Principles of Knowledge Representation and Reasoning

Qualitative Representation and Reasoning II: Allen's Interval Calculus

Bernhard Nebel, Stefan Wölfl, and Felix Lindner
February 8, 2016

Qualitative Temporal Representation and Reasoning

Often we do not want to talk about precise times:
- **NLP** – we do not have precise time points
- **Planning** – we do not want to commit to time points too early
- **Scenario descriptions** – we do not have the exact times or do not want to state them

What are the primitives in our representation system?
- **Time points**: actions and events are instantaneous, or we consider their beginning and ending
- **Time intervals**: actions and events have duration
- Reducibility? Expressiveness? Computational costs for reasoning?

1 Allen’s Interval Calculus

- **Motivation**
- **Intervals and Relations Between Them**
- **Composing Interval Relations**

Motivation: Example

Consider a planning scenario for multimedia generation:

- **P1**: Display Picture1
- **P2**: Say “Put the plug in.”
- **P3**: Say “The device should be shut off.”
- **P4**: Point to Plug-in-Picture1.

Temporal relations between events:

- **P2** should happen during **P1**
- **P3** should happen during **P1**
- **P2** should happen before or directly precede **P3**
- **P4** should happen during or end together with **P2**

⇝ **P4** happens before or directly precedes **P3**
⇝ We could add the statement “**P4** does not overlap with **P3**” without creating an inconsistency.
Allen’s Interval Calculus

- Allen’s interval calculus: time intervals and binary relations over them
- Time intervals: $X = (X^-, X^+)$, where $X^-$ and $X^+$ are interpreted over the reals and $X^- < X^+$ (~ naive approach)
- Relations between concrete intervals, e.g.:
  - $(1.0, 2.0)$ strictly before $(3.0, 5.5)$
  - $(1.0, 3.0)$ meets $(3.0, 5.5)$
  - $(1.0, 4.0)$ overlaps $(3.0, 5.5)$
  - $(1.0, 4.0)$ during $(3.0, 5.5)$
  - $\ldots$

Which relations are conceivable?

The base relations

How many ways are there to order the four points of two intervals?

<table>
<thead>
<tr>
<th>Relation</th>
<th>Symbol</th>
<th>Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>${(X, Y) : X^- &lt; X^+ &lt; Y^- &lt; Y^+}$</td>
<td>$\prec$</td>
<td>before</td>
</tr>
<tr>
<td>${(X, Y) : X^- &lt; X^+ = Y^- &lt; Y^+}$</td>
<td>$m$</td>
<td>meets</td>
</tr>
<tr>
<td>${(X, Y) : X^- &lt; X^+ &lt; Y^- &lt; Y^+}$</td>
<td>$o$</td>
<td>overlaps</td>
</tr>
<tr>
<td>${(X, Y) : Y^- &lt; X^- &lt; X^+ &lt; Y^+}$</td>
<td>$s$</td>
<td>starts</td>
</tr>
<tr>
<td>${(X, Y) : Y^- &lt; X^- &lt; X^+ = Y^+}$</td>
<td>$f$</td>
<td>finishes</td>
</tr>
<tr>
<td>${(X, Y) : Y^- = X^- &lt; X^+ &lt; Y^+}$</td>
<td>$\equiv$</td>
<td>equal</td>
</tr>
</tbody>
</table>

and the converse relations (obtained by exchanging $X$ and $Y$)

These relations are JEPD.

The 13 base relations graphically

Disjunctive descriptions

- Assumption: We don’t have precise information about the relation between $X$ and $Y$, e.g.:
  $$X \circ Y \text{ or } X \mathbin{\smallfrown} Y$$
- ... modelled by sets of base relations (meaning the union of the relations):
  $$X \{o, m\} Y$$

$\Rightarrow$ $2^{13}$ imprecise relations (incl. $\emptyset$ and $B$

Example of an indefinite qualitative description:

$$\{X \{o, m\} Y, Y \{m\} Z, X \{o, m\} Z\}$$
Our example...formally

P1: Display Picture1
P3: Say “The device should be shut off.”
P2: Say “Put the plug in.”
P4: Point to Plug-in-Picture1.

Compose the constraints: P4 \{d, f\} P2 and P2 \{d\} P1
\implies P4 \{d\} P1.

