Principles of Knowledge Representation and Reasoning

Qualitative Representation and Reasoning II:
Allen’s Interval Calculus

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Allen’s Interval Calculus
Qualitative Temporal Representation and Reasoning

Often we do not want to talk about precise times:

- **NLP** – we do not have precise time points
- **Planning** – we do not want to commit to time points too early
- **Scenario descriptions** – we do not have the exact times or do not want to state them

What are the primitives in our representation system?

- **Time points**: actions and events are instantaneous, or we consider their beginning and ending
- **Time intervals**: actions and events have duration
- Reducibility? Expressiveness? Computational costs for reasoning?
Motivation: Example

Consider a planning scenario for multimedia generation:

P1: Display Picture1
P2: Say “Put the plug in.”
P3: Say “The device should be shut off.”
P4: Point to Plug-in-Picture1.

Temporal relations between events:

P2 should happen during P1
P3 should happen during P1
P2 should happen before or directly precede P3
P4 should happen during or end together with P2

⇝ P4 happens before or directly precedes P3
⇝ We could add the statement “P4 does not overlap with P3” without creating an inconsistency.
Allen’s Interval Calculus

- Allen’s interval calculus: time intervals and binary relations over them
- Time intervals: $X = (X^-, X^+)$, where $X^-$ and $X^+$ are interpreted over the reals and $X^- < X^+$ (naïve approach)
- Relations between concrete intervals, e.g.:
  
  - $(1.0,2.0)$ strictly before $(3.0,5.5)$
  - $(1.0,3.0)$ meets $(3.0,5.5)$
  - $(1.0,4.0)$ overlaps $(3.0,5.5)$
  - ...

Which relations are conceivable?
The base relations

How many ways are there to order the four points of two intervals?

<table>
<thead>
<tr>
<th>Relation</th>
<th>Symbol</th>
<th>Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>{(X, Y) : X^− &lt; X^+ &lt; Y^− &lt; Y^+}</td>
<td>&lt;</td>
<td>before</td>
</tr>
<tr>
<td>{(X, Y) : X^− &lt; X^+ = Y^− &lt; Y^+}</td>
<td>m</td>
<td>meets</td>
</tr>
<tr>
<td>{(X, Y) : X^− &lt; Y^− &lt; X^+ &lt; Y^+}</td>
<td>o</td>
<td>overlaps</td>
</tr>
<tr>
<td>{(X, Y) : X^− = Y^− &lt; X^+ &lt; Y^+}</td>
<td>s</td>
<td>starts</td>
</tr>
<tr>
<td>{(X, Y) : Y^− &lt; X^− &lt; X^+ = Y^+}</td>
<td>f</td>
<td>finishes</td>
</tr>
<tr>
<td>{(X, Y) : Y^− &lt; X^− &lt; X^+ &lt; Y^+}</td>
<td>d</td>
<td>during</td>
</tr>
<tr>
<td>{(X, Y) : Y^− = X^− &lt; X^+ = Y^+}</td>
<td>≡</td>
<td>equal</td>
</tr>
</tbody>
</table>

