma)):\ \\
\quad \text{closed} := \text{closed} \cup \{\text{state}(\sigma)\} \\
\quad \text{distance}(\sigma) := g(\sigma) \\
\quad \text{if is-goal}(\text{state}(\sigma)):\ \\
\quad \quad \text{return } \text{extract-solution}(\sigma) \\
\quad \text{for each } \langle o, s \rangle \in \text{succ(state}(\sigma)):\ \\
\quad \quad \sigma' := \text{make-node}(\sigma, o, s) \\
\quad \quad \text{if } h(\sigma') < \infty: \text{open}.\text{insert}(\sigma') \\
\text{return unsolvable}
\]
Properties of weighted A*

The weight $W \in \mathbb{R}^+$ is a parameter of the algorithm.
- for $W = 0$, behaves like breadth-first search
- for $W = 1$, behaves like A$^*$
- for $W \to \infty$, behaves like greedy best-first search

Properties:
- one of the most commonly used algorithms for satisficing planning
- for $W > 1$, can prove similar properties to A$^*$, replacing optimal with bounded suboptimal: generated solutions are at most a factor $W$ as long as optimal ones
Hill-climbing

\[ \sigma := \text{make-root-node}(\text{init}()) \]

\textbf{forever:}

\begin{itemize}
  \item \textbf{if} is-goal(state(\sigma)):
    \begin{itemize}
      \item \textbf{return} extract-solution(\sigma)
    \end{itemize}
  \item \(\Sigma' := \{ \text{make-node}(\sigma, o, s) \mid (o, s) \in \text{succ}(\text{state}(\sigma)) \} \)
  \item \(\sigma := \text{an element of } \Sigma' \text{ minimizing } h \text{ (random tie breaking)}\)
\end{itemize}

- can easily get stuck in \textbf{local minima} where immediate improvements of \(h(\sigma)\) are not possible
- many variations: tie-breaking strategies, restarts
Enforced hill-climbing

Enforced hill-climbing: procedure improve

```python
def improve(σ₀):
    queue := new fifo-queue
    queue.push-back(σ₀)
    closed := Ø
    while not queue.empty():
        σ = queue.pop-front()
        if state(σ) ∉ closed:
            closed := closed ∪ {state(σ)}
            if h(σ) < h(σ₀):
                return σ
            for each ⟨o, s⟩ ∈ succ(state(σ)):
                σ' := make-node(σ, o, s)
                queue.push-back(σ')
        fail
```

≈≈ breadth-first search for more promising node than σ₀
Enforced hill-climbing (ctd.)

Enforced hill-climbing

\[ \sigma := \text{make-root-node}(\text{init}()) \]

while not \text{is-goal}(\text{state}(\sigma)):
    \sigma := \text{improve}(\sigma)

return \text{extract-solution}(\sigma)

- one of the three most commonly used algorithms for satisficing planning
- can fail if procedure \text{improve} fails (when the goal is unreachable from \( \sigma_0 \))
- complete for \textit{undirected} search spaces (where the successor relation is symmetric) if \( h(\sigma) = 0 \) for all goal nodes and only for goal nodes
Summary

- distinguish: planning states, search states, search nodes
  - planning state: situation in the world modelled by the task
  - search state: subproblem remaining to be solved
    - In state-space search (usually progression search), planning states and search states are identical.
    - In regression search, search states usually describe sets of states (“subgoals”).
  - search node: search state + info on “how we got there”

- search algorithms mainly differ in order of node expansion
  - uninformed vs. informed (heuristic) search
  - local vs. systematic search
heuristics: estimators for “distance to goal node”
- usually: the more accurate, the better performance
- desiderata: safe, goal-aware, admissible, consistent
- the ideal: perfect heuristic $h^*$

most common algorithms for satisficing planning:
- greedy best-first search
- weighted $A^*$
- enforced hill-climbing

most common algorithm for optimal planning:
- $A^*$