Composition of base relations

2 Reasoning in Allen’s Interval Calculus

- Enforcing path consistency
- NP-Hardness Example
- The Continuous Endpoint Class
- Completeness for the CEP Class

Outlook

- Using the composition table and the rules about operations on relations, we can deduce new relations between time intervals.
- What would be a systematic approach?
- How costly is that?
- Is that complete?
- If not, could it be complete on a subset of the relation system?
Constraint propagation: The naive algorithm

Enforcing path consistency using the straight-forward method:
Let Table[i,j] be an array of size n x n (n: number of intervals) in which we record the constraints between the intervals.

**EnforcePathConsistency1(C)**

**Input:** a (binary) CSP C = \( \langle V, D, C \rangle \)

**Output:** an equivalent, but path consistent CSP C’

repeat
  for each pair \((i,j)\), \(1 \leq i, j \leq n\)
  for each k with \(1 \leq k \leq n\)
  \(\text{Table}[i,j] := \text{Table}[i,j] \cap (\text{Table}[i,k] \odot \text{Table}[k,j])\)
until no entry in Table is changed

\(\leadsto\) needs \(O(n^2)\) intersections and compositions.

---

An \(O(n^3)\) algorithm

**EnforcePathConsistency2(C)**

**Input:** a (binary) CSP C = \( \langle V, D, C \rangle \)

**Output:** an equivalent, but path consistent CSP C’

Paths\((i,j)\) = \(\{ (i,j,k) : 1 \leq k \leq n \}\) \(\cup\) \(\{ (k,i,j) : 1 \leq k \leq n \}\)

Queue := \(\bigcup_{i,j} \text{Paths}(i,j)\)

while Queue \(\neq\) \(\emptyset\)
  select and delete \((i,k,j)\) from Queue
  \(T := \text{Table}[i,j] \cap (\text{Table}[i,k] \odot \text{Table}[k,j])\)
  if T \(\neq\) \(\text{Table}[i,j]\)
    \(\text{Table}[i,j] := T\)
    \(\text{Queue} := \text{Queue} \cup \text{Paths}(i,j)\)

---

Example for incompleteness

NP-hardness

**Theorem (Kautz & Vilain)**

CSAT is NP-hard for Allen's interval calculus.

**Proof:**
Reduction from 3-colorability (original proof using 3Sat).
Let \(G = (V, E)\), \(V = \{v_1, \ldots, v_n\}\) be an instance of 3-colorability. Then we use the intervals \(\{ v_1, \ldots, v_n, 1, 2, 3 \}\) with the following constraints:

1. \(\{ m \}\) \(\rightarrow\) 2
2. \(\{ m \}\) \(\rightarrow\) 3
3. \(v_i \{ m, \equiv, m^{-1}\} \) \(\forall v_i \in V\)
4. \(v_i \{ m, m^{-1}, <, >\} \) \(v_j \forall (v_i, v_j) \in E\)

This constraint system is satisfiable iff \(G\) can be colored with 3 colors.
Looking for special cases

- Idea: Let us look for polynomial special cases. In particular, let us look for sets of relations (subsets of the entire set of relations) that have an easy CSAT problem.
- Note: Interval formulae $X \, R \, Y$ can be expressed as clauses over atoms of the form $a \, op \, b$, where:
  - $a$ and $b$ are endpoints $X^-, X^+, Y^-$ and $Y^+$ and
  - $op \in \{<,>,=,\leq,\geq\}$.
- Example: All base relations can be expressed as unit clauses.

Lemma

Let $\pi(\Theta)$ be the translation of $\Theta$ to clause form. $\Theta$ is satisfiable over intervals iff $\pi(\Theta)$ is satisfiable over the rational numbers.

The Continuous Endpoint Class

Continuous Endpoint Class $C$: This is a subset of $\mathcal{A}$ such that there exists a clause form for each relation containing only unit clauses where $\neg(a = b)$ is forbidden.

Example: All basic relations and $\{d, o, s\}$, because

$$\pi(X \{d, o, s\} Y) = \{X^- < X^+, Y^- < Y^+, X^- < Y^+, X^+ > Y^-, X^+ < Y^+\}$$

Helly’s Theorem

Definition

A set $M \subseteq \mathbb{R}^n$ is convex iff for all pairs of points $a, b \in M$, all points on the line connecting $a$ and $b$ belong to $M$.

Theorem (Helly)

Let $F$ be a finite family of at least $n+1$ convex sets in $\mathbb{R}^n$. If all sub-families of $F$ with $n+1$ sets have a non-empty intersection, then $\bigcap F \neq \emptyset$.
**Proof (part 1).**

We prove the claim by induction over \( k \) with \( k \leq n \).