and the converse relations (obtained by exchanging \(X\) and \(Y\))

~~~ These relations are JEPD.
The 13 base relations graphically

before
meets
overlaps
during
starts
finishes
equals
before
meets
overlaps
during
starts
finishes

X
—
Y
—
Y
—
Y
—
Y
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Y
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Y
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Y
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Y
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Y
—
Disjunctive descriptions

- Assumption: We don’t have precise information about the relation between $X$ and $Y$, e.g.:

  $$X \circ Y \text{ or } X \mathbin{\mathrm{m}} Y$$

- ...modelled by sets of base relations (meaning the union of the relations):

  $$X \{ \circ, \mathbin{\mathrm{m}} \} Y$$

$$\sim \sim 2^{13}$$ imprecise relations (incl. $\emptyset$ and $\mathbb{B}$)

Example of an indefinite qualitative description:

$$\left\{ X \{ \circ, \mathbin{\mathrm{m}} \} Y, Y \{ \mathbin{\mathrm{m}} \} Z, X \{ \circ, \mathbin{\mathrm{m}} \} Z \right\}$$
Our example...formally

P1: Display Picture1
P3: Say “The device should be shut off.”

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Compose the constraints: $P4 \{ d, f \} P2$ and $P2 \{ d \} P1$
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Compose the constraints: $P4 \{d, f\} P2$ and $P2 \{d\} P1$ $\rightsquigarrow P4 \{d\} P1.$
Composition of base relations

<table>
<thead>
<tr>
<th></th>
<th>&lt;</th>
<th>≥</th>
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<th>d(^{-1})</th>
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Using the **composition table** and the rules about operations on relations, we can **deduce** new relations between time intervals.

- What would be a **systematic** approach?
- How costly is that?
- Is that **complete**?
- If not, could it be complete on a subset of the relation system?
Reasoning in Allen’s Interval Calculus
Constraint propagation: The naive algorithm

Enforcing path consistency using the straight-forward method:
Let $Table[i,j]$ be an array of size $n \times n$ ($n$: number of intervals) in which we record the constraints between the intervals.

\[ EnforcePathConsistency1(C) \]

*Input:* a (binary) CSP $C = \langle V, D, C \rangle$

*Output:* an equivalent, but path consistent CSP $C'$

**repeat**

**for** each pair $(i,j), 1 \leq i,j \leq n$

**for** each $k$ with $1 \leq k \leq n$

\[
Table[i,j] := Table[i,j] \cap (Table[i,k] \circ Table[k,j])
\]

**until** no entry in $Table$ is changed

\[ \Rightarrow \] needs $O(n^5)$ intersections and compositions.
An $O(n^3)$ algorithm

EnforcePathConsistency2($C$)

*Input:* a (binary) CSP $C = \langle V, D, C \rangle$

*Output:* an equivalent, but path consistent CSP $C'$

$Paths(i,j) = \{(i,j,k) : 1 \leq k \leq n\} \cup \{(k,i,j) : 1 \leq k \leq n\}$

$Queue := \bigcup_{i,j} Paths(i,j)$

*while* $Queue \neq \emptyset$

select and delete $(i,k,j)$ from $Queue$

$T := Table[i,j] \cap (Table[i,k] \circ Table[k,j])$

*if* $T \neq Table[i,j]$

$Table[i,j] := T$

$Table[j,i] := T^{-1}$

$Queue := Queue \cup Paths(i,j)$
Example for incompleteness
NP-hardness

Theorem (Kautz & Vilain)

CSAT is NP-hard for Allen's interval calculus.

Proof.

Reduction from 3-colorability (original proof using 3Sat).

Let $G = (V, E)$, $V = \{v_1, \ldots, v_n\}$ be an instance of 3-colorability. Then we use the intervals $\{v_1, \ldots, v_n, 1, 2, 3\}$ with the following constraints:

$$
\begin{align*}
1 & \{m\} & 2 \\
2 & \{m\} & 3 \\
v_i & \{m, \equiv, m^{-1}\} & 2 & \forall v_i \in V \\
v_i & \{m, m^{-1}, <, >\} & v_j & \forall (v_i, v_j) \in E
\end{align*}
$$

This constraint system is satisfiable iff $G$ can be colored with 3 colors.
Looking for special cases

- **Idea:** Let us look for polynomial special cases. In particular, let us look for sets of relations (subsets of the entire set of relations) that have an easy CSAT problem.

- **Note:** Interval formulae $X R Y$ can be expressed as **clauses** over **atoms** of the form $a op b$, where:
  - $a$ and $b$ are endpoints $X^-, X^+, Y^-$ and $Y^+$ and
  - $op \in \{<, >, =, \leq, \geq\}$.

- **Example:** All base relations can be expressed as unit clauses.

---

