Principles of
Knowledge Representation and Reasoning

Qualitative Representation and Reasoning II:
Allen’s Interval Calculus

Bernhard Nebel, Stefan Wölfl, and Felix Lindner

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1 Allen’s Interval Calculus

- Motivation
- Intervals and Relations Between Them
- Composing Interval Relations
Qualitative Temporal Representation and Reasoning

Often we do not want to talk about precise times:

- **NLP** – we do not have precise time points
- **Planning** – we do not want to commit to time points too early
- **Scenario descriptions** – we do not have the exact times or do not want to state them

What are the primitives in our representation system?

- **Time points**: actions and events are instantaneous, or we consider their beginning and ending
- **Time intervals**: actions and events have duration
- Reducibility? Expressiveness? Computational costs for reasoning?
Motivation: Example

Consider a planning scenario for multimedia generation:

P1: Display Picture1
P2: Say “Put the plug in.”
P3: Say “The device should be shut off.”
P4: Point to Plug-in-Picture1.

Temporal relations between events:

<table>
<thead>
<tr>
<th>Relation</th>
<th>Events</th>
</tr>
</thead>
<tbody>
<tr>
<td>P2 should happen during P1</td>
<td>P1</td>
</tr>
<tr>
<td>P3 should happen during P1</td>
<td>P1</td>
</tr>
<tr>
<td>P2 should happen before or directly precede P3</td>
<td>P3</td>
</tr>
<tr>
<td>P4 should happen during or end together with P2</td>
<td>P2</td>
</tr>
</tbody>
</table>

⇝ P4 happens before or directly precedes P3
⇝ We could add the statement “P4 does not overlap with P3” without creating an inconsistency.
Allen’s Interval Calculus

- Allen’s interval calculus: time intervals and binary relations over them

- **Time intervals**: $X = (X^-, X^+)$, where $X^-$ and $X^+$ are interpreted over the reals and $X^- < X^+$ (naïve approach)

- **Relations** between concrete intervals, e.g.:
  
  $$(1.0, 2.0) \text{ \textit{strictly before} } (3.0, 5.5)$$
  
  $$(1.0, 3.0) \text{ \textit{meets} } (3.0, 5.5)$$
  
  $$(1.0, 4.0) \text{ \textit{overlaps} } (3.0, 5.5)$$
  
  …

Which relations are conceivable?
The base relations

How many ways are there to order the four points of two intervals?

<table>
<thead>
<tr>
<th>Relation</th>
<th>Symbol</th>
<th>Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>{(X, Y) : X^- &lt; X^+ &lt; Y^- &lt; Y^+}</td>
<td>(&lt;)</td>
<td>before</td>
</tr>
<tr>
<td>{(X, Y) : X^- &lt; X^+ = Y^- &lt; Y^+}</td>
<td>(m)</td>
<td>meets</td>
</tr>
<tr>
<td>{(X, Y) : X^- &lt; Y^- &lt; X^+ &lt; Y^+}</td>
<td>(o)</td>
<td>overlaps</td>
</tr>
<tr>
<td>{(X, Y) : X^- = Y^- &lt; X^+ &lt; Y^+}</td>
<td>(s)</td>
<td>starts</td>
</tr>
<tr>
<td>{(X, Y) : Y^- &lt; X^- &lt; X^+ = Y^+}</td>
<td>(f)</td>
<td>finishes</td>
</tr>
<tr>
<td>{(X, Y) : Y^- &lt; X^- &lt; X^+ &lt; Y^+}</td>
<td>(d)</td>
<td>during</td>
</tr>
<tr>
<td>{(X, Y) : Y^- = X^- &lt; X^+ = Y^+}</td>
<td>(\equiv)</td>
<td>equal</td>
</tr>
</tbody>
</table>

and the **converse** relations (obtained by exchanging \(X\) and \(Y\))

\[\Leftrightarrow\] These relations are JEPD.
The 13 base relations graphically

before
meets
overlaps
during
starts
finishes
equals
before$^{-1}$
meets$^{-1}$
overlaps$^{-1}$
during$^{-1}$
starts$^{-1}$
finishes$^{-1}$
Disjunctive descriptions

- Assumption: We don’t have precise information about the relation between $X$ and $Y$, e.g.:

  $$X \circ Y \text{ or } X \mathbin{m} Y$$

- ...modelled by sets of base relations (meaning the union of the relations):

  $$X \{ \circ, m \} Y$$

$$\bowtie 2^{13}$$ imprecise relations (incl. $\emptyset$ and $\mathbb{B}$)

Example of an indefinite qualitative description:

$$\left\{ X \{ \circ, m \} Y, Y \mathbin{m} Z, X \{ \circ, m \} Z \right\}$$
Our example...formally

P1: Display Picture1
P3: Say “The device should be shut off.”