**Base case:** \( k = 1, 2, 3 \)  

**Induction assumption:** Assume strong \((k−1)-\)consistency (and non-emptiness of all relations)

**Induction step:** From the assumption, it follows that there is an instantiation of \( k−1 \) variables \( X_i \) to pairs \((s_i, e_i)\) satisfying the constraints \( R_{ij} \) between the \( k−1 \) variables.

We have to show that we can extend the instantiation to any \( k \)th variable.

---

**Proof (part 2).**

The instantiation of the \( k−1 \) variables \( X_i \) to \((s_i, e_i)\) restricts the instantiation of \( X_k \).

**Note:** Since \( R_{ij} \in C \) by assumption, these restrictions can be expressed by inequalities of the form:

\[
s_i < X_k^- \land e_j \geq X_k^- \land \ldots
\]

Such inequalities define convex subsets in \( \mathbb{R}^2 \).

\( \Rightarrow \) Consider sets of 3 inequalities (= 3 convex sets).

---

**Proof (part 3).**

**Case 1:** All 3 inequalities mention only \( X_k^- \) (or mention only \( X_k^+ \)). Then it suffices to consider only 2 of these inequalities (the strongest).

Because of 3-consistency, there exists at least 1 common point satisfying these 2 inequalities.

**Case 2:** The inequalities mention \( X_k^- \) and \( X_k^+ \), but do not contain the inequality \( X_k^- < X_k^+ \). Then there are at most 2 inequalities with the same variable and we have the same situation as in Case 1.

**Case 3:** The set contains the inequality \( X_k^- < X_k^+ \). In this case, only three intervals (incl. \( X_k^- \)) can be involved and by 3-consistency there exists a common point.

\( \Rightarrow \) With Helly’s Theorem, there exists an instantiation consistent with all inequalities.

\( \Rightarrow \) Strong \( k \)-consistency for all \( k \leq n \).

---

**Outlook**

- \( \text{CMIN}(C) \) can be computed in \( O(n^3) \) time (for \( n \) being the number of intervals) using the path consistency algorithm.
- \( C \) is a set of relations occurring “naturally” when observations are uncertain.
- \( C \) contains 83 relations (incl. the impossible and the universal relations).
- Are there larger sets such that path consistency computes minimal CSPs? Probably not.
- Are there larger sets of relations that permit polynomial satisfiability testing? Yes.
The Endpoint Subclass

The Endpoint Subclass $\mathcal{P} \subseteq \mathcal{A}$ is the subclass that permits a clause form containing only unit clauses ($a \neq b$ is allowed).

Example: all basic relations and $\{d, o\}$ since

$$\pi(X \{d, o\} Y) = \{X^- < X^+, Y^- < Y^+, X^- < Y^+, X^+ > Y^-, X^+ < Y^-, X^- \neq Y^-, X^+ \neq Y^+\}$$

The ORD-Horn Subclass

ORD-Horn Subclass: $\mathcal{H} \subseteq \mathcal{A}$ is the subclass that permits a clause form containing only Horn clauses where only the following literals are allowed:

- $a \leq b, a = b, a \neq b$

$\neg a \leq b$ is not allowed!

Example: all $R \in \mathcal{P}$ and $\{o, s, f^{-1}\}$:

$$\pi(X \{o, s, f^{-1}\} Y) = \{X^- \leq X^+, X^- \neq X^+, Y^- \leq Y^+, Y^- \neq Y^+, X^- \leq Y^-, X^- \neq Y^+, Y^- \leq X^+, X^+ > Y^-, X^+ \leq Y^+, X^- \neq Y^- \vee X^+ \neq Y^+\}.$$
Satisfiability over partial orders

Proposition

Let $\Theta$ be a CSP over $\mathcal{H}$. $\Theta$ is satisfiable over interval interpretations iff $\pi(\Theta) \cup \text{ORD}$ is satisfiable over arbitrary interpretations.

Proof.

$\Rightarrow$: Since the reals form a partially ordered set (i.e., satisfy ORD), this direction is trivial.

$\Leftarrow$: Each extension of a partial order to a linear order satisfies all formulae of the form $a \leq b$, $a = b$, and $a \neq b$ which have been satisfiable over the original partial order.

Complexity of CSAT($\mathcal{H}$)

Let $\text{ORD}_{\pi(\Theta)}$ be the propositional theory resulting from instantiating all axioms with the endpoints occurring in $\pi(\Theta)$.

Proposition

$\text{ORD} \cup \pi(\Theta)$ is satisfiable iff $\text{ORD}_{\pi(\Theta)} \cup \pi(\Theta)$ is so.