**Lemma**

*Let $\pi(\Theta)$ be the translation of $\Theta$ to clause form. $\Theta$ is satisfiable over intervals iff $\pi(\Theta)$ is satisfiable over the rational numbers.*
The Continuous Endpoint Class

Continuous Endpoint Class $\mathcal{C}$: This is a subset of $\mathcal{A}$ such that there exists a clause form for each relation containing only unit clauses where $\neg(a = b)$ is forbidden.

Example: All basic relations and $\{d, o, s\}$, because

$$\pi(X \{d, o, s\} Y) = \{ X^- < X^+, Y^- < Y^+, X^- < Y^+, X^+ > Y^-, X^+ < Y^+ \}$$

![Diagram of continuous endpoints](image)
Why do we have completeness?

The set $\mathcal{C}$ is **closed** under intersection, composition, and converse (it is a **sub-algebra** wrt. these three operations on relations). This can be shown by using a computer program.

**Lemma**

*Each 3-consistent interval CSP over $\mathcal{C}$ is globally consistent.*

**Theorem (van Beek)**

*Path consistency solves $\text{CMIN}(\mathcal{C})$ and decides $\text{CSAT}(\mathcal{C})$.*

(Proof: Follows from the above lemma and the fact that a strongly $n$-consistent CSP is minimal.)

**Corollary**

*A path consistent interval CSP consisting of base relations only is satisfiable.*
Helly’s Theorem

**Definition**

A set $M \subseteq \mathbb{R}^n$ is **convex** iff for all pairs of points $a, b \in M$, all points on the line connecting $a$ and $b$ belong to $M$.

**Theorem (Helly)**

Let $F$ be a finite family of at least $n + 1$ convex sets in $\mathbb{R}^n$. If all sub-families of $F$ with $n + 1$ sets have a non-empty intersection, then $\bigcap F \neq \emptyset$. 
Strong $n$-consistency (1)

Proof (part 1).

We prove the claim by induction over $k$ with $k \leq n$. 

Base case: $k = 1, 2, 3\ldots$

Induction assumption: Assume strong $(k-1)$-consistency (and non-emptiness of all relations)

Induction step: From the assumption, it follows that there is an instantiation of $k-1$ variables $X_i$ to pairs $(s_i, e_i)$ satisfying the constraints $R_{ij}$ between the $k-1$ variables.

We have to show that we can extend the instantiation to any $k$th variable.
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**Strong $n$-consistency (1)**

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We have to show that we can extend the instantiation to any $k$th variable.
Strong $n$-consistency (2): Instantiating the $k$th variable

**Proof (part 2).**

The instantiation of the $k-1$ variables $X_i$ to $(s_i, e_i)$ restricts the instantiation of $X_k$. Note: Since $R_{ij} \in C$ by assumption, these restrictions can be expressed by inequalities of the form:

\[ s_i < X_k + k \land e_j \geq X_k - k \land \ldots \]

Such inequalities define convex subsets in $\mathbb{R}^2$. ⇝ Consider sets of 3 inequalities ($=3$ convex sets).
Strong $n$-consistency (2): Instantiating the $k$th variable

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Proof (part 3).

**Case 1:** All 3 inequalities mention only $X_k^-$ (or mention only $X_k^+$). Then it suffices to consider only 2 of these inequalities (the strongest). Because of 3-consistency, there exists at least 1 common point satisfying these 2 inequalities.
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**Case 3:** The set contains the inequality $X_k^- < X_k^+$. In this case, only three intervals (incl. $X_k$) can be involved and by 3-consistency there exists a common point.

⇒ With Helly’s Theorem, there exists an instantiation consistent with all inequalities.
⇒ Strong $k$-consistency for all $k \leq n$. 

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$\Rightarrow$ With Helly’s Theorem, there exists an instantiation consistent with all inequalities.

$\Rightarrow$ Strong $k$-consistency for all $k \leq n$. 
Outlook

- $\text{CMIN}(\mathcal{C})$ can be computed in $O(n^3)$ time (for $n$ being the number of intervals) using the path consistency algorithm.
- $\mathcal{C}$ is a set of relations occurring “naturally” when observations are uncertain.
- $\mathcal{C}$ contains 83 relations (incl. the impossible and the universal relations).
- Are there larger sets such that path consistency computes minimal CSPs? Probably not.
- Are there larger sets of relations that permit polynomial satisfiability testing? Yes.
A Maximal Tractable Sub-Algebra
The EP-subclass