P2: Say “Put the plug in.”
P4: Point to Plug-in-Picture1.

Composing the constraints: $P4 \{d, f\} P2$ and $P2 \{d\} P1$

$\leadsto P4\{d\} P1$. 
Composition of base relations

<table>
<thead>
<tr>
<th></th>
<th>m</th>
<th>s</th>
<th>m^{-1}</th>
<th>s^{-1}</th>
<th>f</th>
<th>f^{-1}</th>
</tr>
</thead>
<tbody>
<tr>
<td>m</td>
<td></td>
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<tr>
<td>s</td>
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<td>f</td>
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<td>f^{-1}</td>
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</tbody>
</table>
Using the **composition table** and the rules about operations on relations, we can **deduce** new relations between time intervals.

What would be a **systematic** approach?

How costly is that?

Is that **complete**?

If not, could it be complete on a subset of the relation system?
2 Reasoning in Allen’s Interval Calculus

- Enforcing path consistency
- NP-Hardness Example
- The Continuous Endpoint Class
- Completeness for the CEP Class
Constraint propagation: The naive algorithm

Enforcing path consistency using the straight-forward method:
Let $Table[i,j]$ be an array of size $n \times n$ ($n$: number of intervals) in which we record the constraints between the intervals.

**EnforcePathConsistency1($C$)**

*Input:* a (binary) CSP $C = \langle V, D, C \rangle$

*Output:* an equivalent, but path consistent CSP $C'$

repeat
    for each pair $(i,j)$, $1 \leq i, j \leq n$
        for each $k$ with $1 \leq k \leq n$
            $Table[i,j] := Table[i,j] \cap (Table[i,k] \circ Table[k,j])$
    until no entry in $Table$ is changed

$\Rightarrow$ needs $O(n^5)$ intersections and compositions.
An $O(n^3)$ algorithm

**EnforcePathConsistency2(\mathcal{C})**

*Input:* a (binary) CSP $\mathcal{C} = \langle V, D, C \rangle$

*Output:* an equivalent, but path consistent CSP $\mathcal{C}'$

$Paths(i,j) = \{(i,j,k) : 1 \leq k \leq n\} \cup \{(k,i,j) : 1 \leq k \leq n\}$

$Queue := \bigcup_{i,j} Paths(i,j)$

**while** $Queue \neq \emptyset$

select and delete $(i,k,j)$ from $Queue$

$T := Table[i,j] \cap (Table[i,k] \circ Table[k,j])$

**if** $T \neq Table[i,j]$

$Table[i,j] := T$

$Table[j,i] := T^{-1}$

$Queue := Queue \cup Paths(i,j)$
Example for incompleteness

- Allen's Interval Calculus
- Reasoning in Allen's Interval Calculus
- Enforcing path consistency
- NP-Hardness Example
- The Continuous Endpoint Class
- Completeness for the CEP Class
- A Maximal Tractable Sub-Algebra
- Literature

\[
\begin{align*}
D &\rightarrow A \\
A &\rightarrow B \\
B &\rightarrow C \\
C &\rightarrow D
\end{align*}
\]
NP-hardness

Theorem (Kautz & Vilain)

CSAT is NP-hard for Allen’s interval calculus.

Proof.

Reduction from 3-colorability (original proof using 3Sat).
Let $G = (V, E)$, $V = \{v_1, \ldots, v_n\}$ be an instance of 3-colorability.
Then we use the intervals $\{v_1, \ldots, v_n, 1, 2, 3\}$ with the following constraints:

1. $\{m\}$
2. $\{m\}$
3. $\{m, \equiv, m^{-1}\}$
$v_i$ $\equiv (m, m^{-1}, \prec, \succ)$

This constraint system is satisfiable iff $G$ can be colored with 3 colors.
Looking for special cases

- **Idea**: Let us look for polynomial special cases. In particular, let us look for sets of relations (subsets of the entire set of relations) that have an easy CSAT problem.