Proof idea: Herbrand expansion!

Theorem

CSAT($\mathcal{H}$) can be decided in polynomial time.

Proof.

CSAT($\mathcal{H}$) instances can be translated into a propositional Horn theory with blowup $O(n^2)$ according to the previous Prop., and such a theory is decidable in polynomial time.

Path consistency and the OH-class

Lemma

Let $\Theta$ be a path-consistent set over $\mathcal{H}$. Then

$$(X\{\}Y) \notin \Theta \iff \Theta \text{ is satisfiable}$$

Proof idea: One can show that $\text{ORD}_{\pi(\Theta)} \cup \pi(\Theta)$ is closed wrt. positive unit resolution. Since this inference rule is refutation complete for Horn theories, the claim follows.

Theorem

Enforcing path consistency decides CSAT($\mathcal{H}$).

$\Rightarrow$ Maximal of $\mathcal{H}$?

$\Rightarrow$ Do we have to check all 8192 – 868 extensions?

Complexity of sub-algebras

Let $\hat{S}$ be the closure of $S \subseteq \mathcal{A}$ under converse, intersection, and composition (i.e., the carrier of the least sub-algebra generated by $S$).

Theorem

CSAT($\hat{S}$) can be polynomially transformed to CSAT($S$).

Proof Idea.

All relations in $\hat{S} - S$ can be modeled by a fixed number of compositions, intersections, and conversions of relations in $S$, introducing perhaps some fresh variables.

$\Rightarrow$ Polynomalit of $S$ extends to $\hat{S}$.

$\Rightarrow$ NP-hardness of $\hat{S}$ is inherited by all generating sets $S$.

$\Rightarrow$ Note: $\mathcal{H} = \hat{H}$. 
Minimal extensions of the $H$-subclass

A computer-aided case analysis leads to the following result:

**Lemma**
There are only two minimal sub-algebras that strictly contain $H$: $X_1, X_2$

$$N_1 = \{d, d^{-1}, o^{-1}, s^{-1}, f\} \in X_1$$
$$N_2 = \{d^{-1}, o, o^{-1}, s^{-1}, f^{-1}\} \in X_2$$

The clause form of these relations contain “proper” disjunctions!

**Theorem**
$CSAT(H \cup \{N_i\})$ is NP-complete.

**Question:** Are there other maximal tractable subclasses?

“Interesting” subclasses

Interesting subclasses of $A$ should contain all basic relations.

A computer-aided case analysis reveals:
For $S \supseteq \{\{B\} : B \in B\}$ it holds that

1. $S \subseteq H$, or
2. $N_1$ or $N_2$ is in $S$.

In case 2, one can show: $CSAT(S)$ is NP-complete.

$\Rightarrow H$ is the only interesting maximal tractable subclass.

If we include non-interesting subalgebras, there exist exactly 18 tractable classes.

Relevance?

Theory: We now know the boundary between polynomial and NP-hard reasoning problems along the dimension expressiveness.

Practice: All known applications either need only $P$ or they need more than $H$!

Backtracking methods might profit from the result by reducing the branching factor.

$\Rightarrow$ How difficult is $CSAT(A)$ in practice?

$\Rightarrow$ What are the relevant branching factors?

Solving general Allen CSPs

- Backtracking algorithm using path consistency as a forward-checking method
- Relies on tractable fragments of Allen’s calculus: split relations into relations of a tractable fragment, and backtrack over these.
- Refinements and evaluation of different heuristics

$\Rightarrow$ Which tractable fragment should one use?
Branching factors

- If the labels are split into base relations, then on average a label is split into **6.5 relations**
- If the labels are split into pointizable relations ($P$), then on average a label is split into **2.955 relations**
- If the labels are split into ORD-Horn relations ($H$), then on average a label is split into **2.533 relations**

$\Rightarrow$ A difference of **0.422**

$\Rightarrow$ This makes a difference for "hard" instances.

Summary

- Allen's interval calculus is often adequate for describing relative orders of events that have duration.
- The satisfiability problem for CSPs using the relations is NP-complete.
- For the continuous endpoint class, minimal CSPs can be computed using the path-consistency method.
- For the larger ORD-Horn class, CSAT is still decided by the path-consistency method.
- Can be used in practice for backtracking algorithms.

Literature

A complete classification of complexity in Allen's algebra in the presence of a non-trivial basic relation.

Reasoning about Temporal Relations: The Tractable Subalgebras of Allen's Interval Algebra.