End-Point Subclass: \( \mathcal{P} \subseteq \mathcal{A} \) is the subclass that permits a clause form containing only unit clauses (\( a \neq b \) is allowed).
The EP-subclass

End-Point Subclass: $\mathcal{P} \subseteq \mathcal{A}$ is the subclass that permits a clause form containing only unit clauses ($a \neq b$ is allowed).

Example: all basic relations and $\{d, o\}$ since

$$\pi(X \{d, o\} Y) = \{ X^- < X^+, Y^- < Y^+, X^- < Y^+, X^+ > Y^-, X^- \neq Y^- \}$$

\[
\begin{array}{cccccc}
\cdot & \cdot & X & \cdot & \cdot & \cdot \\
\cdot & \cdot & X & \cdot & \cdot & \cdot \\
\end{array}
\]

$X$

$Y$

Theorem (Vilain & Kautz 86, Ladkin & Maddux 88) Enforcing path consistency decides CSAT ($P$).
The EP-subclass

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\[
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X^- < Y^+, X^+ > Y^-, X^- \neq Y^- , \\
X^+ < Y^+ \} \]

\[
\begin{array}{c}
\text{X} \\
\text{Y}
\end{array}
\]

Theorem (Vilain & Kautz 86, Ladkin & Maddux 88)

**Enforcing path consistency decides** $\text{CSAT}(\mathcal{P})$. 
The ORD-Horn Subclass

**ORD-Horn Subclass:** \( \mathcal{H} \subseteq \mathcal{A} \) is the subclass that permits a clause form containing only Horn clauses where only the following literals are allowed:

\[
\begin{align*}
  &a \leq b, a = b, a \neq b \\
  &\neg a \leq b \text{ is not allowed!}
\end{align*}
\]
The ORD-Horn Subclass

**ORD-Horn Subclass**: \( \mathcal{H} \subseteq \mathcal{A} \) is the subclass that permits a clause form containing only **Horn clauses** where only the following **literals** are allowed:

\[
a \leq b, a = b, a \neq b
\]

\( \neg a \leq b \) is not allowed!

**Example**: all \( R \in \mathcal{P} \) and \( \{ o, s, f^{-1} \} \):

\[
\pi(X\{o, s, f^{-1}\} Y) = \left\{ X^- \leq X^+, X^- \neq X^+, \right. \\
y^- \leq y^+, y^- \neq y^+, \\
x^- \leq y^-, \\
x^- \leq y^+, x^- \neq y^+, \\
y^- \leq x^+, x^+ \neq y^-, \\
x^+ \leq y^+, \\
x^- \neq y^- \lor x^+ \neq y^+ \right\}.
\]
Partial orders: The **ORD** theory

Let **ORD** be the following theory:

\[
\begin{align*}
\forall x, y, z : & \quad x \leq y \land y \leq z \quad \rightarrow \quad x \leq z & \quad & \text{(transitivity)} \\
\forall x : & \quad x \leq x & \quad & \text{(reflexivity)} \\
\forall x, y : & \quad x \leq y \land y \leq x \quad \rightarrow \quad x = y & \quad & \text{(anti-symmetry)} \\
\forall x, y : & \quad x = y \quad \rightarrow \quad x \leq y & \quad & \text{(weakening of =)} \\
\forall x, y : & \quad x = y \quad \rightarrow \quad y \leq x & \quad & \text{(weakening of =).}
\end{align*}
\]

- **ORD** describes partially ordered sets, \(\leq\) being the ordering relation.
- **ORD** is a Horn theory
- What is missing wrt. **dense** and **linear** orders?
Satisfiability over partial orders

Proposition

Let $\Theta$ be a CSP over $\mathcal{H}$. $\Theta$ is satisfiable over interval interpretations iff $\pi(\Theta) \cup \text{ORD}$ is satisfiable over arbitrary interpretations.

Proof.

$\Rightarrow$: Since the reals form a partially ordered set (i.e., satisfy $\text{ORD}$), this direction is trivial.
Satisfiability over partial orders

**Proposition**

Let $\Theta$ be a CSP over $\mathcal{H}$. $\Theta$ is satisfiable over interval interpretations iff $\pi(\Theta) \cup \text{ORD}$ is satisfiable over arbitrary interpretations.

**Proof.**

$\Rightarrow$: Since the reals form a partially ordered set (i.e., satisfy $\text{ORD}$), this direction is trivial.

$\Leftarrow$: Each extension of a partial order to a linear order satisfies all formulae of the form $a \leq b$, $a = b$, and $a \neq b$ which have been satisfied over the original partial order.
Let $\text{ORD}^{\pi(\Theta)}$ be the propositional theory resulting from instantiating all axioms with the endpoints occurring in $\pi(\Theta)$.

**Proposition**

$\text{ORD} \cup \pi(\Theta)$ is satisfiable iff $\text{ORD}^{\pi(\Theta)} \cup \pi(\Theta)$ is so.

**Proof idea:** Herbrand expansion!
Complexity of CSAT(\(\mathcal{H}\))

Let \(\text{ORD}_{\pi(\Theta)}\) be the propositional theory resulting from instantiating all axioms with the endpoints occurring in \(\pi(\Theta)\).

**Proposition**

\(\text{ORD} \cup \pi(\Theta)\) is satisfiable iff \(\text{ORD}_{\pi(\Theta)} \cup \pi(\Theta)\) is so.