- **Note**: Interval formulae $X R Y$ can be expressed as **clauses** over **atoms** of the form $a \ op b$, where:
  - $a$ and $b$ are endpoints $X^-, X^+, Y^-$ and $Y^+$ and
  - $\ op \in \{<,>,=,\leq,\geq\}$.

- **Example**: All base relations can be expressed as unit clauses.

**Lemma**

*Let $\pi(\Theta)$ be the translation of $\Theta$ to clause form. $\Theta$ is satisfiable over intervals iff $\pi(\Theta)$ is satisfiable over the rational numbers.*
The Continuous Endpoint Class

Continuous Endpoint Class $C$: This is a subset of $A$ such that there exists a clause form for each relation containing only unit clauses where $\neg(a = b)$ is forbidden.

Example: All basic relations and $\{d, o, s\}$, because

$$\pi(X \{d, o, s\} Y) = \{ X^- < X^+, Y^- < Y^+, X^- < Y^+, X^+ > Y^-, X^+ < Y^+ \}$$
Why do we have completeness?

The set $\mathcal{C}$ is closed under intersection, composition, and converse (it is a sub-algebra wrt. these three operations on relations). This can be shown by using a computer program.

**Lemma**

*Each 3-consistent interval CSP over $\mathcal{C}$ is globally consistent.*

**Theorem (van Beek)**

*Path consistency solves CMIN($\mathcal{C}$) and decides CSAT($\mathcal{C}$).*

(Proof: Follows from the above lemma and the fact that a strongly $n$-consistent CSP is minimal.)

**Corollary**

*A path consistent interval CSP consisting of base relations only is satisfiable.*
Helly’s Theorem

Definition

A set \( M \subseteq \mathbb{R}^n \) is convex iff for all pairs of points \( a, b \in M \), all points on the line connecting \( a \) and \( b \) belong to \( M \).

Theorem (Helly)

Let \( F \) be a finite family of at least \( n + 1 \) convex sets in \( \mathbb{R}^n \). If all sub-families of \( F \) with \( n + 1 \) sets have a non-empty intersection, then \( \bigcap F \neq \emptyset \).
Strong $n$-consistency (1)

Proof (part 1).

We prove the claim by induction over $k$ with $k \leq n$.

Base case: $k = 1, 2, 3$ \checkmark

Induction assumption: Assume strong $(k - 1)$-consistency (and non-emptiness of all relations)

Induction step: From the assumption, it follows that there is an instantiation of $k - 1$ variables $X_i$ to pairs $(s_i, e_i)$ satisfying the constraints $R_{ij}$ between the $k - 1$ variables.

We have to show that we can extend the instantiation to any $k$th variable.
Strong \( n \)-consistency (2): Instantiating the \( k \)th variable

**Proof (part 2).**

The instantiation of the \( k - 1 \) variables \( X_i \) to \((s_i, e_i)\) restricts the instantiation of \( X_k \).

**Note:** Since \( R_{ij} \in \mathcal{C} \) by assumption, these restrictions can be expressed by inequalities of the form:

\[
s_i < X_k^+ \land e_j \geq X_k^- \land \ldots
\]

Such inequalities define convex subsets in \( \mathbb{R}^2 \).

\( \implies \) Consider sets of 3 inequalities (= 3 convex sets).
Strong $n$-consistency (3): Using Helly’s Theorem

Proof (part 3).

Case 1: All 3 inequalities mention only $X_k^-$ (or mention only $X_k^+$). Then it suffices to consider only 2 of these inequalities (the strongest). Because of 3-consistency, there exists at least 1 common point satisfying these 2 inequalities.

Case 2: The inequalities mention $X_k^-$ and $X_k^+$, but do not contain the inequality $X_k^- < X_k^+$. Then there are at most 2 inequalities with the same variable and we have the same situation as in Case 1.

Case 3: The set contains the inequality $X_k^- < X_k^+$. In this case, only three intervals (incl. $X_k$) can be involved and by 3-consistency there exists a common point.

$\Rightarrow$ With Helly’s Theorem, there exists an instantiation consistent with all inequalities.

$\Rightarrow$ Strong $k$-consistency for all $k \leq n$.  ■
Outlook

- CMIN(\(C\)) can be computed in \(O(n^3)\) time (for \(n\) being the number of intervals) using the path consistency algorithm.

- \(C\) is a set of relations occurring “naturally” when observations are uncertain.

- \(C\) contains 83 relations (incl. the impossible and the universal relations).

- Are there larger sets such that path consistency computes minimal CSPs? Probably not.

- Are there larger sets of relations that permit polynomial satisfiability testing? Yes.
3 A Maximal Tractable Sub-Algebra

- The Endpoint Subclass
- The ORD-Horn Subclass
- Maximality
- Solving Arbitrary Allen CSPs
The EP-subclass

End-Point Subclass: $\mathcal{P} \subseteq \mathcal{A}$ is the subclass that permits a clause form containing only unit clauses ($a \neq b$ is allowed).

Example: all basic relations and $\{d, o\}$ since

$$\pi(X \{d, o\} Y) = \{ X^- < X^+, Y^- < Y^+, X^- < Y^+, X^+ > Y^-, X^- \neq Y^-, X^+ < Y^+ \}$$

Theorem (Vilain & Kautz 86, Ladkin & Maddux 88)

Enforcing path consistency decides $\text{CSAT}(\mathcal{P})$. 
The ORD-Horn Subclass

ORD-Horn Subclass: $\mathcal{H} \subseteq \mathcal{A}$ is the subclass that permits a clause form containing only Horn clauses where only the following literals are allowed:

$$a \leq b, a = b, a \neq b$$

$\neg a \leq b$ is not allowed!

Example: all $R \in \mathcal{P}$ and $\{o, s, f^{-1}\}$:

$$\pi(X\{o, s, f^{-1}\}Y) = \left\{\begin{array}{l}
x^- \leq x^+, x^- \neq x^+, \\
y^- \leq y^+, y^- \neq y^+, \\
x^- \leq y^-, \\
x^- \leq y^+, x^- \neq y^+, \\
y^- \leq x^+, x^+ \neq y^-, \\
x^+ \leq y^+, \\
x^- \neq y^- \lor x^+ \neq y^+ \end{array}\right\}. $$
Partial orders: The \textit{ORD} theory

Let \textit{ORD} be the following theory:

\begin{align*}
\forall x, y, z : & \quad x \leq y \land y \leq z \rightarrow x \leq z \quad \text{(transitivity)} \\
\forall x : & \quad x \leq x \quad \text{(reflexivity)} \\
\forall x, y : & \quad x \leq y \land y \leq x \rightarrow x = y \quad \text{(anti-symmetry)} \\
\forall x, y : & \quad x = y \rightarrow x \leq y \quad \text{(weakening of =)} \\
\forall x, y : & \quad x = y \rightarrow y \leq x \quad \text{(weakening of =).}
\end{align*}

\begin{itemize}
  \item \textit{ORD} describes partially ordered sets, \( \leq \) being the ordering relation.
  \item \textit{ORD} is a Horn theory
  \item What is missing wrt. dense and linear orders?
\end{itemize}
Satisfiability over partial orders

Proposition

Let \( \Theta \) be a CSP over \( \mathcal{H} \). \( \Theta \) is satisfiable over interval interpretations iff \( \pi(\Theta) \cup \text{ORD} \) is satisfiable over arbitrary interpretations.

Proof.

\( \Rightarrow \): Since the reals form a partially ordered set (i.e., satisfy \( \text{ORD} \)), this direction is trivial.

\( \Leftarrow \): Each extension of a partial order to a linear order satisfies all formulae of the form \( a \leq b \), \( a = b \), and \( a \neq b \) which have been satisfied over the original partial order.
Complexity of CSAT(\(\mathcal{H}\))

Let \(\text{ORD}_{\pi(\Theta)}\) be the propositional theory resulting from instantiating all axioms with the endpoints occurring in \(\pi(\Theta)\).

**Proposition**

\(\text{ORD} \cup \pi(\Theta)\) is satisfiable iff \(\text{ORD}_{\pi(\Theta)} \cup \pi(\Theta)\) is so.

**Proof idea:** Herbrand expansion!

**Theorem**

CSAT(\(\mathcal{H}\)) can be decided in polynomial time.

**Proof.**

CSAT(\(\mathcal{H}\)) instances can be translated into a propositional Horn theory with blowup \(O(n^3)\) according to the previous Prop., and such a theory is decidable in polynomial time.

\(\mathcal{C} \subset \mathcal{P} \subset \mathcal{H}\) with \(|\mathcal{C}| = 83, |\mathcal{P}| = 188, |\mathcal{H}| = 868\)
Path consistency and the OH-class

Lemma

Let $\Theta$ be a path-consistent set over $\mathcal{H}$. Then

$$(X\{\} Y) \notin \Theta \iff \Theta \text{ is satisfiable}$$

Proof idea: One can show that $ORD_\pi(\Theta) \cup \pi(\Theta)$ is closed wrt. positive unit resolution. Since this inference rule is refutation complete for Horn theories, the claim follows.