**Proof idea:** Herbrand expansion!

**Theorem**

CSAT(\(\mathcal{H}\)) can be decided in polynomial time.

**Proof.**

CSAT(\(\mathcal{H}\)) instances can be translated into a propositional Horn theory with blowup \(O(n^3)\) according to the previous Prop., and such a theory is decidable in polynomial time.
Path consistency and the OH-class

Lemma

Let $\Theta$ be a path-consistent set over $\mathcal{H}$. Then

$$(X\{\}Y) \notin \Theta \iff \Theta \text{ is satisfiable}$$

Proof idea: One can show that $\text{ORD}_{\pi(\Theta)} \cup \pi(\Theta)$ is closed wrt. positive unit resolution. Since this inference rule is refutation complete for Horn theories, the claim follows.

Theorem

Enforcing path consistency decides $\text{CSAT}(\mathcal{H})$.

Maximality of $\mathcal{H}$?

Do we have to check all $8192 - 868$ extensions?
Let $\hat{S}$ be the closure of $S \subseteq A$ under converse, intersection, and composition (i.e., the carrier of the least sub-algebra generated by $S$).

**Theorem**

$\text{CSAT}(\hat{S})$ can be polynomially transformed to $\text{CSAT}(S)$.

**Proof Idea.**

All relations in $\hat{S} - S$ can be modeled by a fixed number of compositions, intersections, and conversions of relations in $S$, introducing perhaps some fresh variables.

$\Rightarrow$ Polynominality of $S$ extends to $\hat{S}$.

$\Rightarrow$ NP-hardness of $\hat{S}$ is inherited by all generating sets $S$.

$\Rightarrow$ Note: $\mathcal{H} = \hat{\mathcal{H}}$. 

February 8, 2016 Nebel, Wölfl, Lindner – KR&R
Minimal extensions of the $\mathcal{H}$-subclass

A computer-aided case analysis leads to the following result:

**Lemma**

There are only two minimal sub-algebras that strictly contain $\mathcal{H}$: $\mathcal{X}_1, \mathcal{X}_2$

\[
\mathcal{N}_1 = \{d, d^{-1}, o^{-1}, s^{-1}, f\} \in \mathcal{X}_1
\]
\[
\mathcal{N}_2 = \{d^{-1}, o, o^{-1}, s^{-1}, f^{-1}\} \in \mathcal{X}_2
\]

The clause form of these relations contain “proper” disjunctions!

**Theorem**

$CSAT(\mathcal{H} \cup \{N_i\})$ is NP-complete.

**Question:** Are there other maximal tractable subclasses?
“Interesting” subclasses

Interesting subclasses of $\mathcal{A}$ should contain all basic relations.

A computer-aided case analysis reveals:
For $S \supseteq \{\{B\} : B \in \mathcal{B}\}$ it holds that

1. $\hat{S} \subseteq \mathcal{H}$, or
2. $N_1$ or $N_2$ is in $\hat{S}$.

In case 2, one can show: $\text{CSAT}(S)$ is NP-complete.

$\Rightarrow$ $\mathcal{H}$ is the only interesting maximal tractable subclass.

If we include non-interesting subalgebras, there exist exactly 18 tractable classes.
Relevance?

**Theory:** ⊕ We now know the boundary between polynomial and NP-hard reasoning problems along the dimension **expressiveness**.

**Practice:** ⊖ All known applications either need only $\mathcal{P}$ or they need more than $\mathcal{H}$!

Backtracking methods might profit from the result by reducing the branching factor.

〜〜 How difficult is CSAT($\mathcal{A}$) in practice?

〜〜 What are the relevant branching factors?
Solving general Allen CSPs

- Backtracking algorithm using path consistency as a forward-checking method
- Relies on tractable fragments of Allen’s calculus: split relations into relations of a tractable fragment, and backtrack over these.
- Refinements and evaluation of different heuristics

Which tractable fragment should one use?
Branching factors

- If the labels are split into base relations, then on average a label is split into **6.5 relations**

- If the labels are split into pointizable relations ($\mathcal{P}$), then on average a label is split into **2.955 relations**

- If the labels are split into ORD-Horn relations ($\mathcal{H}$), then on average a label is split into **2.533 relations**

~~~ A difference of **0.422**

~~~ This makes a difference for “hard” instances.
Summary

- Allen’s interval calculus is often adequate for describing relative orders of events that have duration.
- The satisfiability problem for CSPs using the relations is NP-complete.
- For the continuous endpoint class, minimal CSPs can be computed using the path-consistency method.
- For the larger ORD-Horn class, CSAT is still decided by the path-consistency method.
- Can be used in practice for backtracking algorithms.
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