Theorem

Enforcing path consistency decides CSAT($\mathcal{H}$).

Maximality of $\mathcal{H}$?

Do we have to check all $8192 - 868$ extensions?
Complexity of sub-algebras

Let \( \hat{S} \) be the closure of \( S \subseteq \mathcal{A} \) under converse, intersection, and composition (i.e., the carrier of the least sub-algebra generated by \( S \)).

**Theorem**

\( \text{CSAT}(\hat{S}) \) can be polynomially transformed to \( \text{CSAT}(S) \).

**Proof Idea.**

All relations in \( \hat{S} - S \) can be modeled by a fixed number of compositions, intersections, and conversions of relations in \( S \), introducing perhaps some fresh variables.

\( \Rightarrow \) Polynomiality of \( S \) extends to \( \hat{S} \).

\( \Rightarrow \) NP-hardness of \( \hat{S} \) is inherited by all generating sets \( S \).

\( \Rightarrow \) Note: \( \mathcal{H} = \hat{\mathcal{H}} \).
Minimal extensions of the $\mathcal{H}$-subclass

A computer-aided case analysis leads to the following result:

**Lemma**

There are only two minimal sub-algebras that strictly contain $\mathcal{H}$: $\mathcal{X}_1, \mathcal{X}_2$

$$N_1 = \{d, d^{-1}, o^{-1}, s^{-1}, f\} \subseteq \mathcal{X}_1$$

$$N_2 = \{d^{-1}, o, o^{-1}, s^{-1}, f^{-1}\} \subseteq \mathcal{X}_2$$

The clause form of these relations contain “proper” disjunctions!

**Theorem**

$CSAT(\mathcal{H} \cup \{N_i\})$ is NP-complete.

**Question:** Are there other maximal tractable subclasses?
“Interesting” subclasses

Interesting subclasses of $\mathcal{A}$ should contain all basic relations.

A computer-aided case analysis reveals:
For $S \supseteq \{ \{B\} : B \in \mathcal{B} \}$ it holds that

1. $\hat{S} \subseteq \mathcal{H}$, or
2. $N_1$ or $N_2$ is in $\hat{S}$.

In case 2, one can show: CSAT($S$) is NP-complete.

$\implies$ $\mathcal{H}$ is the only interesting maximal tractable subclass.

If we include non-interesting subalgebras, there exist exactly 18 tractable classes.
Relevance?

Theory: ⊕ We now know the boundary between polynomial and NP-hard reasoning problems along the dimension expressiveness.

Practice: ⊗ All known applications either need only $\mathcal{P}$ or they need more than $\mathcal{H}$!

Backtracking methods might profit from the result by reducing the branching factor.

⇝ How difficult is CSAT($\mathcal{A}$) in practice?

⇝ What are the relevant branching factors?
Solving general Allen CSPs

- Backtracking algorithm using **path consistency** as a forward-checking method
- Relies on tractable fragments of Allen’s calculus: split relations into relations of a tractable fragment, and backtrack over these.
- Refinements and evaluation of different heuristics
  - Which tractable fragment should one use?
Branching factors

- If the labels are split into **base relations**, then on average a label is split into **6.5 relations**.

- If the labels are split into **pointizable relations** (\(P\)), then on average a label is split into **2.955 relations**.

- If the labels are split into **ORD-Horn relations** (\(H\)), then on average a label is split into **2.533 relations**.

\[\Rightarrow\] A difference of **0.422**

\[\Rightarrow\] This makes a difference for “hard” instances.
Summary

- Allen’s interval calculus is often adequate for describing relative orders of events that have duration.
- The satisfiability problem for CSPs using the relations is NP-complete.
- For the **continuous endpoint class**, minimal CSPs can be computed using the path-consistency method.
- For the larger **ORD-Horn class**, CSAT is still decided by the path-consistency method.
- Can be used in practice for backtracking algorithms.
4 Literature
Literature I

J. F. Allen.
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efficiency of using the ORD-Horn class.
A complete classification of complexity in Allen’s algebra in the presence of a non-trivial basic relation.

Reasoning about Temporal Relations: The Tractable Subalgebras of Allen’s Interval Algebra.